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STABILITY OF PARABOLIC EQUATIONS WITH UNBOUNDED OPERATORS ACTING ON DELAY TERMS

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Abstract. In this article, we study the stability of the initial value problem for the delay differential equation

$$\begin{gathered}\frac{dv(t)}{dt} + Av(t) = B(t)v(t-\omega) + f(t), \quad t \geq 0, \\ v(t) = g(t) \quad (-\omega \leq t \leq 0)\end{gathered}$$

in a Banach space E with the unbounded linear operators A and $B(t)$ with dense domains $D(A) \subseteq D(B(t))$. We establish stability estimates for the solution of this problem in fractional spaces E_α. Also we obtain stability estimates in Hölder norms for the solutions of the mixed problems for delay parabolic equations with Neumann condition with respect to space variables.

1. Introduction

Stability of delay ordinary differential and difference equations and delay partial differential and difference equations with bounded operators acting on delay terms has been studied extensively and and developed over the previous three decades; see, for example [1, 3, 4, 5, 6, 20, 21, 23, 24, 29, 31, 32, 33] and their references. The theory of stability of delay partial differential and difference equations with unbounded operators acting on delay terms has received less attention than delay ordinary differential and difference equations (see, [2, 7, 8, 9, 22, 25]). It is known that various initial-boundary value problems for linear evolutionary delay partial differential equations can be reduced to initial value problems of the form

$$\begin{gathered}\frac{dv(t)}{dt} + Av(t) = B(t)v(t-\omega) + f(t), \quad t \geq 0, \\ v(t) = g(t) \quad (-\omega \leq t \leq 0),\end{gathered} \tag{1.1}$$

where E is an arbitrary Banach space, A and $B(t)$ are unbounded linear operators in E with dense domains $D(A) \subseteq D(B(t))$. Let A be a strongly positive operator, i.e. $-A$ is the generator of the analytic semigroup $\exp\{-tA\}$ $(t \geq 0)$ of the linear bounded operators with exponentially decreasing norm when $t \to \infty$. That means the following estimates hold:

$$\|\exp\{-tA\}\|_{E\to E} \leq Me^{-\delta t}, \quad \|tA\exp\{-tA\}\|_{E\to E} \leq M, \quad t > 0 \tag{1.2}$$

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for some $M > 1$, $\delta > 0$. Let $B(t)$ be closed operators.

A function $v(t)$ is called a solution of problem (1.1) if the following conditions are satisfied:

(i) $v(t)$ is continuously differentiable on the interval $[-\omega, \infty)$. The derivative at the endpoint $t = -\omega$ is understood as the appropriate unilateral derivative.
(ii) The element $v(t)$ belongs to $D(A)$ for all $t \in [-\omega, \infty)$, and the function $Av(t)$ is continuous on the interval $[-\omega, \infty)$.
(iii) $v(t)$ satisfies the equation and the initial condition (1.1).

A solution $v(t)$ of the initial value problem (1.1) is said to be stable if

$$\|v(t)\|_E \le \max_{-\omega \le t \le 0} \|g(t)\|_E + \int_0^t \|f(s)\|_E ds \tag{1.3}$$

for every t, $-\omega \le t < \infty$. We are interested in studying the stability of solutions of the initial value problem under the assumption that

$$\|B(t)A^{-1}\|_{E \to E} \le 1 \tag{1.4}$$

holds for every $t \ge 0$. We have not been able to obtain the estimate (1.3) in the arbitrary Banach space E. Nevertheless, we can establish the analog of estimates (1.3) where the space E is replaced by the fractional spaces $E_\alpha(0 < \alpha < 1)$ under a strong assumption than (1.4). The stability estimates in Hölder norms for the solutions of the mixed problem of the delay differential equations of the parabolic type are obtained.

The present article is organized as follows. Section 1 provides all necessary background. In Section 2, Theorems on stability estimates for the solution of the initial value value problem (1.1) are established. In Section 3, the stability estimates in Hölder norms for the solutions of the initial-boundary value problem for one dimensional delay parabolic equations with Neumann condition with respect to space variables are obtained. Finally, Section 4 is conclusion.

2. Theorems on stability

The strongly positive operator A defines the fractional spaces $E_\alpha = E_\alpha(E, A)$ $(0 < \alpha < 1)$ consisting of all $u \in E$ for which the following norms are finite:

$$\|u\|_{E_\alpha} = \sup_{\lambda > 0} \|\lambda^{1-\alpha} A \exp\{-\lambda A\} u\|_E.$$

We consider the initial value problem (1.1) for delay differential equations of parabolic type in the space $C(E_\alpha)$ of all continuous functions $v(t)$ defined on the segment $[0, \infty)$ with values in a Banach space E_α. First, we consider the problem (1.1) when A^{-1} and $B(t)$ commute; i.e.,

$$A^{-1}B(t)u = B(t)A^{-1}u, \quad u \in D(A). \tag{2.1}$$

Theorem 2.1. *Assume that the condition*

$$\|B(t)A^{-1}\|_{E \to E} \le \frac{(1-\alpha)}{M2^{2-\alpha}} \tag{2.2}$$

holds for every $t \ge 0$, where M is the constant from (1.2). Then for every $t \ge 0$ we have

$$\|v(t)\|_{E_\alpha} \le \max_{-\omega \le t \le 0} \|g(t)\|_{E_\alpha} + \int_0^t \|f(s)\|_{E_\alpha} ds. \tag{2.3}$$

