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OSCILLATION CRITERIA FOR ODD-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH ADVANCED AND DELAYED ARGUMENTS

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Abstract. This article presents oscillation criteria for n-th order nonlinear neutral mixed type differential equations of the form

$$\big((x(t)+ax(t-\tau_1)-bx(t+\tau_2))^\alpha\big)^{(n)} = q(t)x^\beta(t-\sigma_1)+p(t)x^\gamma(t+\sigma_2),$$

$$\big((x(t)-ax(t-\tau_1)+bx(t+\tau_2))^\alpha\big)^{(n)} = q(t)x^\beta(t-\sigma_1)+p(t)x^\gamma(t+\sigma_2),$$

$$\big((x(t)+ax(t-\tau_1)+bx(t+\tau_2))^\alpha\big)^{(n)} = q(t)x^\beta(t-\sigma_1)+p(t)x^\gamma(t+\sigma_2)$$

where n is an odd positive integer, a and b are nonnegative constants, τ_1,τ_2,σ_1 and σ_2 are positive real constants, $q(t),p(t)\in C([t_0,\infty),(0,\infty))$ and α,β and γ are ratios of odd positive integers with $\beta,\gamma\geq 1$. Some examples are provided to illustrate the main results.

1. Introduction

In this article, we study the oscillatory behavior of all solutions of n-th order nonlinear neutral differential equations of the forms

$$((x(t)+ax(t-\tau_1)-bx(t+\tau_2))^\alpha)^{(n)} = q(t)x^\beta(t-\sigma_1)+p(t)x^\gamma(t+\sigma_2), \quad (1.1)$$

$$((x(t)-ax(t-\tau_1)+bx(t+\tau_2))^\alpha)^{(n)} = q(t)x^\beta(t-\sigma_1)+p(t)x^\gamma(t+\sigma_2), \quad (1.2)$$

$$((x(t)+ax(t-\tau_1)+bx(t+\tau_2))^\alpha)^{(n)} = q(t)x^\beta(t-\sigma_1)+p(t)x^\gamma(t+\sigma_2) \quad (1.3)$$

where n is an odd positive integer, a and b are nonnegative constants, τ_1,τ_2,σ_1 and σ_2 are positive real constants, $q(t),p(t)\in C([t_0,\infty),(0,\infty))$ and α,β and γ are ratios of odd positive integers with $\beta,\gamma\geq 1$.

As is customary, a solution is called oscillatory if it has arbitrarily large zeros and non-oscillatory if it is eventually positive or eventually negative. Equations (1.1), (1.2) and (1.3) are called oscillatory if all its solutions are oscillatory.

Differential equations with advanced and delayed arguments (also called mixed differential equations or equations with mixed arguments) occur in many problems of economy, biology and physics (see for example [3, 7, 11, 12, 19]), because differential equations with mixed arguments are much more suitable than delay differential equations for an adequate treatment of dynamic phenomena. The concept of delay

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is related to a memory of system, the past events are importance for the current behavior, and the concept of advance is related to a potential future events which can be known at the current time which could be useful for decision making. The study of various problems for differential equations with mixed arguments can be seen in [4, 9, 18, 22, 23, 27].

It is well known that the solutions of some of these equations cannot be obtained in closed form. In the absence of closed form solutions a rewarding alternative is to resort to the qualitative study of the solutions of these types of differential equations. But it is not quite clear how to formulate an initial value problem for such equations and existence and uniqueness of solutions becomes a complicated issue. To study the oscillation of solutions of differential equations, we need to assume that there exists a solution of such equation on the half line.

The problem of asymptotic and oscillatory behavior of solutions of n-th order delay and neutral type differential equations has received great attention in recent years see for example [1]–[32], and the references cited therein. However, there are few results regarding the oscillatory properties of neutral differential equations with mixed arguments.

In [25] the author has obtained some oscillation theorems for the odd order neutral differential equation

$$\big(x(t)+p_1x(t-\tau_1)+p_2x(t+\tau_2)\big)^{(n)}=q_1x(t-\sigma_1)+q_2x(t+\sigma_2), t\geq t_0, \tag{1.4}$$

where $n\geq 1$ is odd.

In [16] the authors established some oscillation criteria for the following neutral equations

$$(x(t)+cx(t-h)-c^*x(t+h^*))^{(n)}=qx(t-g)+px(t+g^*), \tag{1.5}$$
$$(x(t)-cx(t-h)+c^*x(t+h^*))^{(n)}=qx(t-g)+px(t+g^*), \tag{1.6}$$
$$(x(t)+cx(t-h)-c^*x(t-h^*))^{(n)}=qx(t-g)+px(t+g^*), \tag{1.7}$$
$$(x(t)+cx(t+h)-c^*x(t+h^*))^{(n)}=qx(t-g)+px(t+g^*), \tag{1.8}$$

where $t\geq t_0$ and n is an odd positive integer, c, c^*, h, h^*, p and q are real numbers and g and g^* are positive constants.

In [30] the author has obtained some oscillation results for third-order nonlinear neutral differential equation

$$\big((x(t)+b(t)x(t-\tau_1)+c(t)x(t+\tau_2))^{\alpha}\big)'''=q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\gamma}(t+\sigma_2), \tag{1.9}$$

for $t\geq t_0$, where α,β and γ are ratios of odd positive integers, τ_1,τ_2,σ_1 and σ_2 are positive constants.

