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EXISTENCE AND NONEXISTENCE OF SOLUTIONS TO NONLINEAR GRADIENT ELLIPTIC SYSTEMS INVOLVING $(p(x), q(x))$-LAPLACIAN OPERATORS

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Abstract. In this article, we establish the existence of nontrivial solutions by employing the fibering method introduced by Pohozaev. We also generalize the well-known Pohozaev and Pucci-Serrin identities to a $(p(x), q(x))$-Laplacian system. A nonexistence result for a such system is then proved.

1. Introduction

After the pioneer work by Kovacik and Rokosnik [31] concerning the $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ spaces, many researches have studied the variable exponent spaces. We refer to [17] for the properties of such spaces and [8, 22] for the applications of variable exponent on partial differential equations. In the recent years, problems with $p(x)$-Laplacian have been applied to a large number of application in nonlinear electrorheological fluids, elastic mechanics, image processing, and flow in porous media (see for instance [1, 5, 9, 10, 23, 32, 39, 48]).

In this article, we study the existence and non-existence of the weak solutions for the following $(p(x), q(x))$-gradient elliptic system:

$$
\begin{gathered}
-\Delta_{p(x)} u = c(x)u|u|^{\alpha-1}|v|^{\beta+1} \quad \text{in } \Omega \\
-\Delta_{q(x)} v = c(x)v|v|^{\beta-1}|u|^{\alpha+1} \quad \text{in } \Omega \\
u = v = 0 \quad \text{on } \Omega.
\end{gathered} \tag{1.1}
$$

Here Ω designates a bounded and open set in $\mathbb{R}^N$, with a smooth boundary $\partial\Omega$. $p, q : \Omega \to \mathbb{R}$ are two measurable functions from Ω to $[1, +\infty)$, and c is a function with changing sign. Concerning the existence and nonexistence results for such systems, we cite the work [6]. There the authors use the fibering method introduced by Pohozeav. They obtained the existence of multiple solutions for a Dirichlet problem associated with a quasilinear system involving a pair of (p, q)-Laplacian operators. Recently, Velin [44, 45], employing the fibering method, proved the existence of multiple positive solutions for a class of (p, q)-gradient elliptic systems including systems like (1.1).

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Systems structured as (1.1) have been investigated for instance in [43]. There the authors presented some results dealing with existence and nonexistence of a non-trivial solution $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ of the system

$$\begin{gathered}
-\Delta_p u = u|u|^{\alpha-1}|v|^{\beta+1} \quad \text{in } \Omega \\
-\Delta_q v = v|v|^{\beta-1}|u|^{\alpha+1} \quad \text{in } \Omega \\
u = v = 0 \quad \text{on } \Omega.
\end{gathered} \tag{1.2}$$

The authors have proved nonexistence results when Ω is a strictly starshaped open domain in $\mathbb{R}^N$ and

$$(\alpha+1)\frac{N-p}{Np} + (\beta+1)\frac{N-q}{Nq} \geq 1\,. \tag{1.3}$$

On the other hand, under the assumptions

$$(\alpha+1)\frac{N-p}{Np} + (\beta+1)\frac{N-q}{Nq} < 1, \quad \frac{\alpha+1}{p} + \frac{\beta+1}{q} \neq 1, \tag{1.4}$$

some existence results have been obtained. In [13], the authors deal with nonexistence for an elliptic Dirichlet equation governed by the $p(x)$-Laplacian operator.

The article has the following structure. Section 2 is devoted to introduce some notation and preliminaries needed for the framework of the paper. We also recall some tools defined by the theory of variable exponents Lebesgue and Sobolev spaces. Section 3 states the main results. In Section 4, following the ideas explained in [13], we establish a Pohozaev-type identity for the system (1.1). By using this identity, we deal with the non-existence results of non trivial solutions. In section 5, after recalling the spirit of the fibering method, we show that (1.1) admits at least one weak non-trivial solution.

2. Preliminaries

Let $\mathcal{P}(\Omega)$ denote the set $\{p; p : \Omega \to [1, +\infty) \text{ is measurable }\}$. $\Omega \subset \mathbb{R}^N$ is an open set. $L^{p(x)}(\Omega)$ designates the generalized Lebesgue space. $L^{p(x)}(\Omega)$ consists of all measurable functions u defined on Ω for which the $p(x)$-modular

$$\rho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)} dx$$

is finite. The Luxemberg norm on this space is defined as

$$\|u\|_p = \inf\{\lambda > 0; \rho_{p(.)}(u) = \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\}.$$

Equipped with this norm, $L^{p(x)}(\Omega)$ is a Banach space. Some basic results on the generalized Lebesgue spaces can be find in [12, 19, 21, 22, 26, 27, 31, 32, 33]. If $p(x)$ is constant, $L^{p(x)}(\Omega)$ is reduced to the standard Lebesgue space.

For any $p \in \mathcal{P}(\Omega)$ and $m \in \mathbb{N}^*$, the generalized Sobolev space $W^{m,p(x)}(\Omega)$ is defined by

$$\begin{gathered}
W^{m,p(.)}(\Omega) = \{u \in L^{p(.)}(\Omega) : D^\alpha u \in L^{p(.)}(\Omega) \text{ for all } |\alpha| \leq m\}, \\
\|u\|_{m,p(.)} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^{p(.)}(\Omega)}\,.
\end{gathered}$$

The pair $(W^{m,p(.)}(\Omega), \|\cdot\|_{m,p(.)})$ is a separable Banach space (reflexive if $p^- > 1$). $W_0^{1,p(.)}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$. On the generalized Sobolev space, we refer to the works due to [16, 17, 19, 20, 24, 25, 31].

We define: $p, q : \Omega \to [1, +\infty)$ as two measurable functions.

For a given measurable function $p : \Omega \to [1, +\infty)$, the conjugate function designated by

$$p'(x) = \frac{p(x)}{p(x) - 1}.$$

A function $p : \Omega \to \mathbb{R}$ is ln-Hölder continuous on Ω (See [19]), provided that there exists a constant $L > 0$ such that

$$|p(x) - p(y)| \leq \frac{L}{-\ln|x - y|}, \quad \text{for all } x, y \in \Omega,\ |x - y| \leq \frac{1}{2}. \tag{2.1}$$

$$\begin{aligned} p^- &= \min_{x\in\overline{\Omega}} p(x), \quad q^- = \min_{x\in\overline{\Omega}} q(x),\\ p^+ &= \max_{x\in\overline{\Omega}} p(x), \quad q^+ = \max_{x\in\overline{\Omega}} q(x). \end{aligned}$$

For $c : \Omega \to \mathbb{I}$, $c_+(x) \neq 0$, $c_-(x) \neq 0$.

3. Main results

Let us now state the main results of this paper:

A non-existence result for the $(p(x), q(x))$-Laplacian system (1.1).

Theorem 3.1. *Let Ω be a bounded open set of $\mathbb{R}^N$, with boundary $\partial\Omega$ of class C^1. Let $p, q : \Omega \to \mathbb{I}$ functions of class $C^1_B(\Omega) \cap C(\overline{\Omega})$, $p^-, q^- > 1$, and $c(.) \in C^1_B(\Omega \setminus \mathcal{C})$, with $\operatorname{meas}(\mathcal{C}) = 0$. Assume that Ω be a bounded domain of class C^1, starshaped with respect to the origin; $(p, q) \in C^1_B(\Omega) \cap C(\bar{\Omega})$; $p^-, q^- > 1$; and $(x \cdot \nabla p) \geq 0$, $(x \cdot \nabla q) \geq 0$,*

$$\langle x, \nabla c(x)\rangle \leq 0 \quad \text{for any } x \text{ in } \Omega, \tag{3.1}$$

$$(\alpha + 1)\frac{N - p^+}{Np^+} + (\beta + 1)\frac{N - q^+}{Nq^+} \geq 1. \tag{3.2}$$

Then (1.1) has not a nontrivial classical solution $(u, v) \in (C^2(\Omega) \cap C^1(\bar{\Omega}))^2$ which satisfies:

$$|\nabla u(x)| \geq e^{1/p(x)}, \quad |\nabla v(x)| \geq e^{1/q(x)} \quad \text{a.e } x \in \Omega, \tag{3.3}$$

and

$$\int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}dx > 0.$$

An existence result for the $(p(x), q(x))$-Laplacian system (1.1).

Theorem 3.2. *Let Ω be a bounded open set of $\mathbb{R}^N$, with boundary $\partial\Omega$ of class C^1. Let $p, q : \Omega \to \mathbb{I}^*_+$ two functions of class $C^1_B(\Omega) \cap\ C(\overline{\Omega})$; $p_-, q_- > 1$. Assume that:*

$$(\alpha + 1)\frac{N - p^-}{Np^-} + (\beta + 1)\frac{N - q^-}{Nq^-} < 1, \tag{3.4}$$

$$\gamma^+ = \frac{\alpha + 1}{p^+} + \frac{\beta + 1}{q^+} - 1 > 0. \tag{3.5}$$

Then system (1.1) *admits at least one nontrivial solution* $(u^*, v^*) \in W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$. *Moreover, one have*

$$\|u^*\|_{1,p(x)}^{p^+} = \|v^*\|_{1,q(x)}^{q^+},$$
$$\int_\Omega c(x)|u^*|^{\alpha+1}|v^*|^{\beta+1}dx > 0.$$

Remark 3.3. Let us remark that conditions (3.2) and (3.4) seem to generalize to $(p(x), q(x))-$ gradient elliptic systems conditions (1.3) and (1.4) well known when $(p,q)-$ gradient elliptic systems are considered. Obviously, conditions (3.2) and (3.4) imply respectively

$$1 \le (\alpha+1)\frac{N-p^-}{Np^-} + (\beta+1)\frac{N-q^-}{Nq^-},$$
$$(\alpha+1)\frac{N-p^+}{Np^+} + (\beta+1)\frac{N-q^+}{Nq^+} < 1.$$

4. A Pohozaev-type identity for $(p(x), q(x))$-Laplacian and a nonexistence result

Consider the elliptic system with Dirichlet boundary condition:

$$\begin{gathered}
-\Delta_{p(x)}u = c(x)u|u|^{\alpha-1}|v|^{\beta+1} \quad \text{in } \Omega \\
-\Delta_{q(x)}v = c(x)|u|^{\alpha+1}v|v|^{\beta-1} \quad \text{in } \Omega \\
u = v = 0 \quad \text{on } \Omega,
\end{gathered}$$

where $\Omega \subset \mathbb{I}^N$ is a bounded open set with a regular boundary $\partial\Omega$; p, q, c are defined as in the previous section.