Proof. It is clear that $v(t)=u(t)+w(t)$, where $u(t)$ is the solution of the problem

$$\begin{gathered}\frac{du(t)}{dt}+Au(t)=B(t)u(t-\omega),\quad t\geq 0,\\ u(t)=g(t)\quad(-\omega\leq t\leq 0)\,,\end{gathered}\tag{2.4}$$

and $w(t)$ is the solution of the problem

$$\begin{gathered}\frac{dw(t)}{dt}+Aw(t)=B(t)w(t-\omega)+f(t),\quad t\geq 0,\\ w(t)=0\quad(-\omega\leq t\leq 0)\end{gathered}\tag{2.5}$$

In [8], under the assumption of this theorem it was established that the stability inequality

$$\|u(t)\|_{E_\alpha}\leq\max_{-\omega\leq t\leq 0}\|g(t)\|_{E_\alpha}\tag{2.6}$$

holds for the solution of the problem (2.4) for every $t\geq 0$. Therefore, to prove the theorem it suffices to establish the stability inequality

$$\|w(t)\|_{E_\alpha}\leq\int_0^t\|f(s)\|_{E_\alpha}ds.\tag{2.7}$$

for the solution of the problem (2.5). Now, we consider the problem (2.5). Using the formula

$$w(t)=\int_0^t\exp\{-(t-s)A\}f(s)ds,\tag{2.8}$$

the semigroup property, and the definition of the spaces E_a, we obtain

$$\begin{aligned}\lambda^{1-\alpha}\|A\exp\{-\lambda A\}w(t)\|_E&\leq\lambda^{1-\alpha}\int_0^t\|A\exp\{-(\lambda+t-s)A\}f(s)\|_E ds\\&\leq\int_0^t\frac{\lambda^{1-\alpha}}{(\lambda+t-s)^{1-\alpha}}\|f(s)\|_{E_\alpha}ds\\&\leq\int_0^t\|f(s)\|_{E_\alpha}ds\end{aligned}$$

for every t with $0\leq t\leq\omega$ and λ with $\lambda>0$. This shows that

$$\|w(t)\|_{E_\alpha}\leq\int_0^t\|f(s)\|_{E_\alpha}ds\tag{2.9}$$

for every t, $0\leq t\leq\omega$. Applying the mathematical induction, one can easily show that it is true for every t. Namely, assume that the inequality (2.9) is true for $t,(n-1)\omega\leq t\leq n\omega$, $n=1,2,3,\dots$, for some n. Using the formula

$$\begin{aligned}w(t)&=\exp\{-(t-n\omega)A\}w(n\omega)+\int_{n\omega}^t\exp\{-(t-s)A\}B(s)w(s-\omega)ds\\&\quad+\int_{n\omega}^t\exp\{-(t-s)A\}f(s)\,ds,\end{aligned}\tag{2.10}$$

the semigroup property, the definition of the spaces E_a, estimate (1.2) and condition (2.2), we obtain

$$\lambda^{1-\alpha}\|A\exp\{-\lambda A\}w(t)\|_E$$

$$\leq\lambda^{1-\alpha}\|A\exp\{-(\lambda+t-n\omega)A\}w(n\omega)\|_E+\lambda^{1-\alpha}\int_{n\omega}^t\|A\exp\{-\frac{\lambda+t-s}{2}A\}\|_{E\to E}$$

$$
\begin{aligned}
&\times \|B(s)A^{-1}\|_{E\to E}\|A\exp\{-\frac{\lambda+t-s}{2}A\}w(s-\omega)\|_E ds\\
&+\lambda^{1-\alpha}\int_{n\omega}^{t}\|A\exp\{-(\lambda+t-s)A\}f(s)\|_E ds\\
&\le \frac{\lambda^{1-\alpha}}{(\lambda+t-n\omega)^{1-\alpha}}\|w(n\omega)\|_{E_\alpha}+\lambda^{1-\alpha}(1-\alpha)\int_{n\omega}^{t}\frac{1}{(\lambda+t-s)^{2-\alpha}}\|w(s-\omega)\|_{E_\alpha}ds\\
&+\int_{n\omega}^{t}\frac{\lambda^{1-\alpha}}{(\lambda+t-s)^{1-\alpha}}\|f(s)\|_{E_\alpha}ds\\
&\le \Big(\frac{\lambda^{1-\alpha}}{(\lambda+t-n\omega)^{1-\alpha}}+\lambda^{1-\alpha}(1-\alpha)\int_{n\omega}^{t}\frac{1}{(\lambda+t-s)^{2-\alpha}}ds\Big)\int_0^{n\omega}\|f(s)\|_{E_\alpha}ds\\
&+\int_{n\omega}^{t}\|f(s)\|_{E_\alpha}ds\\
&=\int_0^t\|f(s)\|_{E_\alpha}ds
\end{aligned}
$$

for every t, $n\omega \le t \le (n+1)\omega$, $n = 1,2,3,\dots$ and $\lambda, \lambda > 0$. This shows that

$$\|w(t)\|_{E_\alpha} \le \int_0^t \|f(s)\|_{E_\alpha} ds$$

for every t, $n\omega \le t \le (n+1)\omega$, $n = 1,2,3,\dots$. This result completes the proof. $\square$