Clearly equations (1.5) and (1.6) with $\alpha=\beta=\gamma=1$ and $q(t)=q$, $p(t)=p$ are special cases of equations (1.1) and (1.2). Moreover equation (1.9) with $n=3$ is special case of equation (1.3). Motivated by the above observations in this paper we study the oscillatory behavior of equations (1.1),(1.2) and (1.3) for different values of $\beta\geq 1$ and $\gamma\geq 1$.

In Section 2 we present some lemmas which are useful for our main results. In Section 3, we present some sufficient conditions for the oscillation of all solutions of equations (1.1),(1.2) and (1.3). Examples are provided in Section 4 to illustrate the main results.

2. Some preliminary lemmas

In this section we state the following lemmas which are essential in the proofs of our oscillation theorems.

Lemma 2.1 ([20]). *Let $x(t)$ be a function such that it and each of its derivative up to order $(n-1)$ inclusive are absolutely continuous and of constant sign in an interval (t_0,∞). If $x^{(n)}(t)$ is of constant sign and not identically zero on any interval of the form $[t_1,\infty)$ for some $t_1 \geq t_0$, then there exists a $t_x \geq t_0$ and an integer m, $0 \leq m \leq n$ with $n+m$ even for $x^{(n)} > 0$, or $n+m$ odd for $x^{(n)} \leq 0$, and such that for every $t \geq t_x$,*

$$m > 0 \text{ implies } x^k(t) > 0 \text{ for } k = 0,1,\dots,m-1; \text{and}$$

$$m \leq n-1 \text{ implies } (-1)^{m+k}x^{(k)}(t) > 0 \text{ for } k = m, m+1, \dots, n-1.$$

Lemma 2.2 ([1, Lemma 2.2.2]). *If $x(t)$ is as in Lemma 2.1 and $x^{(n-1)}(t)x^{(n)}(t) \leq 0$ for all $t \geq t_x$, then for every λ, $0<\lambda<1$, there exists a constant $M>0$ such that*

$$|x(\lambda t)| \geq M t^{n-1} |x^{(n-1)}(t)|$$

for all large t.

Lemma 2.3 ([26]). *Let $x(t)$ be a function as in Lemma 2.2. If $\lim_{t\to\infty} x(t) \neq 0$, then for every $\lambda \in (0,1)$,*

$$x(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} x^{(n-1)}(t)$$

for all large t.

Lemma 2.4. *Let $A \geq 0$, $B \geq 0$ and $\gamma \geq 1$. Then*

$$A^\gamma + B^\gamma \geq \frac{1}{2^{\gamma-1}}(A+B)^\gamma.$$

If $A \geq B$, then $A^\gamma - B^\gamma \geq (A-B)^\gamma$.

A proof of the above lemma can be found in [29].

Lemma 2.5 ([21]). *Suppose $q : [t_0,\infty) \to \mathbb{R}$ is a continuous and eventually non-negative function, and σ is a positive real number. Then the following hold.*

(I) If

$$\limsup_{t\to\infty} \int_t^{t+\sigma} \frac{(s-t)^i (t-s+\sigma)^{n-i-1}}{i!(n-i-1)!} q(s)ds > 1,$$

hold for some $i = 0,1,\dots,n-1$, then the inequality

$$y^{(n)}(t) \geq q(t) y(t+\sigma)$$

has no eventually positive solution $y(t)$ which satisfies $y^{(j)}(t) > 0$ eventually, $j = 0,1,\dots,n$.

(II) If

$$\limsup_{t\to\infty} \int_{t-\sigma}^{t} \frac{(t-s)^i (s-t+\sigma)^{n-i-1}}{i!(n-i-1)!} q(s)ds > 1,$$

hold for some of $i = 0,1,\dots,n-1$, then the inequality

$$(-1)^n z^{(n)}(t) \geq q(t) z(t-\sigma)$$

has no eventually positive solution $z(t)$ which satisfies $(-1)^j z^{(j)}(t) > 0$ eventually, $j = 0,1,\dots,n$.

Lemma 2.6 ([31]). *Assume that for large t,*

$$q(s) \neq 0 \quad \text{for all } s \in [t, t^*],$$

where t^* satisfies $\sigma(t^*) = t$. Then

$$x'(t) + q(t)[x(\sigma(t))]^\alpha = 0, \quad t \geq t_0,$$

has an eventually positive solution if and only if the corresponding inequality

$$x'(t) + q(t)[x(\sigma(t))]^\alpha \leq 0, \quad t \geq t_0,$$

has an eventually positive solution.

In [8, 13, 23, 32], the authors investigated the oscillatory behavior of solutions to

$$x'(t) + q(t)[x(\sigma(t))]^\alpha = 0, \quad t \geq t_0, \tag{2.1}$$

where $q \in C([t_0, \infty), \mathbb{R}^+)$, $\sigma \in C([t_0, \infty), \mathbb{R})$, $\sigma(t) < t$, $\lim_{t\to\infty} \sigma(t) = \infty$ and $\alpha \in (0, \infty)$ is a ratio of odd positive integers.