$$\Delta_{p(x)}u = \frac{\partial}{\partial x_i}\Big(|\nabla u|^{p(x)-2}\frac{\partial u}{\partial x_i}\Big).$$

Proposition 4.1. *Let Ω be a bounded open set of $\mathbb{R}^N$, with boundary $\partial\Omega$ of class C^1. Assume that $p, q : \Omega \to \mathbb{I}$ are two functions of class $C_B^1(\Omega) \cap C(\overline{\Omega})$; $p^-, q^- > 1$; $c(.) \in C_B^1(\Omega \setminus \mathcal{C})$, with $\operatorname{meas}(\mathcal{C}) = 0$ and*

$$\langle x, \nabla c(x)\rangle \le 0 \quad \text{for any } x \text{ in } \Omega.$$

For every classical solution $(u, v) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of (1.1), *the following identity holds:*

$$\begin{aligned}
&\frac{\alpha+1}{N}\int_{\partial\Omega}\frac{1-p(x)}{p(x)}|\nabla u|^{p(x)}\langle x,\nu\rangle \mathrm{d}\sigma + \frac{\beta+1}{N}\int_{\partial\Omega}\frac{1-q(x)}{q(x)}|\nabla v|^{q(x)}\langle x,\nu\rangle \mathrm{d}\sigma \\
&= \frac{\alpha+1}{N}\int_\Omega\big(\frac{N-p(x)}{p(x)} - a_1\big)|\nabla u|^{p(x)}\mathrm{d}x + \frac{\beta+1}{N}\int_\Omega\big(\frac{N-q(x)}{q(x)} - a_2\big)|\nabla v|^{q(x)}\mathrm{d}x \\
&\quad + \int_\Omega\Big[\frac{1}{p^2(x)}\langle x\cdot\nabla p\rangle(\ln|\nabla u|^{p(x)} - 1)|\nabla u|^{p(x)} \\
&\quad + \frac{1}{q^2(x)}\langle x,\nabla q\rangle(\ln|\nabla v|^{q(x)} - 1)|\nabla v|^{q(x)}\Big]\mathrm{d}x \\
&\quad + \int_\Omega\big\{(\alpha+1)a_1 + (\beta+1)a_2 - N\big\}c(x)|u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x \\
&\quad - \int_\Omega\langle x,\nabla c\rangle|u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x.
\end{aligned}$$

for all a_1 *and* $a_2 \in \mathbb{R}^N$*.*

Before proving the proposition 4.1, we present the following result generalizing the variational identity of Pucci-Serrin [38].

Proposition 4.2. *Let* Ω *be a bounded open set of* $\mathbb{R}^N$ *with boundary* $\partial\Omega$ *of class* C^1*. Assume that* $p, q : \Omega \to \mathbb{I}$ *are tow functions of class* $C^1_B(\Omega) \cap C(\overline{\Omega})$*;* $p^-, q^- > 1$*;* $c(\cdot) \in C^1_B(\Omega \setminus \mathcal{C})$*,* $\mathcal{C} \subset \Omega$ *with* $\operatorname{meas}(\mathcal{C}) = 0$*. For every classical solution* $(u, v) \in \big(C^2(\Omega) \cap C^1(\overline{\Omega})\big)^2$ *of problem* (1.1)*, the following equality holds*

$$
\begin{aligned}
&\frac{\partial}{\partial x_i}\Big[x_i\Big(\frac{\alpha+1}{p(x)}|\nabla u|^{p(x)} + \frac{\beta+1}{q(x)}|\nabla v|^{q(x)} - c(x)|u|^{\alpha+1}|v|^{\beta+1}\Big) \\
&- (\alpha+1)\big(x_j\frac{\partial u}{\partial x_j} + a_1 u\big)|\nabla u|^{p(x)-2}\frac{\partial u}{\partial x_i} - (\beta+1)\Big(x_j\frac{\partial v}{\partial x_j} + a_2 v\Big)|\nabla v|^{q(x)-2}\frac{\partial v}{\partial x_i}\Big] \\
&= (\alpha+1)\Big[\frac{N-p(x)}{p(x)} - a_1\Big]|\nabla u|^{p(x)} + (\beta+1)\Big[\frac{N-q(x)}{q(x)} - a_2\Big]|\nabla v|^{q(x)} \\
&\quad + \frac{\langle x, \nabla p\rangle}{p^2(x)}(\ln|\nabla u|^{p(x)} - 1)|\nabla u|^{p(x)} + \frac{\langle x, \nabla q\rangle}{q^2(x)}(\ln|\nabla v|^{q(x)} - 1)|\nabla v|^{q(x)} \\
&\quad + \{(\alpha+1)a_1 + (\beta+1)a_2 - N\}c(x)|u|^{\alpha+1}|v|^{\beta+1} - \langle x, \nabla c\rangle|u|^{\alpha+1}|v|^{\beta+1},
\end{aligned}
\tag{4.1}
$$

for all a_1 *and* a_2 *in* $\mathbb{R}$*.*

The proof of Proposition 4.2 can be established by a simple computation.

Proof of Proposition 4.1. In this proof, for any vectors in $\mathbb{I}^N$ $x = (x_i)_{i=1,\dots,N}$ and $y = (y_i)_{i=1,\dots,N}$, the classical inner product xy is denoted $x_i y_i$ and the notation $\sum_{i=1}^N$ is omitted. Let $(u, v) \in \big(C^2_B \cap C^1(\bar{\Omega})\big)^2$ be a classical solution of the problem (1.1). According to the Proposition 4.2, (u, v) satisfies the identity (4.1). Integrating by part over Ω, we get

$$
\begin{aligned}
&\int_{\partial\Omega}\Big[\Big(\frac{\alpha+1}{p(x)}|\nabla u|^{p(x)} + \frac{\beta+1}{q(x)}|\nabla v|^{q(x)} - c(x)|u|^{\alpha+1}|v|^{\beta+1}\Big) \\
&- (\alpha+1)\big(x_j\frac{\partial u}{\partial x_j} + a_1 u\big)|\nabla u|^{p(x)-2}\frac{\partial u}{\partial x_i} \\
&- (\beta+1)\big(x_j\frac{\partial v}{\partial x_j} + a_2 v\big)|\nabla v|^{q(x)-2}\frac{\partial v}{\partial x_i}\Big]\nu_i \mathrm{d}\sigma \\
&= (\alpha+1)\int_\Omega\big(\frac{N-p(x)}{p(x)} - a_1\big)|\nabla u|^{p(x)}\mathrm{d}x \\
&\quad + (\beta+1)\int_\Omega\big(\frac{N-q(x)}{q(x)} - a_2\big)|\nabla v|^{q(x)}\mathrm{d}x \\
&\quad + \int_\Omega\Big[\frac{1}{p^2(x)}\langle x\cdot\nabla p\rangle(\ln|\nabla u|^{p(x)} - 1)|\nabla u|^{p(x)} \\
&\quad + \frac{1}{q^2(x)}\langle x, \nabla q\rangle(\ln|\nabla v|^{q(x)} - 1)|\nabla v|^{q(x)}\Big]\mathrm{d}x \\
&\quad + \int_\Omega\Big\{(\alpha+1)a_1 + (\beta+1)a_2 - N\Big\}c(x)|u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x \\
&\quad - \int_\Omega\langle x, \nabla c\rangle|u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x,
\end{aligned}
\tag{4.2}
$$

where ν is the unit outer normal to the boundary $\partial\Omega$ Since $u = 0$ on $\partial\Omega$, clearly it follows that $\frac{\partial u}{\partial x_i} = (\nabla u.\nu)\nu_i$ for $i = 1, \dots, N$. Then, for x on $\partial\Omega$, we can write

$$\begin{aligned} x_j \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}|\nabla u|^{p(x)-2}\nu_i &= x_j[(\nabla u.\nu)\nu_j]\frac{\partial u}{\partial x_i}|\nabla u|^{p(x)-2}\nu_i \\ &= \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i}|\nabla u|^{p(x)-2}(x.\nu) \\ &= |\nabla u|^{p(x)}(x.\nu) \quad \text{on } \partial\Omega \end{aligned}$$

Using the relation (4.2) and the fact that $u|_{\partial\Omega} = 0$ in the left hand side of this relation, the statement of the Proposition 4.1 occurs. $\square$

Remark 4.3. Before proving Proposition 4.2, we note that the set of functions c satisfying to hypothesis (3.1), is non-empty. Indeed, let x_0 be in $\partial\Omega$ such that $\operatorname{dist}(0, \partial\Omega) = \operatorname{dist}(0, x_0)$. We set $R_0 = \operatorname{dist}(0, \partial\Omega)$. Obviously, we remark that the ball $B(0, R_0)$ is contained in Ω. We define the set Ω_1 by $\Omega_1 = \{x \in \Omega; 0 \le \|x\| \le R_0/2\}$. For instance, we define the function

$$c(x) = \begin{cases} -e^{\|x\|^2} & \text{if } x \in \Omega_1 \\ e^{-\|x\|^2} & \text{if } x \in \Omega \setminus \Omega_1. \end{cases}$$

This function changes sign in Ω and we also have for any $x \in \Omega$, $\langle x, \nabla c(x)\rangle \le 0$. Moreover, $c \in L^\infty(\Omega)$.

Proof of Theorem 3.1. Suppose that there exists a nontrivial classical solution (u, v) in $C^2(\Omega) \cap C^1(\bar\Omega)$ of the problem (1.1). So that, (u, v) satisfies the statement of Proposition 4.1. Since $\Omega \subset \mathbb{R}^N$ is strictly starshaped with respect to the origin, we have $x \cdot \nu > 0$ on $\partial\Omega$ thus

$$-\frac{\alpha+1}{N}\int_{\partial\Omega}\frac{1}{\tilde p(x)}|\nabla u|^{p(x)}\langle x,\nu\rangle \mathrm{d}\sigma - \frac{\beta+1}{N}\int_{\partial\Omega}\frac{1}{\tilde q(x)}|\nabla v|^{q(x)}\langle x,\nu\rangle \mathrm{d}\sigma < 0,$$

where $\frac{1}{\tilde p(x)} = \frac{p(x)-1}{p(x)}$, $\frac{1}{\tilde q(x)} = \frac{q(x)-1}{q(x)}$.