Now, we consider the problem (1.1) when

$$A^{-1}B(t)x \ne B(t)A^{-1}x, \quad x \in D(A)$$

for some $t \ge 0$. Note that A is a strongly positive operator in a Banach space E if and only if its spectrum $\sigma(A)$ lies in the interior of the sector of angle φ, $0 < 2\varphi < \pi$, symmetric with respect to the real axis, and if on the edges of this sector, $S_1 = [z = \rho\exp(i\varphi) : 0 \le \rho < \infty]$ and $S_2 = [z = \rho\exp(-i\varphi) : 0 \le \rho < \infty]$ and outside it the resolvent $(z-A)^{-1}$ is the subject to the bound

$$\|(z-A)^{-1}\|_{E\to E} \le \frac{M_1}{1+|z|} \tag{2.11}$$

for some $M_1 > 0$. First of all let us give lemmas from [9] that will be needed in the sequel.

Lemma 2.2. *For any z on the edges of the sectors*

$$
\begin{gathered}
S_1 = [z = \rho\exp(i\varphi) : 0 \le \rho < \infty],\\
S_2 = [z = \rho\exp(-i\varphi) : 0 \le \rho < \infty]
\end{gathered}
$$

and outside of it, the estimate

$$\|A(z-A)^{-1}x\|_E \le \frac{M_1^\alpha M^\alpha(1+M_1)^{1-\alpha}2^{(2-\alpha)\alpha}}{\alpha(1-\alpha)(1+|z|)^\alpha}\|x\|_{E_\alpha}$$

holds for any $x \in E_\alpha$. Here and in the future M and M_1 are same constants of the estimates (1.2) *and* (2.11).

Lemma 2.3. *For all $s \ge 0$, let the operator $B(s)A^{-1} - A^{-1}B(s)$ with domain which coincide with $D(A)$, admit a closure $Q = \overline{B(s)A^{-1} - A^{-1}B(s)}$ bounded in E. Then for all $\tau > 0$ the following estimate holds:*

$$\|A^{-1}[A\exp\{-\tau A\}B(s) - B(s)A\exp\{-\tau A\}]x\|_E$$

$$\le \frac{e(\alpha+1)M^{\alpha}M_1^{1+\alpha}(1+2M_1)(1+M_1)^{1-\alpha}2^{(2-\alpha)\alpha}\|Q\|_{E\to E}\|x\|_{E_\alpha}}{\tau^{1-\alpha}\pi\alpha^2(1-\alpha)}.$$

Here $Q = \overline{A^{-1}(AB(s)-B(s)A)A^{-1}}$.

Suppose that

$$\begin{aligned} &\|\overline{A^{-1}(AB(t)-B(t)A)A^{-1}}\|_{E\to E} \\ &\le \frac{\pi(1-\alpha)^2\alpha^2\varepsilon}{eM^{1+\alpha}M_1^{1+\alpha}(1+2M_1)(1+M_1)^{1-\alpha}2^{2+\alpha-\alpha^2}(1+\alpha)} \end{aligned} \tag{2.12}$$

holds for every $t \ge 0$. Here and in the future ε is some constant, $0 \le \varepsilon \le 1$. Applications of Lemmas 2.2 and 2.3 enable us to establish the following fact.

Theorem 2.4. *Assume that the condition*

$$\|\overline{A^{-1}B(t)}\|_{E\to E} \le \frac{(1-\alpha)(1-\varepsilon)}{M2^{2-\alpha}} \tag{2.13}$$

holds for every $t \ge 0$. Then for every $t \ge 0$ estimate (2.3) *holds.*

Proof. In [8], under the assumption of this theorem it was established that the stability inequality (2.6) holds for the solution of the problem (2.4) for every $t \ge 0$. Therefore, to prove the theorem it suffices to establish the stability inequality (2.7) for the solution of the problem (2.5). Now, we consider the problem (2.5). Exactly same manner, using the formula (2.8), the semigroup property, the definition of the spaces E_a, we can obtain (2.9) for every t, $0 \le t \le \omega$. Applying the mathematical induction, one can easily show that it is true for every t. Namely, assume that the inequality (2.9) is true for t, $(n-1)\omega \le t \le n\omega$, $n = 1, 2, 3, \dots$ for some n. Using formula (2.10) and the semigroup property, we can write