Let $\alpha = 1$. Then (2.1) reduces to the linear delay differential equation

$$x'(t) + q(t)x(\sigma(t)) = 0, \quad t \geq t_0, \tag{2.2}$$

and it is shown that every solution of equation (2.2) oscillates if

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} q(s)ds > \frac{1}{e}. \tag{2.3}$$

3. Oscillation results

In this section we shall obtain some sufficient conditions for the oscillation of all solutions of (1.1), (1.2) and (1.3). First we study the oscillation of all solutions of equation (1.1).

Theorem 3.1. *Assume that*

$$\int_{t_0}^{+\infty} (q(t) + p(t))\, dt = +\infty$$

hold, and $\tau_2 > \sigma_2, (1 + a^\beta) > 0, a, b \leq 1$, and $1 \leq \beta \leq \gamma$, and $q(t)$ and $p(t)$ are positive and non-increasing functions for $t \geq t_0$. If the differential inequalities either

$$y^{(n)}(t) + \frac{q(t)}{b^\beta} y^{\beta/\alpha}(t - \sigma_1 - \tau_2) \leq 0, \tag{3.1}$$

or

$$y^{(n)}(t) + \frac{q(t) + p(t)}{b^\gamma} y^{\beta/\alpha}(t - (\tau_2 - \sigma_2)) \leq 0 \tag{3.2}$$

and

$$y^{(n)}(t) - \frac{p(t)}{2^{\gamma-1}(1 + a^\beta)^{\gamma/\alpha}} y^{\gamma/\alpha}(t + \sigma_2) \geq 0, \tag{3.3}$$

have no eventually positive solution and no eventually positive increasing solution respectively then every solution of equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of (1.1). Without loss of generality we may assume that $x(t)$ is eventually positive; i.e., there exists a $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Set

$$z(t) = (x(t) + ax(t-\tau_1) - bx(t+\tau_2))^{\alpha}.$$

Then

$$z^{(n)}(t) = q(t)x^{\beta}(t-\sigma_1) + p(t)x^{\gamma}(t+\sigma_2) > 0 \quad \text{for all } t \geq t_1 \geq t_0. \tag{3.4}$$

Thus $z^{(i)}(t)$, $i = 0, 1, \dots, n$, are of one sign on $[t_2, \infty); t_2 \geq t_1$. There are two possibilities: (a) $z(t) < 0$ for $t \geq t_2$, (b) $z(t) > 0$ for $t \geq t_2$.

Case 1: Assume $z(t) < 0$ for $t \geq t_2$. In this case, we let

$$0 < v(t) = -z(t) = (bx(t+\tau_2) - ax(t-\tau_1) - x(t))^{\alpha} \leq b^{\alpha}x^{\alpha}(t+\tau_2).$$

Then in view of the last inequality, we obtain

$$x(t) \geq \frac{1}{b}v^{1/\alpha}(t-\tau_2) \quad \text{for } t \geq t^* \geq t_2. \tag{3.5}$$

Thus by (3.4) and (3.5),

$$v^{(n)}(t) + \frac{q(t)}{b^{\beta}}v^{\beta/\alpha}(t-\sigma_1-\tau_2) + \frac{p(t)}{b^{\gamma}}v^{\gamma/\alpha}(t+\sigma_2-\tau_2) \leq 0, \quad t \geq t^*. \tag{3.6}$$

By Lemma 2.1, it is easy to check that there exists a $T_0 \geq t^*$ such that $v^{(n-1)}(t) > 0$ for $t \geq T_0$. Now, if $v'(t) > 0$ for $t \geq T_0$ then there exist a constant $k > 0$ and a $T \geq T_0$ such that

$$v(t-\sigma_1-\tau_2) \geq k, \quad v(t+\sigma_2-\tau_2) \geq k \quad \text{for } t \geq T.$$

Thus

$$v^{(n)}(t) \leq -k^{\beta/\alpha}\frac{p(t)+q(t)}{b^{\gamma}}, \quad \text{for } t \geq T,$$

and hence

$$0 < v^{(n-1)}(t) \leq v^{(n-1)}(T) - \frac{k^{\beta/\alpha}}{b^{\gamma}}\int_T^t (p(s)+q(s))ds \to -\infty \quad \text{as } t \to \infty,$$

a contradiction. Thus, $v'(t) < 0$ for $t \geq T$ and the function satisfies $(-1)^i v^{(i)}(t) > 0$ eventually for $i = 0, 1, \dots, n$ and $t \geq T$. From (3.6), we have either

$$v^{(n)}(t) + \frac{q(t)}{b^{\beta}}v^{\beta/\alpha}(t-\sigma_1-\tau_2) \leq 0, \quad t \geq T$$

or

$$v^{(n)}(t) + \frac{q(t)+p(t)}{b^{\gamma}}v^{\beta/\alpha}(t-(\tau_2-\sigma_2)) \leq 0, \quad t \geq T,$$

has a positive solution, which is a contradiction.