On other hand, choosing $a_1 \in \mathbb{I}$ and $a_2 \in \mathbb{I}$ such that

$$(\alpha+1)\frac{a_1}{N} + (\beta+1)\frac{a_2}{N} = 1$$

and using the relations (3.2), (3.3), we obtain

$$\begin{aligned} &\frac{\alpha+1}{N}\int_{\partial\Omega}\frac{1-p(x)}{p(x)}|\nabla u|^{p(x)}\langle x,\nu\rangle \mathrm{d}\sigma + \frac{\beta+1}{N}\int_{\partial\Omega}\frac{1-q(x)}{q(x)}|\nabla v|^{q(x)}\langle x,\nu\rangle \mathrm{d}\sigma \\ &= \frac{\alpha+1}{N}\int_\Omega\Big(\frac{N-p(x)}{p(x)} - a_1\Big)|\nabla u|^{p(x)}\mathrm{d}x + \frac{\beta+1}{N}\int_\Omega\Big(\frac{N-q(x)}{q(x)} - a_2\Big)|\nabla v|^{q(x)}\mathrm{d}x \\ &\quad + \int_\Omega\Big\{(\alpha+1)a_1 + (\beta+1)a_2 - N\Big\}c(x)|u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x \\ &\quad - \int_\Omega\langle x,\nabla c\rangle|u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x \\ &\ge (\alpha+1)\frac{N-p^+}{Np^+}\int_\Omega|\nabla u|^{p(x)}\mathrm{d}x + (\beta+1)\frac{N-q^+}{Nq^+}\int_\Omega|\nabla v|^{q(x)}\mathrm{d}x \\ &\quad - (\alpha+1)\frac{a_1}{N}\int_\Omega|\nabla u|^{p(x)}\mathrm{d}x - (\beta+1)\frac{a_2}{N}\int_\Omega|\nabla v|^{q(x)}\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega \big\{(\alpha+1)a_1+(\beta+1)a_2-N\big\}c(x)|u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x\\
&-\int_\Omega \langle x,\nabla c\rangle |u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x\\
&\geq \{(\alpha+1)\frac{N-p^+}{Np^+}+(\beta+1)\frac{N-q^+}{Nq^+}-(\alpha+1)\frac{a_1}{N}\\
&-(\beta+1)\frac{a_2}{N}\}\int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}dx\\
&\geq \{(\alpha+1)\frac{N-p^+}{Np^+}+(\beta+1)\frac{N-q^+}{Nq+}-1\}\int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}dx\\
&-\int_\Omega \langle x,\nabla c\rangle |u|^{\alpha+1}|v|^{\beta+1}dx.
\end{aligned}
$$

Now we remark that any solution (u,v) of (1.1) satisfies

$$\int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}dx=\int_\Omega |\nabla u|^{p(x)}\mathrm{d}x=\int_\Omega |\nabla v|^{q(x)}\mathrm{d}x.$$

So from the hypothesis (3.1), the right-hand side is positive. A contradiction occurs, then the proof is complete. □

5. Existence results via the fibering method

Throughout this section, Ω denotes a bounded open set in $\mathbb{R}^N$. The generalized Sobolev spaces $W_0^{1,p(x)}(\Omega)$ and $W_0^{1,q(x)}(\Omega)$ are equipped with the Luxembourg norm $\|u\|_{W_0^{1,p(x)}(\Omega)}$ and $\|u\|_{W_0^{1,q(x)}(\Omega)}$ respectively. For a best reading, we denote as $\|u\|_{W_0^{1,p(x)}(\Omega)}=\|u\|_{1,p(x)}$ and $\|u\|_{W_0^{1,q(x)}(\Omega)}=\|u\|_{1,q(x)}$. Before starting this section, we need to make some crucial remarks for the understanding of this article.

Remark 5.1. Assuming that

$$(\alpha+1)\frac{N-p^-}{Np_-}+(\beta+1)\frac{N-q^-}{Nq^-}\leq 1.$$

We can establish that the term $\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx$ is well defined. Indeed, since the functional c is bounded in Ω, it suffices to verify that $|z|^{\alpha+1}|w|^{\beta+1}$ belongs in $L^1(\Omega)$. This fact derives to the condition $\frac{\alpha+1}{p^+}+\frac{\beta+1}{q^+}>1$ and so $\frac{\alpha+1}{p^-}+\frac{\beta+1}{q^-}>1$. So, there exists a pair $(\hat{p},\hat{q})$ such that (1)

$$p^-<\hat{p}<\frac{Np^-}{N-p^-}, \tag{5.1}$$

$$q^-<\hat{q}<\frac{Nq^-}{N-q^-} \tag{5.2}$$

(2) $\frac{\alpha+1}{\hat{p}}+\frac{\beta+1}{\hat{q}}=1$.

Remark 5.2. Since $\frac{Np^-}{N-p^-}<\frac{Np(x)}{N-p(x)}$ and $\frac{Nq^-}{N-q^-}<\frac{Nq(x)}{N-q(x)}$, the assumption $(\alpha+1)\frac{N-p^-}{Np^-}+(\beta+1)\frac{N-q^-}{Nq^-}\leq 1$ implies that for any $x\in\Omega$, inequalities (5.1) and (5.2) become

$$p^-<\hat{p}<\frac{Np(x)}{N-p(x)},$$

$$q- < \hat{q} < \frac{Nq(x)}{N-q(x)}.$$

In particular, the imbeddings $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\hat{p}}(\Omega)$ and $W_0^{1,q(x)}(\Omega) \hookrightarrow L^{\hat{q}}(\Omega)$ are continuous. Consequently, employing the Hölder inequality, the above estimate is fulfilled:

$$\Big| \int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}dx \Big| \le \|c\|_{L^\infty(\Omega)}\|u\|^{\alpha+1}_{L^{\hat{p}}(\Omega)}\|v\|^{\beta+1}_{L^{\hat{q}}(\Omega)} \le Cst\|u\|^{\alpha+1}_{1,p(x)}\|v\|^{\beta+1}_{1,q(x)}.$$

Remark 5.3. (1) Under assumption (3.5), we have

$$\frac{\alpha+1}{p^-} + \frac{\beta+1}{q^-} - 1 > 0. \tag{5.3}$$

(2) When $q(x)$ and $p(x)$ are constant, (3.5) and (5.3) are reduced to the well-known condition

$$1 < \frac{\alpha+1}{p} + \frac{\beta+1}{q}.$$

5.1. Notation and hypotheses.

Notation. $X_0(x)$ denotes $W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$.

For any $(z,w) \in X_0(x)$, we set

$$\begin{gathered} A(z) = \int_\Omega |\nabla z|^{p(x)}dx, \quad B(w) = \int_\Omega |\nabla w|^{q(x)}dx, \\ C(z,w) = \int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx. \end{gathered} \tag{5.4}$$

$$\gamma^+ = \frac{\alpha+1}{p^+} + \frac{\beta+1}{q^+}, \quad \gamma^- = \frac{\alpha+1}{p^-} + \frac{\beta+1}{q^-}.$$

$\mathbf{J}$ designates the functional from $X_0(x)$ to $\mathbb{R}$ and defined by

$$\mathbf{J}(u,v) = (\alpha+1)\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + (\beta+1)\int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}dx - C(u,v)dx. \tag{5.5}$$

Following remarks 5.1-5.2, the functional $\mathbf{J}$ is well defined from $X_0(x)$ to $\mathbb{I}$.

Hypotheses.

$$1 < \gamma^+, \tag{5.6}$$

$$(p,q) \in \big(\mathcal{P}(\Omega) \cup C(\overline{\Omega})\big)^2 \text{ satisfies (2.1).} \tag{5.7}$$

Moreover, assume that

$$1 < p^- \le p^+ < +\infty, \quad 1 < q^- \le q^+ < +\infty. \tag{5.8}$$

Definition of a weak solution for (1.1).

Definition 5.4. A pair $(u,v) \in X_0(x)$ is a weak solution of (1.1) if for any $(\phi,\psi) \in X_0(x)$:

$$\begin{gathered} \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla \phi \, dx = \int_\Omega c(x)u|u|^{\alpha-1}|v|^{\beta+1}u\phi \, dx, \\ \int_\Omega |\nabla v|^{q(x)-2}\nabla v \nabla \psi \, dx = \int_\Omega c(x)|u|^{\alpha+1}v|v|^{\beta-1}u\psi \, dx. \end{gathered}$$

Fibering Method for quasilinear systems. Pohozaev introduced the fibering method in [34] (see also [36, 37]). For more details about various applications, we refer the reader to [2, 3, 4, 6, 7, 14, 28, 29, 40, 41, 46, 47]). The fibering method applied to this problem consists in seeking the the pair $(u,v) \in X$ in the form

$$u = rz, \quad v = \rho w \tag{5.9}$$

where the functions z and w belong to $W_0^{1,p(x)}(\Omega) \setminus \{0\}$ and $W_0^{1,q(x)}(\Omega) \setminus \{0\}$ respectively, where r and ρ are real numbers. Moreover, since we look for nontrivial solutions (i.e: $u \neq 0$, $v \neq 0$), we must assume that $r \neq 0$ and $\rho \neq 0$. The fibering method ensures the existence. However, compared to other well known methods, we obtain the specific form (5.9).

Remark 5.5. In [6], the authors applied the fibering method to obtain the existence of multiple solutions for a problem like (1.1) when the exponents $p(x)$ and $q(x)$ are constant. In their studies, we note that the fibering parameters r and ρ depending on z and w verify $r^p = \rho^q$ for (z,w) such that $A(z) = 1$ and $B(z) = 1$ for instance. Inspired by this point of view, here we propose to seek a couple $(u,v) = (rz, \rho w)$, with $r = t^{1/p^+}$ and $\rho = t^{1/q^+}$, for $t > 0$.