$$\begin{aligned} &\lambda^{1-\alpha}A\exp\{-\lambda A\}w(t) \\ &= \lambda^{1-\alpha}A\exp\{-(\lambda+t-n\omega)A\}w(n\omega) \\ &\quad + \lambda^{1-\alpha}\int_{n\omega}^{t}\exp\{-\frac{\lambda+t-s}{2}A\}B(s)A\exp\{-\frac{\lambda+t-s}{2}A\}w(s-\omega)ds \\ &\quad + \lambda^{1-\alpha}\int_{n\omega}^{t}\exp\{-\frac{\lambda+t-s}{2}A\}[A\exp\{-\frac{\lambda+t-s}{2}A\}B(s)-B(s)A \\ &\quad \times \exp\{-\frac{\lambda+t-s}{2}A\}]w(s-\omega)ds + \lambda^{1-\alpha}\int_{n\omega}^{t}A\exp\{-(\lambda+t-s)A\}f(s)ds \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where

$$\begin{gathered} I_1 = \lambda^{1-\alpha}A\exp\{-(\lambda+t-n\omega)A\}w(n\omega), \\ I_2 = \lambda^{1-\alpha}\int_{n\omega}^{t}\exp\{-\frac{\lambda+t-s}{2}A\}B(s)A\exp\{-\frac{\lambda+t-s}{2}A\}w(s-\omega)ds, \\ \begin{aligned} I_3 = \lambda^{1-\alpha}\int_{n\omega}^{t}&\exp\{-\frac{\lambda+t-s}{2}A\}[A\exp\{-\frac{\lambda+t-s}{2}A\}B(s)-B(s)A \\ &\times\exp\{-\frac{\lambda+t-s}{2}A\}]w(s-\omega)ds, \end{aligned} \\ I_4 = \lambda^{1-\alpha}\int_{n\omega}^{t}A\exp\{-(\lambda+t-s)A\}f(s)ds. \end{gathered}$$

Using estimate (1.2) and condition (2.13), we obtain

$$\begin{aligned}\|I_1\|_E &= \lambda^{1-\alpha}\|A\exp\{-(\lambda+t-n\omega)A\}w(n\omega)\|_E\\ &\le \frac{\lambda^{1-\alpha}}{(\lambda+t-n\omega)^{1-\alpha}}\|w(n\omega)\|_{E_\alpha}\\ &\le \frac{\lambda^{1-\alpha}}{(\lambda+t-n\omega)^{1-\alpha}}\int_0^{n\omega}\|f(s)\|_{E_\alpha}ds,\end{aligned}$$

$$\begin{aligned}\|I_2\|_E &\le \lambda^{1-\alpha}\int_{n\omega}^{t}\|A\exp\{-\frac{\lambda+t-s}{2}A\}\|_{E\to E}\|\overline{A^{-1}B(s)}\|_{E\to E}\\ &\quad\times\|A\exp\{-\frac{\lambda+t-s}{2}A\}w(s-\omega)\|_E ds\\ &\le \max_{n\omega\le t\le\omega}\|\overline{A^{-1}B(t)}\|_{E\to E}\int_{n\omega}^{t}\frac{M\lambda^{1-\alpha}2^{2-\alpha}}{(\lambda+t-s)^{2-\alpha}}ds\max_{n\omega\le s\le\omega}\|w(s-\omega)\|_{E_\alpha}\\ &\le \big(1-\frac{\lambda^{1-\alpha}}{(\lambda+t-n\omega)^{1-\alpha}}\big)(1-\varepsilon)\int_0^{n\omega}\|f(s)\|_{E_\alpha}ds,\end{aligned}$$

$$\|I_4\|_E \le \int_{n\omega}^{t}\frac{\lambda^{1-\alpha}}{(\lambda+t-s)^{1-\alpha}}\|f(s)\|_{E_\alpha}ds \le \int_{n\omega}^{t}\|f(s)\|_{E_\alpha}ds$$

for every t, $n\omega\le t\le(n+1)\omega$, $n=1,2,3,\dots$ and $\lambda,\lambda>0$. Now let us estimate I_3. By Lemma 2.3 and using the estimate (1.2) and condition (2.12), we obtain