Case 2: Assume $z(t) > 0$ for $t \geq t_2$. By the Lemma 2.1, there exists a $t_3 \geq t_2$ such that $z'(t) > 0$ for $t \geq t_3$. Next, we let

$$y(t) = z(t) + a^{\beta}z(t-\tau_1) - \frac{b^{\gamma}}{2^{\gamma-1}}z(t+\tau_2), \quad t \geq t_3. \tag{3.7}$$

Then

$$\begin{aligned} y^{(n)}(t) &= z^{(n)}(t) + a^{\beta}z^{(n)}(t-\tau_1) - \frac{b^{\gamma}}{2^{\gamma-1}}z^{(n)}(t+\tau_2) \\ &= q(t)x^{\beta}(t-\sigma_1) + p(t)x^{\gamma}(t+\sigma_2) + a^{\beta}\Big(q(t-\tau_1)x^{\beta}(t-\sigma_1-\tau_1) \end{aligned}$$

$$+ p(t-\tau_1)x^\gamma(t+\sigma_2-\tau_1)\Big) - \frac{b^\gamma}{2^{\gamma-1}}\Big(q(t+\tau_2)x^\beta(t-\sigma_1+\tau_2) + p(t+\tau_2)x^\gamma(t+\sigma_2+\tau_2)\Big).$$

Using the monotonicity of $q(t)$ and $p(t)$, $a, b \le 1, 1 \le \beta \le \gamma$ and Lemma 2.4 in the above inequality, we obtain

$$y^{(n)}(t) \ge \frac{q(t)}{2^{\beta-1}}\left(x(t-\sigma_1) + ax(t-\sigma_1-\tau_1) - bx(t-\sigma_1+\tau_2)\right)^\beta + \frac{p(t)}{2^{\gamma-1}}\left(x(t+\sigma_2) + ax(t+\sigma_2-\tau_1) - bx(t+\sigma_2+\tau_2)\right)^\gamma.$$

Now using $z(t) > 0$ for $t \ge t_2$ in the above inequality, we obtain

$$y^{(n)}(t) \ge \frac{q(t)}{2^{\beta-1}}z^{\beta/\alpha}(t-\sigma_1) + \frac{p(t)}{2^{\gamma-1}}z^{\gamma/\alpha}(t+\sigma_2) > 0, \quad t \ge t_3. \tag{3.8}$$

If $y(t) < 0$ eventually, we can get same conclusion as in Case 1. Thus we observe that $y(t) > 0$ eventually. Now, if $z'(t) > 0$ eventually for $t \ge t_2$ then there exist a positive constant c and a $T \ge t_2$ such that, $z(t-\sigma_1) \ge c$, $z(t+\sigma_2) \ge c$. Thus using last inequality in (3.8), we obtain

$$y^{(n)}(t) \ge \frac{q(t)}{2^{\beta-1}}c^{\beta/\alpha} + \frac{p(t)}{2^{\gamma-1}}c^{\gamma/\alpha} > 0.$$

Then $y^{(n-1)}(t) \to \infty$ and $y^{(i)}(t) \to \infty$ for $i = 0, 1, \dots, n-2$ as $t \to \infty$. Therefore, one can conclude that

$$y^{(i)}(t) > 0 \quad \text{eventually for } i = 0, 1, \dots, n. \tag{3.9}$$

Now, using the monotonicity of $z(t)$, we obtain

$$y(t) = z(t) + a^\beta z(t-\tau_1) - \frac{b^\gamma}{2^{\gamma-1}}z(t+\tau_2) \le (1+a^\beta)z(t).$$

then from the above inequality and (3.8), we have

$$y^{(n)}(t) \ge \frac{p(t)}{2^{\gamma-1}(1+a^\beta)^{\gamma/\alpha}}y^{\gamma/\alpha}(t+\sigma_2), \quad t \ge t_3. \tag{3.10}$$

This inequality admits a solution that satisfies (3.9), thus $y(t)$ is a positive increasing solution of the inequality (3.3), which is a contradiction. The proof is now complete. □

Corollary 3.2. *Assume that*

$$\int_{t_0}^{+\infty} (q(t) + p(t))\, dt = +\infty$$

hold, and $\tau_2 > \sigma_2, (1+a^\alpha) > 0, a, b \le 1$ *and* $\alpha = \beta = \gamma \ge 1$ *and* $q(t)$ *and* $p(t)$ *are non-increasing functions for* $t \ge t_0$. *If either*

$$\liminf_{t\to\infty} \int_{t-(\sigma_1+\tau_2)}^{t} (s-\sigma_1-\tau_2)^{n-1}q(s)ds > \frac{b^\alpha(n-1)!}{\lambda e}, \quad \lambda \in (0,1), \tag{3.11}$$

or

$$\liminf_{t\to\infty} \int_{t-\tau_2+\sigma_2}^{t} (s-\tau_2+\sigma_2)^{n-1}(p(s)+q(s))ds > \frac{b^\alpha(n-1)!}{\lambda e}, \quad \lambda \in (0,1), \tag{3.12}$$

and

$$\limsup_{t\to\infty}\int_t^{t+\sigma_2}\frac{(s-t)^i(t-s+\sigma_2)^{n-i-1}}{i!(n-i-1)!}p(s)ds>2^{\alpha-1}(1+a^\alpha),\quad i=0,1,\dots,n-1, \tag{3.13}$$

then every solution of (1.1) *is oscillatory.*

Proof. Let $y(t)$ be a positive solution of (3.2), for $t\ge t_1\ge t_0$. Then we have $y^{(n)}(t)\le 0$ for all $t\ge t_1$. More over, $(-1)^iy^{(i)}(t)>0$ for $i=1,2,\dots,n$ for all $t\ge t_1$. Then from Lemma 2.3 we obtain

$$y(t)\ge\frac{\lambda}{(n-1)!}t^{n-1}y^{(n-1)}(t),\quad \lambda\in(0,1).$$

From (3.2), we have

$$y^{(n)}(t)+\frac{p(t)+q(t)}{b^\alpha}y(t-\tau_2+\sigma_2)\le 0,\quad t\ge t_2.$$