Existence of a fibering parameter $t(z,w)$**.** Existence and properties: Since $\frac{\partial \mathbf{J}}{\partial u}(u,v)$ and $\frac{\partial \mathbf{J}}{\partial v}(u,v)$ exist, a weak solution of (1.1) corresponds to a critical point of the energy functional $\mathbf{J}$ associated to the system (1.1). Hence, assuming that $(u,v) \in X_0(x)$ is a critical point of $\mathbf{J}$, (u,v) satisfies $\big(\frac{\partial \mathbf{J}}{\partial u}(u,v), \frac{\partial \mathbf{J}}{\partial v}(u,v)\big) = (0,0)$. So, according to remark 5.5, a fibering parameter $t(z,w)$ associated to (z,w) is characterized as

$$\frac{d\mathbf{J}}{dt}\big(t^{1/p^+} z, t^{1/q^+} w\big) = 0. \tag{5.10}$$

More precisely, $t(z,w)$ is defined by the following Proposition.

Proposition 5.6. *Let (z,w) be fixed in $X_0(x)$ such that $C(z,w) > 0$.*

(1) Assuming (5.6), there is $t(z,w) \in \mathbb{R}_+^*$ depending on (z,w) such that

$$\frac{\alpha+1}{p^+\gamma^+}\int_\Omega t(z,w)^{\frac{p(x)}{p^+}} |\nabla z|^{p(x)} dx + \frac{\beta+1}{q^+\gamma^+}\int_\Omega t(z,w)^{\frac{q(x)}{q^+}} |\nabla z|^{q(x)} dx = t(z,w)^{\gamma^+} C(z,w). \tag{5.11}$$

(2) Location of $t(z,w)$: for $t(z,w) > 1$ (respectively, $t(z,w) \leq 1$) if $Q(z,w) > 1$ (respectively $Q(z,w) \leq 1$), for any (z,w) such that $C(z,w) > 0$, we have

$$Q(z,w) = \frac{\frac{\alpha+1}{p^+\gamma^+}\int_\Omega |\nabla z|^{p(x)} dx + \frac{\beta+1}{q^+\gamma^+}\int_\Omega |\nabla z|^{q(x)} dx}{C(z,w)}.$$

Moreover, the following two estimates hold: (a) If $0 < t(z,w) < 1$, then

$$Q(z,w)^{1/\gamma^+ - 1} \leq t(z,w) \leq Q(z,w)^{1/\gamma^+} \tag{5.12}$$

(b) If $1 \geq t(z,w)$, then

$$Q(z,w)^{1/\gamma^+} \leq t(z,w) \leq Q(z,w)^{1/\gamma^+ - 1}. \tag{5.13}$$

Proof. We divide the proof in three steps

Step 1: Existence of $t(z,w)$**.** Using the definition of $\mathbf{J}$ (see (5.5)), solving (5.10) is equivalent to solving the equation

$$\frac{\alpha+1}{p^+\gamma^+}\int_\Omega t^{\frac{p(x)}{p^+}} |\nabla z|^{p(x)} dx + \frac{\beta+1}{q^+\gamma^+}\int_\Omega t^{\frac{q(x)}{q^+}} |\nabla z|^{q(x)} dx = t^{\gamma^+} C(z,w).$$

To do this, consider a function $\tilde{f}$ defined on $[0,+\infty[$ by

$$\tilde{f}(t)=\frac{\alpha+1}{p^+\gamma^+}\int_\Omega t^{\frac{p(x)}{p^+}}|\nabla z|^{p(x)}dx+\frac{\beta+1}{q^+\gamma^+}\int_\Omega t^{\frac{q(x)}{q^+}}|\nabla z|^{q(x)}dx-t^{\gamma^+}C(z,w).$$

Choosing $0<t<1$, it follows that

$$t[Q(z,w)-t^{\gamma^+-1}]C(z,w)\le \tilde{f}(t). \tag{5.14}$$

Now, for $1\le t$, we obtain

$$\tilde{f}(t)\le t[Q(z,w)-t^{\gamma^+-1}]C(z,w). \tag{5.15}$$

Consequently, on one side we have $\lim_{t\to 0}\tilde{f}(t)\ge 0$, and on the other side we have $\lim_{t\to+\infty}\tilde{f}(t)=-\infty$. So, using the Mean Value Theorem, we deduce that there exists $t(z,w)\in\mathbb{R}_+^*$ depending on z, w such that $\tilde{f}(t(z,w))=0$. Moreover, $t(z,w)$ obeys to (5.11).

Step 2: Location of $t(z,w)$. We distinguish the cases $Q(z,w)<1$ and $Q(z,w)\ge 1$.

(a) Assume that $Q(z,w)<1$, it follows that $0<t(z,w)<1$. Arguing by opposite, if the assert $1\le t(z,w)$ holds, then from (5.15), we obtain $t(z,w)\le Q(z,w)$. So, $Q(z,w)$ is greater than 1. This is contradicts the hypothesis $Q(z,w)<1$.

(b) Conversely, assuming $Q(z,w)\ge 1$, from (5.14), we get $t(z,w)\ge 1$.

From (5.11) and the hypothesis (5.6), it is easy to deduce that for any (z,w) fixed in $X_0(x)$ such that $C(z,w)>0$, the location of the fibering parameter $t(z,w)$. $\square$

Lemma 5.7. *Assume* (5.6). *Let* (z,w) *in* $X_0(x)\setminus\{(0,0)\}$ *and* $t(z,w)$ *defined as in* (5.11). *The function* $(z,w)\longmapsto t(z,w)$ *is* C^1 *on* $X_0(x)\setminus\{(0,0)\}$.

Proof. From (5.11), we consider on the open set $X_0(x)\setminus\{(0,0)\}\times(]0,1[\cup]1,+\infty[)$ of $X_0(x)\times\mathbb{I}$, the functional η defined as follows:

$$\eta(z,w,t)=\frac{\alpha+1}{p^+\gamma^+}\int_\Omega t^{\frac{p(x)}{p^+}-\gamma^+}|\nabla z|^{p(x)}dx+\frac{\beta+1}{q^+\gamma^+}\int_\Omega t^{\frac{q(x)}{q^+}-\gamma^+}|\nabla w|^{q(x)}dx-C(z,w).$$

Obviously, we note that $\eta(z,w,t(z,w))=0$ and $\frac{\partial\eta}{\partial t}(z,w,t(z,w))<0$. We used the implicit function theorem for the function η. Then $(z,w)\longmapsto t(z,w)$ is C^1 function on $X_0(x)\setminus\{(0,0)\}$. $\square$

A new definition for the enegy functional J derived from Proposition 5.6 and Lemma 5.7. On $X_0(x)\setminus\{(0,0)\}$, we define the function

$$\begin{aligned}\mathcal{J}(z,w)=&\int_\Omega\frac{\alpha+1}{p(x)}t(z,w)^{\frac{p(x)}{p^+}}|\nabla z|^{p(x)}dx+\int_\Omega\frac{\beta+1}{q(x)}t(z,w)^{\frac{q(x)}{q^+}}|\nabla w|^{q(x)}dx\\&-t(z,w)^{\gamma^+}C(z,w).\end{aligned} \tag{5.16}$$

5.2. **A conditional critical point of $\mathcal{J}$.** We start by giving some lemmas.

Lemma 5.8. *Let* $(z_0,w_0)\in X_0(x)\setminus\{(0,0)\}$ *such that* $C(z_0,w_0)\ne 0$. *Then, there exists* $Z_0\in W_0^{1,p(x)}(\Omega)\setminus\{0\}$ *satisfying* $C(Z_0,w_0)>0$.

Proof. We fix $(z_0,w_0)\in X_0(x)\setminus\{(0,0)\}$ for which $C(z_0,w_0)\ne 0$. Then distinguish two cases: (1) $C(z_0,w_0)>0$. Then, the assertion of Lemma 5.8 holds by taking $Z_0=z_0$.

(2) If $C(z_0,w_0)<0$. In this context, we note that
$$\int_\Omega c_+(x)|z_0|^{\alpha+1}|w_0|^{\beta+1}dx<\int_\Omega c_-(x)|z_0|^{\alpha+1}|w_0|^{\beta+1}dx.$$
Assuming that $c_+(x)>0$ and $c_-(x)\geq 0$, we put
$$Z_0=z_0\,\chi_{\{h>0\}}-\hat{\varepsilon}z_0\,\chi_{\{h\leq 0\}}$$
and
$$0<\hat{\varepsilon}<\Big[\frac{\int_{\{c>0\}}h_+(x)|z_0|^{\alpha+1}|w_0|^{\beta+1}dx}{\int_{\{c\leq 0\}}h_-(x)|z_0|^{\alpha+1}|w_1|^{\beta+1}dx+1}\Big]^{1/\alpha+1}.$$
From easy calculations, it follows that $\int_\Omega c(x)|Z_0|^{\alpha+1}|w_0|^{\beta+1}dx>0$. The proof is complete. □

Consequently, we define the set
$$E=\big\{(z,w)\in X;\int_\Omega|\nabla z|^{p(x)}dx=1,\ \int_\Omega|\nabla w|^{q(x)}dx=1\big\}. \tag{5.17}$$
It is obvious that E is a nonempty set (see [19, 31]). We then have the next lemma.