$$\begin{aligned}\|I_3\|_E &\le \lambda^{1-\alpha}\int_{n\omega}^{t}\|A\exp\{-\frac{\lambda+t-s}{2}A\}\|_{E\to E}\|A^{-1}\Big[A\exp\{-\frac{\lambda+t-s}{2}A\}B(s)\\ &\quad - B(s)A\exp\{-\frac{\lambda+t-s}{2}A\}\Big]w(s-\omega)\|_E ds\\ &\le \lambda^{1-\alpha}e(1+\alpha)M^{1+\alpha}M_1^{1+\alpha}(1+2M_1)(1+M_1)^{1-\alpha}2^{(2-\alpha)\alpha}\\ &\quad\times\int_{n\omega}^{t}\frac{\|\overline{A^{-1}(AB(s)-B(s)A)A^{-1}}\|_{E\to E}2^{2-\alpha}\|w(s-\omega)\|_{E_\alpha}}{(\lambda+t-s)^{2-\alpha}\pi\alpha^2(1-\alpha)}ds\\ &\le \max_{0\le s\le\omega}\overline{\|A^{-1}(AB(s)-B(s)A)A^{-1}\|}_{E\to E}\\ &\quad\times\int_{n\omega}^{t}\frac{\lambda^{1-\alpha}e(1+\alpha)M^{1+\alpha}M_1^{1+\alpha}(1+2M_1)(1+M_1)^{1-\alpha}2^{(2-\alpha)\alpha}2^{2-\alpha}}{(\lambda+t-s)^{2-\alpha}\pi\alpha^2(1-\alpha)}ds\\ &\quad\times\int_0^{n\omega}\|f(s)\|_{E_\alpha}ds\\ &\le \Big(1-\frac{\lambda^{1-\alpha}}{(\lambda+t-n\omega)^{1-\alpha}}\Big)\varepsilon\int_0^{n\omega}\|f(s)\|_{E_\alpha}ds\end{aligned}$$

for every t, $n\omega\le t\le(n+1)\omega$, $n=1,2,3,\dots$ and λ, $\lambda>0$. Using the triangle inequality and estimates for all $\|I_k\|_E$, $k=1,2,3,4$, we obtain

$$\lambda^{1-\alpha}\|A\exp\{-\lambda A\}w(t)\|_E \le \int_0^t\|f(s)\|_{E_\alpha}ds$$

for every t, $n\omega\le t\le(n+1)\omega$, $n=1,2,3,\dots$ and λ, $\lambda>0$. This shows that

$$\|w(t)\|_{E_\alpha} \le \int_0^t\|f(s)\|_{E_\alpha}ds$$

for every t, $n\omega\le t\le(n+1)\omega$, $n=1,2,3,\dots$. This result completes the proof. $\square$

Note that these abstract results are applicable to study of stability of various delay parabolic equations with local and nonlocal boundary conditions with respect to space variable. However, it is important to study structure of E_α for space operators in Banach spaces. The structure of E_α for some space differential and difference operators in Banach spaces has been investigated in papers (see, [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 30]). In Section 3, one application of Theorem 2.1 to study the stability of initial-boundary value problem for one dimensional delay parabolic equations with Neumann condition with respect to space variable is given. It is based on the abstract result of this section and structure of E_α for one dimensional differential operator with the Neumann condition with respect to space variables in the Banach space.

3. An application

We consider the initial-boundary value problem for one dimensional delay differential equations of parabolic type

$$\begin{gathered}\frac{\partial u(t,x)}{\partial t}-a(x)\frac{\partial^2 u(t,x)}{\partial x^2}+\delta u(t,x)\\ =b(t)\Big(-a(x)\frac{\partial^2 u(t-\omega,x)}{\partial x^2}+\delta u(t-\omega,x)\Big)+f(t,x),\quad 0<t<\infty, x\in(0,l),\\ u(t,x)=g(t,x),\quad -\omega\le t\le 0,\quad x\in[0,l],\\ u_x(t,0)=u_x(t,l)=0,\quad -\omega\le t<\infty,\end{gathered}\tag{3.1}$$

where $a(x)$, $b(t)$, $g(t,x)$, $f(t,x)$ are sufficiently smooth functions and $\delta>0$ is the sufficiently large number. We will assume that $a(x)\ge a>0$. The problem (3.1) has a unique smooth solution. This allows us to reduce the initial-boundary value problem (3.1) to the initial value problem (1.1) in Banach space $E=C[0,l]$ with a differential operator A^x defined by the formula

$$A^x u=-a(x)\frac{d^2u}{dx^2}+\delta u \tag{3.2}$$

with domain $D(A^x)=\{u\in C^{(2)}[0,1]:u'(0)=u'(1)=0\}$. Let us give a number of corollaries of the abstract Theorem 2.1.

Theorem 3.1. *Assume that*

$$\sup_{0\le t<\infty}|b(t)|\le\frac{1-\alpha}{M2^{2-\alpha}}.$$

Then for all $t\ge0$ the solutions of the initial-boundary value problem (3.1) *satisfy the stability estimates*

$$\|u(t,\cdot)\|_{C^{2\alpha}[0,l]}\le M(\alpha)\Big[\max_{-\omega\le t\le 0}\|g(t)\|_{C^{2\alpha}[0,l]}+\int_0^t\|f(s,\cdot)\|_{C^{2\alpha}[0,l]}ds\Big],$$

for $0<\alpha<1/2$, where $M(\alpha)$ does not dependent on $g(t,x)$ and $f(t,x)$. Here $C^\beta[0,l]$ is the space of functions satisfying a Hölder condition with the indicator $\beta\in(0,1)$.

The proof of Theorem 3.1 is based on the estimate

$$\|\exp\{-tA^x\}\|_{C[0,l]\to C[0,l]}\le M,\quad t\ge0,$$

and on the abstract Theorem 2.1, on the strongly positivity of the operator A^x in $C[0,l]$ (see, [26, 27]) and on Theorem 3.2, on the structure of the fractional space $E_\alpha = E_\alpha(C[0,l], A^x)$ for $0 < \alpha < 1/2$.