Combining the last two inequalities, we obtain

$$y^{(n)}(t)+(p(t)+q(t))\frac{\lambda}{b^\alpha(n-1)!}(t-\tau_2+\sigma_2)^{n-1}y^{(n-1)}(t-\tau_2+\sigma_2)\le 0,\quad t\ge t_2.$$

Let $w(t)=y^{(n-1)}(t)$. Then we see that $w(t)$ is a positive solution of

$$w'(t)+(p(t)+q(t))\frac{\lambda}{b^\alpha(n-1)!}(t-\tau_2+\sigma_2)^{n-1}w(t-\tau_2+\sigma_2)\le 0,\quad t\ge t_2 \tag{3.14}$$

But according to the Lemma 2.6 and the condition (2.3), condition (3.12) guarantees that inequality (3.14) has no positive solution, which is a contradiction. Hence (3.2) has no eventually positive solution. Moreover condition (3.11) is sufficient for the inequality (3.1) has no eventually positive solution,which is a contradiction. Moreover in view of Lemma 2.5 (I) and the condition (3.13), inequality (3.10) has no eventually positive solution which satisfies (3.9), which is a contradiction. Hence (3.3) has no eventually positive increasing solution. □

Next we consider (1.2), and present sufficient conditions for the oscillation of all solutions.

Theorem 3.3. *Assume that*

$$\int_{t_0}^{+\infty}(q(t)+p(t))\,dt=+\infty$$

hold, and $\sigma_i>\tau_i$ *for* $i=1,2,(1+b^\gamma)>0,a,b\le 1$, *and* $1\le\gamma\le\beta$, *and* $q(t)$ *and* $p(t)$ *are positive and nondecreasing functions for* $t\ge t_0$. *If the differential inequalities*

$$y^{(n)}(t)+\frac{q(t)}{a^\beta}y^{\beta/\alpha}(t-\sigma_1+\tau_1)\le 0, \tag{3.15}$$

and

$$y^{(n)}(t)-\frac{p(t)}{2^{\gamma-1}(1+b^\gamma)^{\gamma/\alpha}}y^{\gamma/\alpha}(t+\sigma_2-\tau_2)\ge 0, \tag{3.16}$$

have no eventually positive solution and no eventually positive increasing solution respectively. Then every solution of equation (1.2) *is oscillatory.*

Proof. Let $x(t)$ be a non-oscillatory solution of (1.2). Without loss of generality we may assume that $x(t)$ is eventually positive; i.e., there exists a $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Set

$$z_1(t) = (x(t) - ax(t-\tau_1) + bx(t+\tau_2))^\alpha$$

and proceeding as in the proof of Theorem 3.1, we that the function $z_1^{(i)}(t)$, $i = 0, 1, \dots, n$, are of one sign on $[t_2, \infty)$, $t_2 \geq t_1$. There are two possibilities: (1) $z_1(t) < 0$ for $t \geq t_2$, (2) $z_1(t) > 0$ for $t \geq t_2$.

Case 1: Assume $z_1(t) < 0$ for $t \geq t_2$. In this case, we let

$$0 < v_1(t) = -z_1(t) = (ax(t-\tau_1) - bx(t+\tau_2) - x(t))^\alpha \leq a^\alpha x^\alpha(t-\tau_1).$$

Then in view of the last inequality, we obtain

$$x(t) \geq \frac{1}{a} v_1^{1/\alpha}(t+\tau_1) \quad \text{for } t \geq t^* \geq t_2. \tag{3.17}$$

Thus by (1.2) and (3.17),

$$v_1^{(n)}(t) + \frac{q(t)}{a^\beta} v_1^{\beta/\alpha}(t-\sigma_1+\tau_1) + \frac{p(t)}{a^\gamma} v_1^{\gamma/\alpha}(t+\sigma_2+\tau_1) \leq 0, \quad t \geq t^*. \tag{3.18}$$

By Lemma 2.1, it is easy to check that there exists a $T_0 \geq t^*$ such that $v_1^{(n-1)}(t) > 0$ for $t \geq T_0$. Now, if $v_1'(t) > 0$ for $t \geq T_0$ then there exist a constant $k_1 > 0$ and a $T \geq T_0$ such that

$$v_1(t-\sigma_1+\tau_1) \geq k_1, \quad v_1(t+\sigma_2+\tau_1) \geq k_1 \quad \text{for } t \geq T.$$

Thus

$$v_1^{(n)}(t) \leq -k_1^{\gamma/\alpha} \frac{p(t)+q(t)}{a^\beta}, \quad \text{for } t \geq T,$$

and hence

$$v_1^{(n-1)}(t) \leq v_1^{(n-1)}(T) - \frac{k_1^{\gamma/\alpha}}{a^\beta} \int_T^t (p(s)+q(s))ds \to -\infty \quad \text{as } t \to \infty,$$

a contradiction. Thus, $v_1'(t) < 0$ for $t \geq T$ and the function satisfies $(-1)^i v_1^{(i)}(t) > 0$ eventually for $i = 0, 1, \dots, n$ and $t \geq T$. From (3.18), we have

$$v_1^{(n)}(t) + \frac{q(t)}{a^\beta} v_1^{\beta/\alpha}(t-\sigma_1+\tau_1) \leq 0, \quad t \geq T,$$

has a positive solution, which is a contradiction.