Lemma 5.9. *The set $\{(z,w)\in E;C(z,w)>0\}$ is nonempty.*

Proof. Let (z_0,w_0) be in $X_0(x)\setminus\{(0,0)\}$ such that $C(z_0,w_0)\neq 0$. According to the lemma 5.8, there is $(\mathcal{Z},w_0)\in X_0(x)\setminus\{(0,0)\}$ such that $C(\mathcal{Z},w_0)>0$. The assert of the lemma is holds if for instance $(\mathcal{Z},w_0)\in E$. Now, assume that $(\mathcal{Z},w_0)$ is not in E. Assume that for instance $\int_\Omega|\nabla\mathcal{Z}|^{p(x)}dx>1$ and $\int_\Omega|\nabla w_0|^{q(x)}dx<1$. Applying the mean value theorem to the functions $t\to 1-\int_\Omega|\nabla t\mathcal{Z}|^{p(x)}dx$ and $s\to\int_\Omega|\nabla s w_0|^{q(x)}dx-1$, we get a pair $(t_p,s_q)\in]0,1[\times]1,+\infty[$ such that
$$\int_\Omega|\nabla t_p\mathcal{Z}|^{p(x)}dx=1=\int_\Omega|\nabla s_q w_0|^{q(x)}dx.$$
Moreover, since $C(\mathcal{Z},w_0)>0$, we also have $C(t_p\mathcal{Z},s_q w_0)>0$. The proof is complete. □

Proposition 5.10. *Let the functional $\mathcal{J}$ be defined by* (5.16)*, and let (z,w) be in E. Under hypothesis* (5.6)–(5.8)*, the following estimates hold:*
$$\frac{\gamma^+}{C(z,w)^{1/\gamma^+-1}}-1\leq\mathcal{J}(z,w)\leq\frac{\gamma^--1}{C(z,w)^{\min(\frac{p^-}{p^+},\frac{q^-}{q^+})}},\quad \textit{if } c(z,w)\geq 1,$$
$$\frac{\gamma^+-1}{C(z,w)^{\min(\frac{p^-}{p^+},\frac{q^-}{q^+})}}\leq\mathcal{J}(z,w)\leq\frac{\gamma^-}{C(z,w)^{1/\gamma^+-1}}-1,\quad \textit{if } c(z,w)<1.$$

Proof. Estimates (5.12) and (5.13) imply the following lower and upper bounds for the functional $\mathcal{J}(z,w)$. Indeed: (1) Consider $t(z,w)\geq 1$: after combining (5.16) and (5.11), it follows that
$$\begin{aligned}&\mathcal{J}(z,w)\\&=(\alpha+1)\int_\Omega\big(\frac{1}{p(x)}-\frac{1}{p^+\gamma^+}\big)t(z,w)^{\frac{p(x)}{p^+}}|\nabla z|^{p(x)}dx\\&\quad+(\beta+1)\int_\Omega\big(\frac{1}{q(x)}-\frac{1}{q^+\gamma^+}\big)t(z,w)^{\frac{q(x)}{q^+}}|\nabla w|^{q(x)}dx\end{aligned}$$

$$
\begin{aligned}
&\geq \Big[(\alpha+1)\frac{\gamma^+-1}{p^+\gamma^+}\int_\Omega |\nabla z|^{p(x)}dx \\
&\quad + (\beta+1)\frac{\gamma^+-1}{q^+\gamma^+}\int_\Omega |\nabla w|^{q(x)}dx\Big] t(z,w)^{\min(\frac{p^-}{p^+},\frac{q^-}{q^+})} \\
&\geq \Big[\frac{\alpha+1}{p^+\gamma^+}\int_\Omega |\nabla z|^{p(x)}dx + \frac{\beta+1}{q^+\gamma^+}\int_\Omega |\nabla w|^{q(x)}dx\Big](\gamma^+-1)Q(z,w)^{\min(\frac{p^-}{p^+},\frac{q^-}{q^+})}.
\end{aligned}
$$

The functional $\mathcal{J}(z,w)$ is bounded as follows:

$$
\begin{aligned}
\mathcal{J}(z,w) &\leq t(z,w)\Big[\frac{\alpha+1}{p^-}\int_\Omega |\nabla z|^{p(x)}dx + \frac{\beta+1}{q^-}\int_\Omega |\nabla w|^{q(x)}dx\Big] - t(z,w)^{\gamma^+}C(z,w) \\
&\leq \Big[\frac{\alpha+1}{p^-}\int_\Omega |\nabla z|^{p(x)}dx + \frac{\beta+1}{q^-}\int_\Omega |\nabla w|^{q(x)}dx\Big] Q(z,w)^{1/\gamma^+-1} \\
&\quad - \Big[\frac{\alpha+1}{p^+\gamma^+}\int_\Omega |\nabla z|^{p(x)}dx + \frac{\beta+1}{q^+\gamma^+}\int_\Omega |\nabla w|^{q(x)}dx\Big].
\end{aligned}
$$

(2) Now, consider $t(z,w) < 1$:

$$
\begin{aligned}
&\mathcal{J}(z,w) \\
&\geq t(z,w)\Big[\frac{\alpha+1}{p^+}\int_\Omega |\nabla z|^{p(x)}dx + \frac{\beta+1}{q^+}\int_\Omega |\nabla w|^{q(x)}dx\Big] - t(z,w)^{\gamma^+}C(z,w) \\
&\geq \Big[\frac{\alpha+1}{p^+}\int_\Omega |\nabla z|^{p(x)}dx + \frac{\beta+1}{q^+}\int_\Omega |\nabla w|^{q(x)}dx\Big] Q(z,w)^{1/\gamma^+-1} \\
&\quad - \Big[\frac{\alpha+1}{p^+\gamma^+}\int_\Omega |\nabla z|^{p(x)}dx + \frac{\beta+1}{q^+\gamma^+}\int_\Omega |\nabla w|^{q(x)}dx\Big].
\end{aligned}
$$

Also we have

$$
\begin{aligned}
\mathcal{J}(z,w) &= (\alpha+1)\int_\Omega \big(\frac{1}{p(x)} - \frac{1}{p^+\gamma^+}\big) t(z,w)^{\frac{p(x)}{p^+}} |\nabla z|^{p(x)}dx \\
&\quad + (\beta+1)\int_\Omega \big(\frac{1}{q(x)} - \frac{1}{q^+\gamma^+}\big) t(z,w)^{\frac{q(x)}{q^+}} |\nabla w|^{q(x)}dx \\
&\leq \Big[(\alpha+1)\big(\frac{1}{p^-} - \frac{1}{p^+\gamma^+}\big)\int_\Omega |\nabla z|^{p(x)}dx \\
&\quad + (\beta+1)\big(\frac{1}{q^-} - \frac{1}{q^+\gamma^+}\big)\int_\Omega |\nabla w|^{q(x)}dx\Big] t(z,w)^{\min(\frac{p^-}{p^+},\frac{q^-}{q^+})} \\
&\leq \Big[(\alpha+1)\big(\frac{1}{p^-} - \frac{1}{p^+\gamma^+}\big)\int_\Omega |\nabla z|^{p(x)}dx \\
&\quad + (\beta+1)\big(\frac{1}{q^-} - \frac{1}{q^+\gamma^+}\big)\int_\Omega |\nabla w|^{q(x)}dx\Big] Q(z,w)^{\min(\frac{p^-}{p^+\gamma^+},\frac{q^-}{q^+\gamma^+})}.
\end{aligned}
$$

We choose $(z,w) \in E$, then $Q(z,w)$ is reduced to become $Q(z,w) = \frac{1}{C(z,w)}$. Thus, the assert of the Proposition 5.10 follows. $\square$

Consider the optimal problem

$$
\inf_{\{(z,w)\in E;\ c(z,w)>0\}} \frac{1}{C(z,w)}. \tag{5.18}
$$

We claim that the infimum value is attained in E. To assert this claim, we need the following lemma.

Lemma 5.11. *Under assumption* (5.7)*, the optimal problem* (5.18) *possesses at least one solution.*

Proof. Solving (5.18) is equivalent to solving the maximizing problem:

$$\sup\Big\{\int_\Omega c(x)|z|^{\alpha+1}|w|^{\beta+1}dx;\ (z,w)\in E,\ C(z,w)>0\Big\} =: M. \tag{5.19}$$

Firstly, from Remarks 5.1 and 5.2, we observe that M is finite. Indeed, from the end of Remark 5.2 and the use of [19] or [31], for any $(z,w)\in E$, $0<C(z,w)\le\|c\|_\infty K$, (the constants $\|c\|_\infty$ and K are not depending on (z,w)). We follow the ideas of [6] and we show that there exists $(z_M,w_M)\in E$ such that $C(z,w)\le C(z_M,w_M)$ for any $(z,w)\in E$.

Let (z_n,w_n) be a maximizing sequence of (5.19) (i.e (z_n,w_n) is such that $A(z_n)=1$, $B(w_n)=1$ and $C(z_n,w_n)\to M>0$). It is easy to see that (z_n,w_n) is bounded in $X_0(x)$. It follows that $z_n\rightharpoonup\overline{z}$ weakly in $W_0^{1.p(x)}(\Omega)$ and $z_n\to\overline{z}$ strongly in $L^{\hat p}(\Omega)$. Similarly, $w_n\rightharpoonup\overline{w}$ weakly in $W_0^{1.q(x)}(\Omega)$ and $z_n\to\overline{z}$ strongly in $L^{\hat q}(\Omega)$. Consequently

$$C(z_n,w_n)\to C(\bar z,\bar w).$$

Moreover, since $z\mapsto\int_\Omega|\nabla z|^{p(x)}dx$ is a semimodular in the sense of [12, Definition 2.1.1], applying [12, Theorem 2.2.8]), we obtain that $p(x)$- and $q(x)$-modular functions $\rho_p(\cdot)$ and $\rho_q(\cdot)$ are weakly lower semicontinuous. So, since $z_n\rightharpoonup\bar z$ weakly in $W_0^{1.p(x)}(\Omega)$, we deduce that

$$\int_\Omega|\nabla\bar z|^{p(x)}\,dx\le\liminf_n\int_\Omega|\nabla\bar{z_n}|^{p(x)}\,dx=1$$

and

$$\int_\Omega|\nabla\bar w|^{q(x)}\,dx\le\liminf_n\int_\Omega|\nabla\bar{w_n}|^{q(x)}\,dx=1.$$

Now assume by contradiction that $\int_\Omega|\nabla\bar z|^{p(x)}\,dx<1$ and $\int_\Omega|\nabla\bar w|^{p(x)}\,dx<1$. Then we have $\|\bar z\|_{1,p(x)}<1$ and $\|\bar w\|_{1,p(x)}<1$. We set

$$a=\|\bar z\|_{1,p(x)}=\|\nabla\bar z\|_{L^{p(x)}},\quad b=\|\bar w\|_{1,q(x)}=\|\nabla\bar w\|_{L^{q(x)}}.$$

Using again the properties of the functions ρ_p and ρ_q, it follows that

$$\rho_p\big(|\nabla(\frac{1}{a}\bar z)|\big)=\int_\Omega|\nabla(\frac{1}{a}\bar z)|^{p(x)}\,dx=1$$

and

$$\rho_q\big(|\nabla(\frac{1}{b}\bar w)|\big)=\int_\Omega|\nabla(\frac{1}{b}\bar w)|^{q(x)}\,dx=1.$$

Obviously, we see that $\big(\frac1a\bar z,\frac1b\bar w\big)\in E$. On the other hand,

$$C\big(\frac1a z_n,\frac1b w_n\big)\to C\big(\frac1a\bar z,\frac1b\bar w\big)\quad\text{as } n\to+\infty.$$

However, we remark that

$$C\big(\frac1a\bar z,\frac1b\bar w\big)=\big(\frac1a\big)^{\alpha+1}\big(\frac1b\big)^{\beta+1}C(\bar z,\bar w)=\big(\frac1a\big)^{\alpha+1}\big(\frac1b\big)^{\beta+1}M.$$

Since $a<1$ and $b<1$, we obtain $C\big(\frac1a\bar z,\frac1b\bar w\big)>M$. A contradiction occurs. $\square$

Consequently, combining Proposition 5.10 and Lemma 5.11, we deduce that

$$\inf_{\{(z,w)\in E;\ C(z,w)>0\}} \mathcal{J}(z,w)$$

exists. In the next section, we will show that the infimum of the functional $\mathcal{J}(z,w)$ is attained on E.