Theorem 3.2. *For $\alpha \in (0, 1/2)$, the norms of the space $E_\alpha(C[0,l], A^x)$ and the Hölder space $C^{2\alpha}[0,l]$ are equivalent.*

Proof. First, we prove this statement for the differential operator A^x defined by the formula (3.2) in the case when $a(x) = 1$. It is easy to see that for all $\delta > 0$ and $\lambda \geq 0$ the resolvent equation

$$A^x u + \lambda u = \varphi \tag{3.3}$$

is uniquely solvable and the following formula holds:

$$u(x) = (A^x + \lambda)^{-1}\varphi(x) = \int_0^1 G(x,s;\lambda) f(s) ds. \tag{3.4}$$

Here

$$G(x,s;\lambda) = \begin{cases} \frac{1}{2\sqrt{\lambda+\delta}(1-e^{-\sqrt{\lambda+\delta}2l})^2}\Big\{e^{-\sqrt{\lambda+\delta}(s+x)} + e^{-\sqrt{\lambda+\delta}(x-s)} \\ +e^{-\sqrt{\lambda+\delta}(2l-s-x)} + e^{-\sqrt{\lambda+\delta}(2l+s-x)} - e^{-\sqrt{\lambda+\delta}(2l-s+x)} \\ -e^{-\sqrt{\lambda+\delta}(2l+s+x)} - e^{-\sqrt{\lambda+\delta}(4l-s-x)} + e^{-\sqrt{\lambda+\delta}(4l+s-x)}\Big\} \\ \quad \text{if } 0 \leq s \leq x, \\ \frac{1}{2\sqrt{\lambda+\delta}(1-e^{-\sqrt{\lambda+\delta}2l})^2}\Big\{e^{-\sqrt{\lambda+\delta}(s+x)} + e^{-\sqrt{\lambda+\delta}(s-x)} \\ +e^{-\sqrt{\lambda+\delta}(2l-s-x)} + e^{-\sqrt{\lambda+\delta}(2l+s-x)} - e^{-\sqrt{\lambda+\delta}(2l-s+x)} \\ -e^{-\sqrt{\lambda+\delta}(2l+s+x)} - e^{-\sqrt{\lambda+\delta}(4l-s-x)} + e^{-\sqrt{\lambda+\delta}(4l-s+x)}\Big\} \\ \quad \text{if } x \leq s \leq l. \end{cases} \tag{3.5}$$

We have that

$$\int_0^1 G(x,s;\lambda) ds = \frac{1}{\lambda+\delta}. \tag{3.6}$$

Applying the triangle inequality, formula (3.5), we obtain the following pointwise estimates for the Green's function $G(x,s;\lambda)$ of the operator A^x defined by (3.2) in the case when $a(x) = 1$,

$$|G(x,s;\lambda)| \leq \frac{M(\delta)}{\sqrt{\delta+\lambda}} \begin{cases} e^{-\sqrt{\delta+\lambda}(x-s)}, & 0 \leq s \leq x, \\ e^{-\sqrt{\delta+\lambda}(s-x)}, & x \leq s \leq l, \end{cases} \tag{3.7}$$

$$|G_x(x,s;\lambda)| \leq M(\delta) \begin{cases} e^{-\sqrt{\delta+\lambda}(x-s)}, & 0 \leq s \leq x, \\ e^{-\sqrt{\delta+\lambda}(s-x)}, & x \leq s \leq l. \end{cases} \tag{3.8}$$

Using formula (3.4) and identity (3.6), we obtain

$$\lambda^\alpha A^x (A^x + \lambda)^{-1}\varphi(x) = \frac{\delta\lambda^\alpha}{\delta+\lambda}\varphi(x) + \lambda^{\alpha+1}\int_0^1 G(x,s;\lambda)(\varphi(x) - \varphi(s)) ds. \tag{3.9}$$

Let $\varphi(x) \in C^{2\alpha}[0,l]$. Then, applying formula (3.9), the triangle inequality and estimate (3.7), we obtain

$$\begin{aligned} &\lambda^\alpha |A^x (A^x + \lambda)^{-1}\varphi(x)| \\ &\leq \frac{\delta\lambda^\alpha}{\delta+\lambda}|\varphi(x)| + \lambda^{\alpha+1}\int_0^1 |G(x,s;\lambda)||\varphi(x) - \varphi(s)| ds \end{aligned}$$

$$
\begin{aligned}
&\le \Big(\frac{\delta\lambda^{\alpha}}{\delta+\lambda} + M(\delta)\frac{\lambda^{\alpha+1}}{\sqrt{\delta+\lambda}}\int_0^1 e^{-\sqrt{\delta+\lambda}|s-x|}|s-x|^{2\alpha}ds\Big)\|\varphi\|_{C^{2\alpha}[0,l]}\\
&\le M_1(\delta)\|\varphi\|_{C^{2\alpha}[0,l]}
\end{aligned}
$$

for any $\lambda > 0$ and $x \in [0,l]$. Therefore $\varphi(x) \in E_\alpha(C[0,l], A^x)$ and the following estimate holds:

$$
\|\varphi\|_{E_\alpha(C[0,l],A^x)} \le M_1(\delta)\|\varphi\|_{C^{2\alpha}[0,l]}.
$$