Case 2: Assume $z_1(t) > 0$ for $t \geq t_2$. By the Lemma 2.1, there exists a $t_3 \geq t_2$ such that $z_1'(t) > 0$ for $t \geq t_3$. Next, we let

$$y_1(t) = z_1(t) - \frac{a^\beta}{2^{\beta-1}} z_1(t-\tau_1) + b^\gamma z_1(t+\tau_2), \quad t \geq t_3. \tag{3.19}$$

Then

$$\begin{aligned}
y_1^{(n)}(t) &= z_1^{(n)}(t) - \frac{a^\beta}{2^{\beta-1}} z_1^{(n)}(t-\tau_1) + b^\gamma z_1^{(n)}(t+\tau_2) \\
&= q(t)x^\beta(t-\sigma_1) + p(t)x^\gamma(t+\sigma_2) - \frac{a^\beta}{2^{\beta-1}}\Big(q(t-\tau_1)x^\beta(t-\sigma_1-\tau_1) \\
&\quad + p(t-\tau_1)x^\gamma(t+\sigma_2-\tau_1)\Big) + b^\gamma\Big(q(t+\tau_2)x^\beta(t-\sigma_1+\tau_2) \\
&\quad + p(t+\tau_2)x^\gamma(t+\sigma_2+\tau_2)\Big).
\end{aligned}$$

Using the monotonicity of $q(t)$ and $p(t)$, $a, b \leq 1, 1 \leq \gamma \leq \beta$ and Lemma 2.4 in the above inequality, we obtain

$$
\begin{aligned}
y_1^{(n)}(t) \geq\ & \frac{q(t)}{2^{\beta-1}} \left(x(t-\sigma_1) - ax(t-\sigma_1-\tau_1) + bx(t-\sigma_1+\tau_2)\right)^{\beta} \\
& + \frac{p(t)}{2^{\gamma-1}} \left(x(t+\sigma_2) - ax(t+\sigma_2-\tau_1) + bx(t+\sigma_2+\tau_2)\right)^{\gamma}.
\end{aligned}
$$

Now using $z_1(t) > 0$ for $t \geq t_2$ in the above inequality, we obtain

$$
y_1^{(n)}(t) \geq \frac{q(t)}{2^{\beta-1}} z_1^{\beta/\alpha}(t-\sigma_1) + \frac{p(t)}{2^{\gamma-1}} z_1^{\gamma/\alpha}(t+\sigma_2) > 0, \quad t \geq t_3. \tag{3.20}
$$

If $y(t) < 0$ eventually, we can get same conclusion as in Case 1. Thus we observe that $y(t) > 0$ eventually. Now, if $z'(t) > 0$ eventually for $t \geq t_2$ then there exist a positive constant c_1 and a $T \geq t_2$ such that, $z(t-\sigma_1) \geq c_1$, $z(t+\sigma_2) \geq c_1$. Thus using last inequality in (3.20), we obtain

$$
y^{(n)}(t) \geq \frac{q(t)}{2^{\beta-1}} c_1^{\beta/\alpha} + \frac{p(t)}{2^{\gamma-1}} c_1^{\gamma/\alpha} > 0.
$$

Then $y^{(n-1)}(t) \to \infty$ and $y^{(i)}(t) \to \infty$ for $i = 0, 1, \dots, n-2$ as $t \to \infty$. Therefore one can conclude that

$$
y_1^{(i)}(t) > 0 \quad \text{eventually for } i = 0, 1, \dots, n,\ t \geq t_3. \tag{3.21}
$$

Now,

$$
y_1(t) = z_1(t) - \frac{a^{\beta}}{2^{\beta-1}} z_1(t-\tau_1) + b^{\gamma} z_1(t+\tau_2) \leq (1+b^{\gamma}) z_1(t+\tau_2).
$$

then from the above inequality and (3.20), we have

$$
y_1^{(n)}(t) \geq \frac{p(t)}{2^{\gamma-1}(1+b^{\gamma})^{\gamma/\alpha}} y_1^{\gamma/\alpha}(t+\sigma_2-\tau_2), \quad t \geq t_3. \tag{3.22}
$$

Inequality (3.22) admits a solution that satisfies (3.21), thus $y_1(t)$ is a positive increasing solution of the inequality (3.16), which is a contradiction. The proof is now complete. $\square$

Corollary 3.4. *Let $\sigma_i > \tau_i$ for $i = 1, 2, (1+b^{\alpha}) > 0, a, b \leq 1$ and $\alpha = \beta = \gamma \geq 1$. If*

$$
\liminf_{t\to\infty} \int_{t-(\sigma_1-\tau_1)}^{t} (s-\sigma_1+\tau_1)^{n-1} q(s)ds > \frac{a^{\alpha}(n-1)!}{\lambda_1 e}, \quad \lambda_1 \in (0,1) \tag{3.23}
$$

and

$$
\limsup_{t\to\infty} \int_{t}^{t+\sigma_2-\tau_2} \frac{(s-t)^i (t-s+\sigma_2-\tau_2)^{n-i-1}}{i!(n-i-1)!} p(s)ds > 2^{\alpha-1}(1+b^{\alpha}), \tag{3.24}
$$

for $i = 0, 1, \dots, n-1$, then every solution of (1.2) *is oscillatory.*

The proof of the above corollary is similar to that of Corollary 3.2 and hence it is omitted. Next we consider equation (1.3) and present sufficient conditions for the oscillation of all solutions.