Existence result for the optimal problem (5.18)**.** We are looking for $(z,w)\in E$ satisfying

$$\inf \mathcal{J}(z,w) : A(z)=1,\ B(w)=1. \tag{5.20}$$

To investigate (5.20), we give some lemmas and remarks.

Lemma 5.12. *Let E be the set defined as in* (5.17)*. Assume that the functions p and q satisfy hypothesis* (5.7)*. Then, for any $(z,w)\in X_0(x)$, there exit $\delta(z)>0$ and $\theta(w)>0$ such that*

$$\big(\frac{1}{\delta(z)}z, \frac{1}{\theta(w)}w\big)\in E.$$

Proof. For any fixed z in $W_0^{1,p(x)}(\Omega)\setminus\{0\}$, we define a function f on $]0,+\infty[$ by

$$f(z,\delta)=\int_\Omega (\frac{1}{\delta})^{p(x)}|\nabla z|^{p(x)}dx-1.$$

For any $\delta>1$, we have

$$(\frac{1}{\delta})^{p_+}\int_\Omega |\nabla z|^{p(x)}dx-1\le f(z,\delta)\le (\frac{1}{\delta})^{p^-}\int_\Omega |\nabla z|^{p(x)}dx-1.$$

Now, taking $\delta<1$, we obtain

$$(\frac{1}{\delta})^{p^-}\int_\Omega |\nabla z|^{p(x)}dx-1\le f(z,\delta)\le \big(\frac{1}{\delta}\big)^{p^+}\int_\Omega |\nabla z|^{p(x)}dx-1.$$

It follows from the above inequality that

- for δ large enough, $f(z,\delta)\to -1$ as $\delta\to+\infty$,
- for δ small enough, $f(z,\delta)\to+\infty$ as $\delta\to 0$.

By applying the Mean Value Theorem, we conclude that there exists $\delta_z\in]0,+\infty[$ such that

$$\int_\Omega \frac{1}{\delta_z^{p(x)}}|\nabla z|^{p(x)}dx=1.$$

Similarly, we can prove that, there exists $\theta_w>0$ such that:

$$\int_\Omega \frac{1}{\theta_w^{q(x)}}|\nabla w|^{q(x)}dx=1.$$

The proof is complete. $\square$

Lemma 5.13. *Let $(z,w)\in X_0(x)$ be fixed. The functions $z\mapsto\delta(z)$ defined in Lemma 5.12 possess C^1-regularity respectively from $\mathcal{U}_{z,\delta_z}$ to $\mathbb{I}$ and $\mathcal{V}_{w,\theta_w}$ to $\mathbb{I}$. Here, $\mathcal{U}_{z,\delta_z}$ is a neighborhood of (z,δ_z) lying on the open set $\mathcal{U}=W_0^{1,p(x)}(\Omega)\setminus\{0\}\times]0,+\infty[$ and $\mathcal{V}_{w,\theta_w}$ is a neighborhood of (w,θ_w) lying on the open set $\mathcal{V}=W_0^{1,q(x)}(\Omega)\setminus\{0\}\times]0,+\infty[$.*

Proof. After making a simple computation, it is easily to see that

$$\frac{\partial f}{\partial \delta}(z,\delta) = -\frac{1}{\delta}\int_\Omega p(x)\frac{1}{\delta^{p(x)}}|\nabla z|^{p(x)}dx.$$

Replacing δ by δ_z, we have

$$|\frac{\partial f}{\partial \delta}(z,\delta_z)| > \frac{p^-}{\delta_z} > 0.$$

Hence, the implicit function theorem implies that there exist a neighborhood of $(z,\delta_z), \mathcal{U}_{z,\delta_z} \subset \mathcal{U}$ and a function of class $C^1 : z \longmapsto \delta(z)$ from $\mathcal{U}_{z,\delta_z}$ to $\mathbb{I}$. Particularly, for all z in $W_0^{1,q(x)}(\Omega)$, we have

$$\delta'(z)\cdot\phi = -\frac{\frac{\partial f}{\partial z}(z,\delta_z)\cdot\phi}{\frac{\partial f}{\partial \delta}(z,\delta_z)}. \tag{5.21}$$

Since we have

$$\frac{\partial f}{\partial z}(z,\delta_z)\cdot\phi = \int_\Omega p(x)\frac{1}{\delta_z^{p(x)}}|\nabla z|^{p(x)-2}\nabla z\cdot\nabla\phi dx,$$

the definition (5.21) then becomes

$$\delta'(z)\cdot\phi = -\frac{\int_\Omega p(x)\frac{1}{\delta_z^{p(x)}}|\nabla z|^{p(x)-2}\nabla z\cdot\nabla\phi dx}{\frac{1}{\delta}_z\int_\Omega p(x)\frac{1}{\delta_z^{p(x)}}|\nabla z|^{p(x)}dx}. \tag{5.22}$$

In the same way, we have

$$\theta'(w)\cdot\psi = -\frac{\int_\Omega q(x)\frac{1}{\theta_w^{q(x)}}|\nabla w|^{q(x)-2}\nabla w\cdot\nabla\psi dx}{\frac{1}{\theta}_w\int_\Omega q(x)\frac{1}{\theta_w^{q(x)}}|\nabla w|^{q(x)}dx.}. \tag{5.23}$$

$\square$

Remark 5.14. We introduce the functional $\widetilde{\mathbf{J}}$ defined on $W_0^{1,p(x)} \times W_0^{1,q(x)} \times \mathbb{I}$ by

$$\widetilde{\mathbf{J}}(z,w,t) = \mathbf{J}(t^{1/p^+}z, t^{1/q^+}w). \tag{5.24}$$

Thus, for any $(z,w) \in X_0(x) \setminus \{(0,0)\}$ and $t(z,w)$ given by (5.11), this definition implies that

$$\widetilde{\mathbf{J}}(z,w,t(z,w)) = \mathcal{J}(z,w) \tag{5.25}$$

where the functional $\mathcal{J}$ is given by (5.16).

Lemma 5.15. *Let $(z_n,w_n) \in E$ be a minimizing sequence of* (5.20)*, the sequence (u_n,v_n) with*

$$u_n = t(z_n,w_n)^{1/p^+}z_n, \quad v_n = t(z_n,w_n)^{1/q^+}w_n$$

is then a Palais-Smale sequence for the functional $\mathbf{J}$*. i.e.,*

$$\mathbf{J}(u_n,v_n) \le m, \tag{5.26}$$

$$\mathbf{J}'(u_n,v_n) \to 0, \text{ in the meaning of the norm } \|\cdot\|_{X_0^*(x)}. \tag{5.27}$$

Proof. We follow the ideas of [3]. For a best understanding, some of the notation used here remain unchanged. Generalizing [3], we define $\pi : W_0^{1,p(x)}(\Omega)\setminus\{0\} \to \mathbb{I}$ by

$$\pi(z) = (\pi_1(z),\pi_2(z)) = \big(\delta(z), \frac{z}{\delta(z)}\big)$$

and $\tau : W_0^{1,q(x)}(\Omega) \setminus \{0\} \to \mathbb{I}$ by

$$\tau(w) = (\tau_1(w), \tau_2(w)) = \big(\theta(w), \frac{w}{\theta(w)}\big).$$

Before continuing, let us designate by $T_{(z,w)}E$ the tangent space to E. Denote

$$E_p = \{z \in W_0^{1,p(x)}(\Omega); A(z) = 1\}$$

(respectively, $E_q = \{w \in W_0^{1,q(x)}(\Omega); B(z) = 1\}$), hence, it is clear that $T_{(z,w)}E = T_zE_p \times T_wE_q$. Moreover, for any $(z,w) \in X_0(x)$, for any $(\Phi,\Psi) \in T_{(z,w)}E$, we have

$$\mathcal{J}'(z,w)(\Phi,\Psi) = \frac{\partial \widetilde{\mathbf{J}}}{\partial z}(z,w,t(z,w))(\Phi) + \frac{\partial \widetilde{\mathbf{J}}}{\partial w}(z,w,t(z,w))(\Psi).$$

Now, we consider a minimizing sequence $(z_n, w_n) \in E$. For any $(\phi,\psi) \in X_0(x)$, it is obvious that $(\pi_2'(z_n)\cdot\phi, \tau_2'(w_n)\cdot\psi) \in T_{(z,w)}E$.