Let us prove the opposite estimate. For any positive operator A^x in the Banach space, we can write formula

$$
\varphi(x) = \int_0^\infty A^x(\lambda + A^x)^{-2}\varphi(x)\,d\lambda.
$$

From this relation and formula (3.4), it follows that

$$
\begin{aligned}
\varphi(x) &= \int_0^\infty (A^x+\lambda)^{-1}A^x(A^x+\lambda)^{-1}\varphi(x)d\lambda\\
&= \int_0^\infty\int_0^1 G(x,s;\lambda)A^x(A^x+\lambda)^{-1}\varphi(s)\,ds\,d\lambda.
\end{aligned}
\tag{3.10}
$$

Consequently,

$$
\begin{aligned}
&\varphi(x_1)-\varphi(x_2)\\
&= \int_0^\infty\int_0^1 (G(x_1,s;\lambda)-G(x_2,s;\lambda))A^x(A^x+\lambda)^{-1}\varphi(s)\,ds\,d\lambda\\
&= \int_0^\infty \lambda^{-\alpha}\int_0^1 (G(x_1,s;\lambda)-G(x_2,s;\lambda))\lambda^{\alpha}A^x(A^x+\lambda)^{-1}\varphi(s)\,ds\,d\lambda.
\end{aligned}
\tag{3.11}
$$

Let $\varphi(x) \in E_\alpha(C[0,l], A^x)$. Then, using formula (3.10), estimate (3.7) and the definition of the space $E_\alpha(C[0,l], A^x)$, we obtain

$$
\begin{aligned}
|\varphi(x)| &\le \int_0^\infty \lambda^{-\alpha}\int_0^1 |G(x,s;\lambda)|\lambda^{\alpha}|A^x(A^x+\lambda)^{-1}\varphi(s)|\,ds\,d\lambda\\
&\le M(\delta)\int_0^\infty \lambda^{-\alpha}\frac{1}{\sqrt{\delta+\lambda}}\int_0^1 e^{-\sqrt{\delta+\lambda}|s-x|}\,ds\,d\lambda\|\varphi\|_{E_\alpha(C[0,l],A^x)}\\
&\le \frac{M_1(\delta)}{\alpha}\|\varphi\|_{E_\alpha(C[0,l],A^x)}
\end{aligned}
$$

for any $x \in [0,l]$. Therefore $\varphi(x) \in C[0,l]$ and

$$
\|\varphi\|_{C[0,l]} \le \frac{M_1(\delta)}{\alpha}\|\varphi\|_{E_\alpha(C[0,l],A^x)}
\tag{3.12}
$$

Moreover, using (3.11) and the definition of the space $E_\alpha(C[0,l], A^x)$, we obtain

$$
\begin{aligned}
&\frac{|\varphi(x_1)-\varphi(x_2)|}{|x_1-x_2|^{2\alpha}}\\
&\le \frac{1}{|x_1-x_2|^{2\alpha}}\int_0^\infty \lambda^{-\alpha}\int_0^1 |G(x_1,s;\lambda)-G(x_2,s;\lambda)|\lambda^{\alpha}|A^x(A^x+\lambda)^{-1}\varphi(s)|\,ds\,d\lambda\\
&\le \frac{1}{|x_1-x_2|^{2\alpha}}\int_0^\infty \lambda^{-\alpha}\int_0^1 |G(x_1,s;\lambda)-G(x_2,s;\lambda)|\,ds\,d\lambda\|\varphi\|_{E_\alpha(C[0,l],A^x)}.
\end{aligned}
$$

Let

$$
P = \frac{1}{|x_1-x_2|^{2\alpha}}\int_0^\infty \lambda^{-\alpha}\int_0^1 |G(x_1,s;\lambda)-G(x_2,s;\lambda)|\,ds\,d\lambda.
$$

Then

$$\frac{|\varphi(x_1)-\varphi(x_2)|}{|x_1-x_2|^{2\alpha}} \le P\|\varphi\|_{E_\alpha(C[0,l],A^x)} \tag{3.13}$$

for any $x_1, x_2 \in [0,l]$ and $x_1 \neq x_2$.

