Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 16, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

MEROMORPHIC SOLUTIONS TO COMPLEX DIFFERENCE AND q-DIFFERENCE EQUATIONS OF MALMQUIST TYPE

JIE ZHANG, JIANJUN ZHANG

Abstract. In this article, we study the zeros, poles and fixed points of finite order transcendental meromorphic solutions of complex difference and q-difference equations of Malmquist type respectively. Some examples are structured to show that our results are sharp.

1. Introduction

In this article, a meromorphic function always means it is meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations in the Nevanlinna theory. We use the following standard notations in value distribution theory (see [4, 8, 9]):

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \dots.$$

And we denote by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$, as $r\to\infty$, possibly outside of a set E with finite linear or logarithmic measure, not necessarily the same at each occurrence. We also use the notation $\tau(f)$ to denote the exponent of convergence of fixed points of f, namely

$$\tau(f)=\limsup_{r\to\infty}\frac{\log N(r,\frac{1}{f-z})}{\log r}.$$

The deficiency of a with respect to $f(z)$ is defined by

$$\delta(a,f)=1-\limsup_{r\to\infty}\frac{N(r,\frac{1}{f-a})}{T(r,f)}.$$

We use $\lambda(f)$ and $\bar{\lambda}(f)$ to denote the exponent of convergence of zeros of f counting multiplicities and ignoring multiplicities respectively, namely

$$\lambda(f)=\limsup_{r\to\infty}\frac{\log N(r,\frac{1}{f})}{\log r},\quad \bar{\lambda}(f)=\limsup_{r\to\infty}\frac{\log \overline{N}(r,\frac{1}{f})}{\log r}.$$

A polynomial $Q(z,f)$ is called a differential-difference polynomial in f if Q is a polynomial in f, its derivatives and shifts with small meromorphic coefficients, say $\{a_\lambda|\lambda\in I\}$, such that $T(r,a_\lambda)=S(r,f)$ for all $\lambda\in I$. We define the difference operator $\Delta f=f(z+1)-f(z)$.

2000 *Mathematics Subject Classification.* 30D35, 34M10.
Key words and phrases. Meromorphic; differential-difference equation; fixed point; order.

Submitted September 13, 2013. Published January 10, 2014.

Recently, a large number of researches focusing on complex difference and q-difference equation emerged. For example, Gundersen et al [3] considered the complex q-difference equation of Malmquist type and obtained the following result.

Theorem 1.1. *Let f be a transcendental meromorphic solution of the q-difference equation*

$$f(qz)=R(z,f)=\frac{a_0(z)+a_1(z)f(z)+\cdots+a_p(z)f^p(z)}{b_0(z)+b_1(z)f(z)+\cdots+b_t(z)f^t(z)}, \tag{1.1}$$

where $q\in\mathbb{C}$, $|q|\geq 1$, $a_p(z)\not\equiv 0$, $b_t(z)\equiv 1$, and meromorphic coefficients $a_i(z)$ $(i=0,1,\dots,p)$ and $b_j(z)$ $(j=0,1,\dots,t-1)$ are of growth $S(r,f)$. If

$$\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})=S(r,f),$$

then (1.1) is either the form

$$f(qz)=a_p(z)f^p(z)\quad or\quad f(qz)=\frac{a_0(z)}{f^t(z)}.$$

Here we consider a q-difference equation whose form is more general than in Equation (1.1) under a condition similar to Theorem 1.1 and obtain some results as follows.