From the above, setting $B_n = (z_n, w_n, t(z_n,w_n))$ and following the spirit of the proof of the [3, Lemma 3.1], we have:

$$\begin{gathered}
\frac{\partial \mathbf{J}}{\partial u}(u_n,v_n)(\phi) = \frac{\partial \widetilde{\mathbf{J}}}{\partial z}(B_n)(\pi_2'(z_n)\cdot\phi),\\
\frac{\partial \mathbf{J}}{\partial v}(u_n,v_n)(\phi) = \frac{\partial \widetilde{\mathbf{J}}}{\partial w}(B_n)(\pi_2'(w_n)\cdot\psi),\\
\mathcal{J}'(z_n,w_n)(\pi_2'(z_n)\cdot\phi, \tau_2'(w_n)\cdot\psi) = \frac{\partial \widetilde{\mathbf{J}}}{\partial z}(B_n)(\pi_2'(z_n)\cdot\phi) + \frac{\partial \widetilde{\mathbf{J}}}{\partial w}(B_n)(\tau_2'(w_n)\cdot\psi).
\end{gathered}$$

Then, since

$$\mathbf{J}'(u_n,v_n)(\phi,\psi) = \frac{\partial \mathbf{J}}{\partial u}(u_n,v_n)(\phi) + \frac{\partial \mathbf{J}}{\partial v}(u_n,v_n)(\psi)$$

for any $(\phi,\psi) \in X_0(x)$, it follows that

$$\mathbf{J}'(u_n,v_n)(\phi,\psi) = \mathcal{J}'(z_n,w_n)(\pi_2'(z_n)\cdot\phi, \tau_2'(w_n)\cdot\psi).$$

However, applying the Ekeland variational principle, we have

$$|\mathcal{J}'(z_n,w_n)(\pi_2'(z_n)\cdot\phi, \tau_2'(w_n)\cdot\psi)| \le \frac{1}{n}\|(\pi_2'(z_n)\cdot\phi, \tau_2'(w_n)\cdot\psi)\|_{X_0(x)},$$

for all $(\phi,\psi) \in X_0(x)$. Therefore,

$$|\mathbf{J}'(u_n,v_n)\cdot(\phi,\psi)| \le \frac{1}{n}\big\|\big(\pi_2'(z_n)\cdot\phi, \tau_2'(w_n)\cdot\psi\big)\big\|_{X_0(x)}, \quad \forall (\phi,\psi) \in X_0(x).$$

The space $X_0(x)$ is equipped with the cartesian norm $\|\cdot\|_{X_0(x)} = \|\cdot\|_{1,p(x)} + \|\cdot\|_{1,q(x)}$. Then the following estimate holds

$$|\mathbf{J}'(u_n,v_n)\cdot(\phi,\psi)| \le \frac{1}{n}\Big(\|(\pi_2'(z_n)\cdot\phi\|_{1,p(x)} + \|\tau_2'(w_n)\cdot\psi)\|_{1,q(x)}\Big). \tag{5.28}$$

To simplify notation, we set $\tilde{\delta}_n = \delta(z_n)$. So, from the definition of π_2, we check that

$$\pi_2'(z_n)\cdot\phi = \frac{\phi}{\tilde{\delta}_n} - \frac{z_n \int_\Omega p(x)\frac{1}{\tilde{\delta}_n^{p(x)}}|\nabla z_n|^{p(x)-2}\nabla z_n\cdot\nabla\phi dx}{\frac{1}{\tilde{\delta}}_n \int_\Omega p(x)\frac{1}{\tilde{\delta}_n^{p(x)}}|\nabla z_n|^{p(x)}dx}.$$

Thus,

$$\|\pi_2'(z_n)\cdot\phi\|_{1,p(x)} \le \frac{\|\phi\|_{1,p(x)}}{\tilde{\delta}_n} + \frac{\|z_n\|_{1,p(x)}|\int_\Omega p(x)\frac{1}{\tilde{\delta}_n^{p(x)}}|\nabla z_n|^{p(x)-2}\nabla z_n\cdot\nabla\phi dx|}{\frac{1}{\tilde{\delta}}_n \int_\Omega p(x)\frac{1}{\tilde{\delta}_n^{p(x)}}|\nabla z_n|^{p(x)}dx}$$

$$\le \frac{\|\phi\|_{1,p(x)}}{\tilde{\delta}_n} + \frac{|\int_\Omega p(x)\frac{1}{\tilde{\delta}_n^{p(x)}}|\nabla z_n|^{p(x)-2}\nabla z_n \cdot \nabla\phi dx|}{\int_\Omega p(x)\frac{1}{\tilde{\delta}_n^{p(x)}}|\nabla z_n|^{p(x)}dx}.$$

Particularly, applying successively the Hölder inequality for $p(x)$-Lebesgue space [30, 31, 19], we find

$$\begin{aligned} \Big|\int_\Omega p(x)\frac{|\nabla z_n|^{p(x)-2}}{z_n^{p(x)-2}}\frac{\nabla z_n}{\tilde{\delta}_n}\cdot\frac{\nabla\phi}{\tilde{\delta}_n}dx\Big| &\le p+\Big|\frac{|\nabla z_n|^{p(x)-1}}{\tilde{\delta}_n^{p(x)-1}}\Big|_{L^{\frac{p(x)}{p(x)-1}}(\Omega)}\frac{\|\phi\|_{1,p(x)}}{\tilde{\delta}_n} \\ &= p^+\frac{\|\phi\|_{1,p(x)}}{\tilde{\delta}_n}. \end{aligned} \tag{5.29}$$

$$\int_\Omega p(x)\frac{1}{\tilde{\delta}_n^{p(x)}}|\nabla z_n|^{p(x)}dx \ge p^-\int_\Omega \frac{1}{\tilde{\delta}_n^{p(x)}}|\nabla z_n|^{p(x)}dx \ge p^-. \tag{5.30}$$

The above remarks allow us to obtain the new estimate:

$$\|\pi_2'(z_n)\cdot\phi\|_{1,p(x)} \le \big(1+\frac{p+}{p^-}\big)\frac{\|\phi\|_{1,p(x)}}{\tilde{\delta}_n}.$$

From the properties on the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ spaces (see for instance [19]), and because $\int_\Omega \frac{|\nabla z_n|^{p(x)}}{\tilde{\delta}_n^{p(x)}}dx = 1$ and $\int_\Omega |\nabla z_n|^{p(x)}dx = 1$, we have $\|z_n\|_{1,p(x)} = \tilde{\delta}_n = 1$. Therefore

$$\|\pi_2'(z_n)\cdot\phi\|_{1,p(x)} \le \big(1+\frac{p^+}{p^-}\big)\|\phi\|_{1,p(x)}.$$

Similarly,

$$\|\tau_2'(w_n)\cdot\psi\|_{1,q(x)} \le \big(1+\frac{q^+}{q^-}\big)\|\psi\|_{1,q(x)}.$$

Taking into account the estimate (5.28), we conclude that

$$\lim_{n\to+\infty}\|\mathbf{J}'(u_n,v_n)\|_{X_0^*(x)} = 0.$$

This completes the proof. □

Lemma 5.16. *Assume that* (5.6) *holds. Let* (z_n, w_n) *be a minimizing sequence of* $\mathbf{J}$ *on the manifold* E. *The sequence* $(u_n, v_n) = (t(z_n, w_n)^{1/p^+}z_n, t(z_n, w_n)^{1/q^+}w_n)$ *is bounded in* $X_0(x)$.

Proof. Since $u_n = t(z_n, w_n)^{1/p^+}z_n$, $v_n = t(z_n, w_n)^{1/q^+}w_n$, by the characterization (5.11), it follows that

$$\int_\Omega |\nabla u_n|^{p(x)}dx + \int_\Omega |\nabla v_n|^{q(x)}dx - 2\int_\Omega c(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}dx = 0. \tag{5.31}$$

On the other hand, because (z_n, w_n) is a minimizing sequence for $\inf_{(z,w)\in E}\mathcal{J}(z,w)$, we have

$$m \le (\alpha+1)\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + (\beta+1)\int_\Omega \frac{1}{q(x)}|\nabla v_n|^{q(x)}dx - C(u_n,v_n) < m+\frac{1}{n}. \tag{5.32}$$

Combining (5.31) and (5.32), one concludes that

$$m \le \int_\Omega \big(\frac{\alpha+1}{p(x)}-1\big)|\nabla u_n|^{p(x)}dx + \int_\Omega \big(\frac{\beta+1}{q(x)}-1\big)|\nabla v_n|^{q(x)}dx + C(u_n,v_n) < m+\frac{1}{n}.$$

Recall that

$$\int_{\Omega}|\nabla u_n|^{p(x)}dx=\int_{\Omega}|\nabla v_n|^{q(x)}dx=\int_{\Omega}c(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}dx,$$

hence

$$m\le\int_{\Omega}\frac{\alpha+1}{p(x)}|\nabla u_n|^{p(x)}dx+\int_{\Omega}\big(\frac{\beta+1}{q(x)}-1\big)|\nabla v_n|^{q(x)}dx<m+\frac{1}{n}.$$

After making some easy calculations, we obtain

$$\int_{\Omega}|\nabla u_n|^{p(x)}dx<\frac{m+1}{\gamma^+-1}.$$

Arguing similarly, we find that

$$\int_{\Omega}|\nabla v_n|^{q(x)}dx<\frac{m+1}{\gamma^+-1}.$$

We have proved that the sequence is bounded in $X_0(x)$. □

Lemma 5.17. *Under hypothesis* (5.7)*, problem* (5.20) *possesses at least one solution.*

Proof. We divide the proof in three steps.
Step 1: Weak convergence of u_n and v_n. Let $(z_n,w_n)\in E$ be a minimizing sequence. From Lemma 5.15, it is known that $\lim_{n\to+\infty}\mathbf{J}(u_n,v_n)=m$ and $\lim_{n\to+\infty}\|\mathbf{J}'(u_n,v_n)\|_{X_0^*(x)}=0$ and that $(u_n,v_n)$is bounded in $X_0(x)$. Extracting if necessary to a subsequence, there exists a pair (u^*,v^*) in $X_0(x)$ such that

$$\begin{gathered}u_n\rightharpoonup u^*\quad\text{in } W_0^{1.p(x)}(\Omega),\\ v_n\rightharpoonup v^*\quad\text{in } W_0^{1.q(x)}(\Omega).\end{gathered}$$

Step 2: Strong convergence of u_n and v_n in $W_0^{1,p(x)}(\Omega)$ (resp. $W_0^{1,q(x)}(\Omega)$). To do this, we establish that u_n and v_n are two Cauchy sequences. Firstly, easy calculations ensure that for any $m\in\mathbb{N}$ and $l\in\mathbb{N}$,

$$\begin{aligned}&\big[\mathbf{J}'(u_m,v_m)-\mathbf{J}'(u_l,v_l)\big](u_m-u_l,0)\\&=(\alpha+1)\int_{\Omega}(|\nabla u_m|^{p(x)-2}\nabla u_m-|\nabla u_l|^{p(x)-2}\nabla u_l)(\nabla u_m-u_l)dx\\&\quad-(\alpha+1)\int_{\Omega}c(x)\big[|v_m|^{(\beta+1}|u_m|^{\alpha-1}u_m-|v_l|^{(\beta+1}|u_l|^{\alpha-1}u_l\big](u_m-u_l)dx.\end{aligned}$$