Theorem 3.5. *Let $\sigma_2 > \tau_2, a \leq 1, b \geq 1$ and $1 \leq \beta \leq \gamma$, and*

$$
\begin{aligned}
Q(t) &= \min\{q(t-\tau_1), q(t), q(t+\tau_2)\}, \\
P(t) &= \min\{p(t-\tau_1), p(t), p(t+\tau_2)\},
\end{aligned}
$$

be positive functions for $t \geq t_0$. If the differential inequality

$$y^{(n)}(t) - \frac{P(t)}{4^{\gamma-1}(1+a^\beta+b^\gamma)^{\gamma/\alpha}} y^{\gamma/\alpha}(t+\sigma_2-\tau_2) \geq 0, \tag{3.25}$$

has no eventually positive solution. Then every solution of (1.3) *is oscillatory.*

Proof. Let $x(t)$ be an eventually positive solution of equation (1.3), then there exists a $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Set

$$z_2(t) = (x(t) + ax(t-\tau_1) + bx(t+\tau_2))^\alpha, \quad t \geq t_1.$$

and proceeding as in the proof of Theorem 3.1, we see that the function $z_2^{(i)}(t), i = 0, 1, \dots, n$ are of one sign on $[t_2, \infty)$, for some $t_2 \geq t_1$. Now we define

$$y_2(t) = z_2(t) + a^\beta z_2(t-\tau_1) + b^\gamma z_2(t+\tau_2), \quad t \geq t_2. \tag{3.26}$$

Then $y_2(t) > 0$ for $t \geq t_2$ and then

$$\begin{aligned} y_2^{(n)}(t) &= z_2^{(n)}(t) + a^\beta z_2^{(n)}(t-\tau_1) + b^\gamma z_2^{(n)}(t+\tau_2) \\ &= q(t)x^\beta(t-\sigma_1) + p(t)x^\gamma(t+\sigma_2) + a^\beta \Big(q(t-\tau_1)x^\beta(t-\sigma_1-\tau_1) \\ &\quad + p(t-\tau_1)x^\gamma(t+\sigma_2-\tau_1)\Big) + b^\gamma \Big(q(t+\tau_2)x^\beta(t-\sigma_1+\tau_2) \\ &\quad + p(t+\tau_2)x^\gamma(t+\sigma_2+\tau_2)\Big). \end{aligned}$$

Since $a \leq 1, b \geq 1, 1 \leq \beta \leq \gamma$ and using Lemma 2.4 in the above inequality, we obtain

$$\begin{aligned} y_2^{(n)}(t) &\geq \frac{Q(t)}{4^{\beta-1}} \left(x(t-\sigma_1) + ax(t-\sigma_1-\tau_1) + bx(t-\sigma_1+\tau_2)\right)^\beta \\ &\quad + \frac{P(t)}{4^{\gamma-1}} \left(x(t+\sigma_2) + ax(t+\sigma_2-\tau_1) + bx(t+\sigma_2+\tau_2)\right)^\gamma. \end{aligned}$$

Now using $z_2(t) > 0$ for $t \geq t_2$ in the above inequality, we obtain

$$y_2^{(n)}(t) \geq \frac{Q(t)}{4^{\beta-1}} z_2^{\beta/\alpha}(t-\sigma_1) + \frac{P(t)}{4^{\gamma-1}} z_2^{\gamma/\alpha}(t+\sigma_2) > 0, \quad t \geq t_2. \tag{3.27}$$

Since $z_2(t) > 0$ and $z_2'(t) > 0$ are eventually positive increasing functions. From (3.26) we see that $y_2(t) > 0$ and $y_2'(t) > 0$ and also from inequality (3.27), $y_2^{(n)}(t) > 0$ for $t \geq t_2$. As a result of this

$$y_2^{(i)}(t) > 0, \quad \text{for } t \geq t_2 \text{ and } i = 0, 1, \dots, n. \tag{3.28}$$

Using the monotonicity of $z_2(t)$, we obtain

$$y_2(t) = z_2(t) + a^\beta z_2(t-\tau_1) + b^\gamma z_2(t+\tau_2) \leq (1+a^\beta+b^\gamma) z_2(t+\tau_2).$$

Then from the above inequality and (3.27), we have

$$\begin{aligned} y_2^{(n)}(t) &\geq \frac{P(t)}{4^{\gamma-1}} z_2^{\gamma/\alpha}(t+\sigma_2) \\ &\geq \frac{P(t)}{4^{\gamma-1}(1+a^\beta+b^\gamma)^{\gamma/\alpha}} y_2^{\gamma/\alpha}(t-\tau_2+\sigma_2), \quad t \geq t_2. \end{aligned}$$

This inequality admits a solution that satisfies (3.28), thus $y_2(t)$ is a positive increasing solution of the inequality (3.25), which is a contradiction. The proof is now complete. $\square$

Corollary 3.6. *Let $\sigma_2 > \tau_2, a \le 1, b \ge 1$ and $\alpha = \beta = \gamma \ge 1$. If*

$$\limsup_{t\to\infty} \int_t^{t+\sigma_2-\tau_2} \frac{(s-t)^i(t-s+\sigma_2-\tau_2)^{n-i-1}}{i!(n-i-1)!} P(s)ds > 4^{\alpha-1}(1+a^\alpha+b^\alpha), \tag{3.29}$$

where $i = 0, 1, \dots, n-1$, then every solution of equation (1.3) *is oscillatory.*

The proof of the above corollary is similar to that of Corollary 3.2 and hence it is omitted.