Theorem 1.2. *Let f be a transcendental meromorphic solution of a q-difference equation of the following form*

$$\prod_{i=1}^{n}f(q_iz)=R(z,f)=\frac{a_0(z)+a_1(z)f(z)+\cdots+a_p(z)f^p(z)}{b_0(z)+b_1(z)f(z)+\cdots+b_t(z)f^t(z)}, \tag{1.2}$$

where $q_i\neq 0,1$, $i=1,2\dots n$, and $R(z,f)$ is an irreducible rational function in f with meromorphic coefficients $a_i(z)$ $(i=0,1,\dots p)$ and $b_j(z)$ $(j=0,1,\dots t)$ of growth $S(r,f)$ such that $a_p(z)\not\equiv 0, b_t(z)\equiv 1$. If

$$\max\{\overline{\lambda}(f),\overline{\lambda}(\frac{1}{f})\}<\sigma(f)=\sigma\leq\infty,$$

then (1.2) is reduced to the form

$$\prod_{i=1}^{n}f(q_iz)=a_p(z)f^p(z)\quad or\quad \prod_{i=1}^{n}f(q_iz)=\frac{a_0(z)}{f^t(z)}.$$

The author in [10] considered a special complex difference equation of Malmquist type and obtained the following result.

Theorem 1.3. *Let $R(z)$ be a non-constant rational function. For the difference equation*

$$f(z+1)=R\circ f(z),$$

(1) suppose it admits a non-constant rational solution $f(z)$, then both $R(z)$ and $f(z)$ are fractional linear functions;
(2) suppose it admits a transcendental meromorphic function $f(z)$ of finite order $\sigma(f)$, then $R(z)$ is a fractional linear function, and suppose that it is denoted by $R(z)=\frac{az+b}{cz+d}$, where $ad-bc\neq 0$, furthermore:
(2.1) if $bc\neq 0$, then $\lambda(f)=\lambda(\frac{1}{f})=\tau(f)=\sigma(f)$;
(2.2) if $R\neq id$ and $\sigma(f)>0$, then
(2.2.1) $f(z)$ has at most one finite Borel exceptional value provided $(d-a)^2+4b=0$

when $c \neq 0$*;*
(2.2.2) if $f(z)$ *has Borel exceptional value* ∞*, then* $f(z)$ *has at most one finite Borel exceptional value* $\frac{b}{1-a}$ *.*

Here we consider a type of difference equation more general than in Theorem 1.3 and obtain some results as follows.

Theorem 1.4. *Suppose that* $c_1, c_2, \dots, c_n$ *are distinct nonzero constants. If complex difference equation of Malmquist type*

$$\sum_{j=1}^{n} f(z+c_j) = R(f(z)) = \frac{P(f(z))}{Q(f(z))} = \frac{a_p f^p(z) + a_{p-1} f^{p-1}(z) + \dots + a_0}{b_q f^q(z) + b_{q-1} f^{q-1}(z) + \dots + b_0} \quad (1.3)$$

admits a transcendental meromorphic solution $f(z)$ *of finite order, where* $P(f)$ *and* $Q(f)$ *are relatively prime polynomials in* f *with constant coefficients* a_s $(s = 0, 1, \dots, p)$ *and* b_t $(t = 0, 1, \dots, q)$ *such that* $a_0 a_p b_q \neq 0$*, and*

$$d = \deg R(z) = \max\{\deg P(z), \deg Q(z)\} \geq 2,$$

then
(1) $f(z)$ *has infinitely many zeros and satisfies* $\delta(0, f) = 0$*;*
(2) $f(z)$ *has infinitely many fixed points and satisfies* $\tau(f) = \sigma(f)$*;*
(3) $f(z)$ *has infinitely many poles and satisfies* $\lambda(\frac{1}{f}) = \sigma(f)$*;*
(4) $f(z)$ *has no deficiency value* b *except that the value* b *satisfies*

$$a_p b^p + a_{p-1} b^{p-1} + \dots + a_0 = nb(b_q b^q + b_{q-1} b^{q-1} + \dots + b_0).$$

Example 1.5. Let $f(z) = 1/(e^{\pi i z} - 1)$, then we see the following identical equation holds.

$$(f(z+1) + f(z+2))(2f(z) + 1) = 2f^2(z),$$

which means $f(z)$ satisfies the complex difference equation of Malmquist type

$$\sum_{j=1}^{2} f(z+c_j) = R(f(z)),$$

where $c_1 = 1, c_2 = 2$ and $R(z) = 2z^2/(2z+1)$. But $f(z) \neq 0$.