Now, we estimate P. Let $|x_1-x_2| \le 1$. Then, using the triangle inequality and estimate (3.7), we obtain

$$P \le M(\delta)\int_0^\infty \lambda^{-\alpha}\frac{1}{\sqrt{\delta+\lambda}}\int_0^1 \Big(e^{-\sqrt{\delta+\lambda}|s-x_2|}+e^{-\sqrt{\delta+\lambda}|s-x_1|}\Big)\,ds\,d\lambda \le \frac{M_2(\delta)}{\alpha}. \tag{3.14}$$

Let $|x_1-x_2|>1$. For more definitely we put $x_1<x_2$. Then, using estimates (3.7) and (3.8), we obtain

$$\begin{aligned}
P &\le M(\delta)\frac{1}{|x_1-x_2|^{2\alpha}}\int_{|x_1-x_2|^2}^\infty \lambda^{-\alpha}\frac{1}{\sqrt{\delta+\lambda}}\int_0^1 (e^{-\sqrt{\delta+\lambda}|s-x_2|}+e^{-\sqrt{\delta+\lambda}|s-x_1|})\,ds\,d\lambda\\
&\quad + M(\delta)\frac{1}{|x_1-x_2|^{2\alpha}}\int_0^{|x_1-x_2|^2}\lambda^{-\alpha}\int_0^1\int_{x_1}^{x_2} e^{-\sqrt{\delta+\lambda}|s-x|}dx\,ds\,d\lambda\\
&\le M(\delta)\frac{1}{|x_1-x_2|^{2\alpha}}\int_{|x_1-x_2|^2}^\infty \lambda^{-\alpha-1}d\lambda\\
&\quad + M(\delta)\frac{1}{|x_1-x_2|^{2\alpha}}\int_0^{|x_1-x_2|^2}\lambda^{-\alpha-\frac12}d\lambda|x_1-x_2|\\
&\le \frac{M_3(\delta)}{\alpha(1-2\alpha)}.
\end{aligned} \tag{3.15}$$

Therefore $\varphi(x)\in C^{2\alpha}[0,l]$ and from estimates (3.12), (3.13) and (3.15) it follows that

$$\|\varphi\|_{C^{2\alpha}[0,l]} \le \frac{M(\delta)}{\alpha(1-2\alpha)}\|\varphi\|_{E_\alpha(C[0,l],A^x)}.$$

Second, let $a(x)$ be the smooth function defined on the segment $[0,l]$ and $a(x)\ge a>0$. We prove this statement for the differential operator A^x defined by the formula (3.2). It is easy to see that if $a(x)=$ constant, the resolvent equation (3.3) can be transformed in the last case by dividing both sides of resolvent equation (3.3) to a. We have the following estimates for Green's function

$$|G^x(x,s;\lambda)| \le \frac{M(\delta,a)}{\sqrt{\delta+\lambda}}\begin{cases} e^{-\sqrt{\frac{\delta+\lambda}{a}}(x-s)}, & 0\le s\le x,\\ e^{-\sqrt{\frac{\delta+\lambda}{a}}(s-x)}, & x\le s\le l,\end{cases}$$

$$|G^x_x(x,s;\lambda)| \le M(\delta,a)\begin{cases} e^{-\sqrt{\frac{\delta+\lambda}{a}}(x-s)}, & 0\le s\le x,\\ e^{-\sqrt{\frac{\delta+\lambda}{a}}(s-x)}, & x\le s\le l.\end{cases}$$

Since the proof of theorem is based on the estimates of Green's function, it is true also for this case. Under one more assumption that $\delta>0$ is the sufficiently large number, applying a fixed point theorem and last estimates and the formula

$$\begin{aligned}
&G^x(x,x_0;\lambda)\\
&= G^{x_0}(x,x_0;\lambda)+(\lambda+\delta)\int_0^1 G^{x_0}(x,y;\lambda)\big(\frac{1}{a(y)}-\frac{1}{a(x_0)}\big)G^y(y,x_0;\lambda)dy,
\end{aligned}$$

we obtain the estimates

$$|G^x(x,x_0;\lambda)| \le \frac{M(\delta,a)}{\sqrt{\delta+\lambda}} \begin{cases} e^{-\frac{1}{2}\sqrt{\frac{\delta+\lambda}{a}}(x-x_0)}, & 0 \le x_0 \le x, \\ e^{-\frac{1}{2}\sqrt{\frac{\delta+\lambda}{a}}(x_0-x)}, & x \le x_0 \le l, \end{cases}$$

$$|G^x_x(x,x_0;\lambda)| \le M(\delta,a) \begin{cases} e^{-\frac{1}{2}\sqrt{\frac{\delta+\lambda}{a}}(x-x_0)}, & 0 \le x_0 \le x, \\ e^{-\frac{1}{2}\sqrt{\frac{\delta+\lambda}{a}}(x_0-x)}, & x \le x_0 \le l \end{cases}$$

for the Green's function of the differential operator A^x defined by the formula (3.2). Therefore, the statement of theorem is true also for the differential operator A^x defined by the formula (3.2). Theorem 3.2 is proved. □

Conclusion. In the present paper, two theorems on the stability of the initial value problem for the delay parabolic differential equations with unbounded operators acting on delay terms in fractional spaces E_α are established. Theorem on the structure of fractional spaces E_α generated by the differential operator A^x defined by the formula (3.2) in $C[0,l]$ space is proved. In practice, the stability estimates in Hölder norms for the solutions of the mixed problems for delay parabolic equations with Neumann condition with respect to space variable are obtained.

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