4. Examples

In this section we present some examples to illustrate the main results.

Example 4.1. Consider the differential equation

$$\Big((x(t) + \frac{1}{4}x(t-\pi) - \frac{1}{4}x(t+2\pi))^3\Big)^{(v)} = \frac{1}{4}x^3(t-3\pi/2) + \frac{1}{8}x^3(t+3\pi/2), \tag{4.1}$$

for $t \ge 0$. Here $a = 1/4$, $b = 1/4$, $\alpha = \beta = \gamma = 3$, $\tau_1 = \pi$, $\tau_2 = 2\pi$, $\sigma_1 = 3\pi/2$, $\sigma_2 = 3\pi/2$, $q(t) = 1/4$, $p(t) = 1/8$. Then one can see that all conditions of Corollary 3.2 are satisfied. Therefore all the solutions of equation (4.1) are oscillatory. In fact $x(t) = \sin^{1/3} t$ is one such oscillatory solution of equation (4.1).

Example 4.2. Consider the differential equation

$$\Big((x(t) - \frac{e^{\pi/3}}{9}x(t-\pi) + \frac{1}{e^{\pi/3}}x(t+\pi))^3\Big)^{(v)} = \frac{4e^{5\pi/2}}{729}x^3(t-5\pi/2) + \frac{4}{729e^{3\pi}}x^3(t+3\pi), \tag{4.2}$$

where $t \ge 0$. Here $a = e^{\pi/3}/9$, $b = 1/e^{\pi/3}$, $\alpha = \beta = \gamma = 3$, $\tau_1 = \pi$, $\tau_2 = \pi$, $\sigma_1 = 5\pi/2$, $\sigma_2 = 3\pi$, $q(t) = 4e^{5\pi/2}/729$, $p(t) = 4/(729e^{3\pi})$. Then one can see that all conditions of Corollary 3.4 are satisfied. Therefore, all the solutions of equation (4.2) are oscillatory. In fact $x(t) = e^{t/3}\sin^{1/3} t$ is one such oscillatory solution of equation (4.2).

Example 4.3. Consider the differential equation

$$(x(t) + x(t-\pi) + x(t+\pi))^{(v)} = \frac{5}{t-\pi}x(t-\pi) + \frac{t}{t+3\pi/2}x(t+3\pi/2), \tag{4.3}$$

for $t \ge 0$. Here $a = b = 1$, $\alpha = \beta = \gamma = 1$, $\tau_1 = \tau_2 = \pi$, $\sigma_1 = \pi$, $\sigma_2 = 3\pi/2$, $q(t) = \frac{5}{t-\pi}, p(t) = \frac{t}{t+3\pi/2}$. Then one can see that all conditions of Corollary 3.6 are satisfied. Therefore, all the solutions of equation (4.3) are oscillatory. In fact $x(t) = t\sin t$ is one such oscillatory solution of equation (4.3).

Example 4.4. Consider the differential equation

$$\Big(x(t) + \frac{1}{2}x(t-\pi/2) - \frac{1}{2}x(t+2\pi)\Big)^{(vii)} = \frac{1}{2}x(t-7\pi/2) + \frac{1}{2}x(t+4\pi), \tag{4.4}$$

for $t \ge 0$. Here $a = 1/2$, $b = 1/2$, $\alpha = \beta = \gamma = 1$, $\tau_1 = \pi/2$, $\tau_2 = 2\pi$, $\sigma_1 = 5\pi/2$, $\sigma_2 = 4\pi$, $q(t) = 1/2$, $p(t) = 1/2$. Then one can see that all conditions of Corollary 3.2 are satisfied. Therefore, all the solutions of equation (4.4) are oscillatory. In fact $x(t) = \sin t + \cos t$ is one such oscillatory solution of equation (4.4).

Example 4.5. Consider the differential equation

$$\Big((x(t) - \frac{e}{3}x(t-1) + \frac{1}{e^2}x(t+2))^3\Big)^{(v)} = 1000e^9x^3(t-3) + \frac{125}{e^9}x^3(t+3), \quad (4.5)$$

for $t \geq 0$. Here $a = e/3$, $b = 1/e^2$, $\alpha = \beta = \gamma = 3$, $\tau_1 = 1$, $\tau_2 = 2$, $\sigma_1 = \sigma_2 = 3$, $q(t) = 1000e^9$, $p(t) = \frac{e^9}{125}$. Then one can see that all conditions of Corollary 3.4 are satisfied except the condition (3.24). Therefore, not all solutions of (4.5) are oscillatory. In fact $x(t) = e^t$ is one such non-oscillatory solution, since it satisfies equation (4.5).

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