This example shows that the assumption $a_0 \neq 0$ is necessary for our result (1) in Theorem 1.4.

Example 1.6. Let $f(z) = e^{\pi i z} + z$, then we see the following identical equation holds.

$$f(z+2) + f(z+4) = 2f(z) + 6,$$

which means $f(z)$ satisfies the complex difference equation of Malmquist type

$$\sum_{j=1}^{2} f(z+c_j) = R(f(z)),$$

where $c_1 = 2, c_2 = 4$ and $R(z) = 2z + 6$. But $f(z) \neq z, \infty$.

This example shows that the assumption $\deg R(z) \geq 2$ is necessary for our results (2)-(3) in Theorem 1.4.

Example 1.7. Let $f(z)=e^{\pi iz}+1$, then we see the following identical equation holds.

$$f(z-\frac{\log 2}{i\pi})+f(z-\frac{\log 2}{i\pi}+2)=f(z)+1,$$

which means $f(z)$ satisfies the complex difference equation of Malmquist type

$$\sum_{j=1}^{2} f(z+c_j)=R(f(z)),$$

where $c_1=-\frac{\log 2}{i\pi}$, $c_2=2+c_1$ and $R(z)=z+1$. But $f(z)\neq 1,\infty$.

This example shows that the assumption $\deg R(z)\geq 2$ is necessary for our result (3) and the assumption $a_pb^p+a_{p-1}b^{p-1}+\cdots+a_0=nb(b_qb^q+b_{q-1}b^{q-1}+\cdots+b_0)$ is necessary for our result (4) in Theorem 1.4.

Example 1.8. Let $f(z)=2+\frac{1}{e^{\pi iz}-1}$, then by a simple calculation, we see the following identical equation holds.

$$(f(z+2)+f(z+1))(f(z)-\frac{3}{2})=f^2(z)-2,$$

which means $f(z)$ satisfies the complex difference equation of Malmquist type

$$\sum_{j=1}^{2} f(z+c_j)=R(f(z))=\frac{f^2(z)-2}{f(z)-\frac{3}{2}},$$

where $c_1=1,c_2=2$ and $R(z)=\frac{z^2-2}{z-\frac{3}{2}}$. But $f(z)\neq 2$.

This example shows that the assumption $a_pb^p+a_{p-1}b^{p-1}+\cdots+a_0=nb(b_qb^q+b_{q-1}b^{q-1}+\cdots+b_0)$ is necessary for our result (4) in Theorem 1.4 even when $\deg R(z)\geq 2$.

In 2007, Laine and Yang [6] considered zeros of one certain type of difference polynomials and obtained the following classic theorem.

Theorem 1.9. *Let f be a transcendental entire function of finite order and c be a nonzero complex constant. If $n\geq 2$, then $f^n(z)f(z+c)-a$ has infinitely many zeros, where $a\in\mathbb{C}\setminus\{0\}$.*

At last, we also consider one special difference polynomial $f^n(\Delta f)^s-\alpha(z)$ corresponding to Theorem 1.9 as follows.

Theorem 1.10. *Let f be a transcendental entire function of finite order, $\alpha(z)(\not\equiv 0)$ be a small function of f and $\Delta f\not\equiv 0$. Then $f^n(\Delta f)^s-\alpha(z)(n\geq 2)$ has infinitely many zeros.*

2. Some lemmas

To prove our results, we need some lemmas as follows.

Lemma 2.1 ([2]). *Let f be a meromorphic function with a finite order σ, and η be a nonzero constant. For any $\varepsilon>0$, we have*

$$m(r,\frac{f(z+\eta)}{f(z)})=O(r^{\sigma-1+\varepsilon}).$$

Lemma 2.2 ([2]). *Let $f(z)$ be a transcendental meromorphic function with finite order σ and η be a nonzero complex number. Then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r);$$

i.e., $T(r, f(z+\eta)) = T(r, f) + S(r, f)$.