Thus, after making a suitable rearrangement, we obtain

$$\begin{aligned}&\int_{\Omega}(|\nabla u_m|^{p(x)-2}\nabla u_m-|\nabla u_l|^{p(x)-2}\nabla u_l)(\nabla u_m-u_l)dx\\&=\frac{1}{\alpha+1}\big[\mathbf{J}'(u_m,v_m)-\mathbf{J}'(u_l,v_l)\big](u_m-u_l,0)dx\\&\quad+\int_{\Omega}c(x)\big[|v_m|^{\beta+1}|u_m|^{\alpha-1}u_m-|v_l|^{\beta+1}|u_l|^{\alpha-1}u_l\big](u_m-u_l)dx.\end{aligned}$$

We claim that

$$\int_{\Omega}c(x)\big[|v_m|^{(\beta+1}|u_m|^{\alpha-1}u_m-|v_l|^{(\beta+1}|u_l|^{\alpha-1}u_l\big](u_m-u_l)dx\to0,\tag{5.33}$$

as $m, l \to +\infty$. Indeed in view of (5.33) and according to Remarks 5.1 and 5.2 (the notation used remains the same), we observe that

$$
\begin{aligned}
&\Big| \int_\Omega c(x)\big[|v_m|^{\beta+1}|u_m|^{\alpha-1}v_m - |v_l|^{\beta+1}|u_l|^{\alpha-1}u_l\big](u_m - u_l)dx\Big| \\
&\leq \int_\Omega c(x)|v_m|^{\beta+1}|u_m|^{\alpha}|u_m - u_l|dx + \int_\Omega c(x)|v_l|^{\beta+1}|u_l|^{\alpha}|u_m - u_l|dx \\
&\leq \|c\|_\infty \|v_m\|^{\beta+1}_{L^{\hat q}(\Omega)} \|u_m\|^{\alpha}_{L^{\hat p}(\Omega)} \|u_m - u_l\|_{L^{\hat p}(\Omega)} \\
&\quad + \|c\|_\infty \|v_l\|^{\beta+1}_{L^{\hat q}(\Omega)} \|u_l\|^{\alpha}_{L^{\hat p}(\Omega)} \|u_m - u_l\|_{L^{\hat p}(\Omega)} \\
&\leq C\|u_m - u_l\|_{L^{\hat p}(\Omega)}.
\end{aligned}
\tag{5.34}
$$

Before continuing, we recall a fundamental convergence property. It is well known that the imbedding $W_0^{1.p(x)}(\Omega) \hookrightarrow L^{\delta(x)}(\Omega)$ (resp $W_0^{1.q(x)}(\Omega) \hookrightarrow L^{\gamma(x)}(\Omega)$) with $\delta(x) < \frac{Np(x)}{N-p(x)}$ (resp. $\gamma(x) < \frac{Nq(x)}{N-q(x)}$) is compact (see [19]).

Choose $\gamma(x) = \hat p$, it follows that u_n converges strongly to u^* in $L^{\hat p}(\Omega)$ and so on, u_n is a Cauchy sequence in sense of the $L^{\hat p}(\Omega)$ norm. Consequently, (5.34) occurs. Furthermore, following [30], there exist constants C_1, C_2, C_3, C_4 such that

$$
\langle F(\nabla u_m) - F(\nabla u_l), u_m - u_l\rangle \geq
\begin{cases}
C_1\|u_m - u_l\|^2_{1,p(x)}, & \text{if } 1 < p(x) < 2,\\
C_2\|u_m - u_l\|^{2p_{0,1}/p_{1,1}}_{1,p(x)}, & \text{if } 1 < p(x) < 2,\\
C_3\|u_m - u_l\|^{p_{0,2}}_{1,p(x)}, & \text{if } 2 \geq p(x),\\
C_4\|u_m - u_l\|^{p_{1,2}}_{1,p(x)}, & \text{if } 2 \geq p(x),
\end{cases}
\tag{5.35}
$$

where $F(\xi) = |\xi|^{p(x)-2}\xi$ for all $\xi \in \mathbb{I}^N$; $p_{0,j} = \inf_{x\in\Omega_j} p(x)$ and $p_{1,j} = \sup_{x\in\Omega_j} p(x)$ for $j = 1, 2$; $\Omega_1 = \{x \in \Omega; 1 < p(x) < 2\}$; and $\Omega_2 = \{x \in \Omega;\quad 2 \geq p(x)\}$.

Then, from (5.27), (5.33) and (5.35), we conclude that u_n converges strongly to $u*$ in $W_0^{1,p(x)}(\Omega)$. Similar argues allow to prove that the sequence v_n converges to v^* strongly in $W_0^{1,q(x)}(\Omega)$.

Step 3: (u^*, v^*) is a solution of (1.1) involving a fibering decomposition. We show that $u^* = \bar r \bar z$ and $v^* = \bar\rho \bar w$ involve a solution of (1.1) via the fibering method. Let us recall that $\bar z$ and $\bar w$ are respectively the weak limit of z_n and w_n is the weak limit of w_n. The sequence $t_n = t(z_n, w_n)$ is defined as in (5.11). To simplify notation, we set $r_n = t_n^{1/p^+}$ and $\rho_n = t_n^{1/q^+}$. Moreover, using (5.12) and (5.13), by extracting subsequences, if necessary, we can assume that t_n converges in $\mathbb{I}$. We designate as $\bar t = \lim_{n\to+\infty} t_n$. So, it follows $r_n \to \bar t^{1/p^+}$ and $\rho_n \to \bar t^{1/q^+}$ when n tends to $+\infty$. We set $\bar r = \bar t^{1/p^+}$ and $\bar\rho = \bar t^{1/q^+}$.

Because of the formulation

$$
\frac{1}{r_n}u_n - \frac{1}{\bar r}u^* = \frac{1}{r_n \bar r}[(\bar r - r_n)u^* + \bar r(u_n - u^*)],
$$

and the convergence results announced above, it is clear that

$$
\|\frac{u_n}{r_n} - \frac{u^*}{\bar r}\|_{1,p(x)} \to 0, \quad \text{as } n \text{ tends to } +\infty.
$$

In other words, since $\frac{u_n}{r_n} = z_n$, we deduce that z_n converges strongly to $\frac{u^*}{\bar r}$ in $W_0^{1,p(x)}(\Omega)$.

Thus, since z_n converges weakly to $\bar{z}$ in $W_0^{1,p(x)}(\Omega)$, we deduce from above and also from uniqueness $\bar{z} = \frac{u^*}{\bar{r}}$. On the other hand

$$\|u^*\|_{1,p(x)} \leq \liminf_n \|u_n\|_{1,p(x)} \leq \limsup_n \|u_n\|_{1,p(x)}.$$

So

$$\|\bar{z}\|_{1,p(x)}\bar{r} \leq \liminf_n \|u_n\|_{1,p(x)} \leq \limsup_n \|u_n\|_{1,p(x)}$$

thus

$$\|\bar{z}\|_{1,p(x)}r \leq \liminf_n r_n\|z_n\| \leq \limsup_n \|u_n\|_{1,p(x)}.$$

Since $\|z_n\|_{1,p(x)} = 1$ and $\|u_n\|_{1,p(x)} \leq \|u_n - u^*\|_{1,p(x)} + \|u^*\|_{1,p(x)}$, we obtain

$$\|\bar{z}\|_{1,p(x)}\bar{r} \leq \bar{r} \leq \|\bar{z}\|_{1,p(x)}\bar{r}$$

thus after dividing by $\bar{r} > 0$, it occurs $\|\bar{z}\|_{1,p(x)} = 1$. In the same manner, we obtain $\|\bar{w}\|_{1,q(x)} = 1$. We can conclude that $(\bar{z}, \bar{w})$ is solution of the conditional problem (5.20). Furthermore, since $\|\bar{z}\|_{1,p(x)} = \|\bar{w}\|_{1,q(x)} = 1$, using [19], we deduce the second part of the Theorem 3.2. The proof is complete. □

The material needed to prove Theorem 3.2 is complete. Next, we establish that the boundary value problem (1.1) admits at least one solution.

5.3. **Proof of Theorem 3.2.** Existence of a critical point for $\mathbf{J}$.

Proof. The previous lemmas imply that $(\bar{z}, \bar{w})$ is a conditional critical point for $\mathcal{J}$. From the Euler-Lagrange characterization, we deduce that there is a pair (m_1, m_2) in $\mathbb{I}^2$ such that for any $(h, k) \in X_0(x)$,

$$\nabla\mathcal{J}(\bar{z}, \bar{w}) \cdot (h, k) = m_1\nabla A(\bar{z}, \bar{w}) \cdot (h, k) + m_2\nabla B(\bar{z}, \bar{w}) \cdot (h, k). \tag{5.36}$$

In (5.36), we choose $h = \bar{z}$, $k = \bar{w}$, we obtain

$$\mathcal{J}'(\bar{z}, \bar{w})(\bar{z}, \bar{w}) = 0. \tag{5.37}$$

Combining (5.36) and (5.37), we obtain

$$\begin{gathered} m_1A^{(1)} \cdot (\bar{z}, \bar{w}) + m_2B^{(1)} \cdot (\bar{z}, \bar{w}) = 0 \\ m_1A^{(2)} \cdot (\bar{z}, \bar{w}) + m_2B^{(2)} \cdot (\bar{z}, \bar{w}) = 0. \end{gathered}$$

Here, $A^{(1)}$, $B^{(1)}$ (resp. $A^{(2)}$ and $B^{(2)}$) denote the first derivatives with respect to z (resp. w). Note taht

$$\det\begin{pmatrix} A^{(1)} \cdot (\bar{z}, \bar{w}) & B^{(1)} \cdot (\bar{z}, \bar{w}) \\ A^{(2)} \cdot (\bar{z}, \bar{w}) & B^{(2)} \cdot (\bar{z}, \bar{w}) \end{pmatrix} \geq p^-q^-A(\bar{z})B(\bar{w}) = p^-q^- > 0.$$

It follows that $m_1 = m_2 = 0$. Consequently, $\mathcal{J}'(\bar{z}, \bar{w}) = 0$, or again,

$$\mathbf{J}'(\bar{r}\bar{z}, \bar{\rho}\bar{w}) = 0$$

Finally, we can conclude that $(u^*, v^*) = (\bar{r}\bar{z}, \bar{\rho}\bar{w})$ is a critical point of $\mathbf{J}$. □

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