Lemma 2.3 ([2]). *Let $f(z)$ be a meromorphic function with finite exponent of convergence of poles $\lambda = \lambda(\frac{1}{f}) < \infty$, and η be a fixed number. Then for each $\varepsilon > 0$, we have*

$$N(r, f(z+\eta)) = N(r, f) + O(r^{\lambda-1+\varepsilon}) + O(\log r).$$

Lemma 2.4 ([5]). *Let $w(z)$ be a finite order transcendental meromorphic solution of the equation*

$$P(z, w) = 0,$$

where $P(z, w)$ is a differential-difference polynomial in w and its shifts. If $P(z, a) \not\equiv 0$ for a meromorphic function $a \in S(r, w)$, then

$$m(r, \frac{1}{w-a}) = S(r, w).$$

Lemma 2.5 ([8]). *Let $f(z)$ be a non-constant meromorphic function in the complex plane and*

$$R(f) = \frac{p(f)}{q(f)},$$

where $p(f) = \sum_{k=0}^{p} a_k f^k$ and $q(f) = \sum_{j=0}^{q} b_j f^j$ are two mutually prime polynomials in $f(z)$. If the coefficients a_k, b_j are small functions of $f(z)$ and $a_p(z) \not\equiv 0, b_q(z) \not\equiv 0$, then

$$T(r, R(f)) = \max\{p, q\} T(r, f) + S(r, f).$$

Lemma 2.6 ([7]). *Let f be a transcendental meromorphic function and*

$$F = a_n f^n + a_{n-1} f^{n-1} + \dots + a_0 \quad (a_n \not\equiv 0)$$

be a polynomial in f with coefficients being small functions of f. Then either

$$F = a_n \Big(f + \frac{a_{n-1}}{n a_n}\Big)^n \quad \text{or} \quad T(r, f) \le \overline{N}(r, \frac{1}{F}) + \overline{N}(r, f) + S(r, f).$$

Remark 2.7. From the definition, we have (see [1])

$$m(r, f(cz)) = m(|c|r, f(z)) \quad \text{and} \quad N(|c|r, f(z)) = N(r, f(cz)) + n(0, f(cz)) \log |c|,$$

i.e.,

$$N(|c|r, f(z)) = N(r, f(cz)) + O(1) \quad \text{and} \quad T(|c|r, f(z)) = T(r, f(cz)) + O(1).$$

3. Proofs of theorems

Proof of Theorem 1.2. First of all, we suppose that both p and t are positive integers, and rewrite Equation (1.2) as

$$H(z) := \prod_{i=1}^{n} f(q_i z) = A(z) \frac{P(z, f)}{T(z, f)}, \tag{3.1}$$

where

$$A(z) = \frac{a_p(z)}{b_t(z)}, \quad P(z, f) = \frac{a_0(z)}{a_p(z)} + \frac{a_1(z)}{a_p(z)} f(z) + \dots + f^p(z)$$

and

$$T(z,f)=\frac{b_0(z)}{b_t(z)}+\frac{b_1(z)}{b_t(z)}f(z)+\cdots+f^t(z).$$

Then from the definitions of H,P and T in Equation (3.1), we obtain

$$\frac{H'}{H}=\sum_{i=1}^{n}\frac{q_if'(q_iz)}{f(q_iz)}, \tag{3.2}$$

$$(\frac{H'}{H}-\frac{A'}{A})PT=P'T-PT'. \tag{3.3}$$

Fixing constants β,γ such that

$$\max\{\overline{\lambda}(f),\overline{\lambda}(\frac{1}{f})\}<\beta<\gamma<\sigma,$$

then we obtain

$$T(r,\frac{f'}{f})=m(r,\frac{f'}{f})+\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})=O(r^{\beta})+S(r,f). \tag{3.4}$$

By choosing subsequence r_n such that $T(r_n,f)>{r_n}^{\gamma}$, and noticing Equation (3.4) simultaneously, we obtain

$$T(r_n,\frac{f'}{f})=S(r_n,f). \tag{3.5}$$

Using the same method which leads to (3.4), and taking note of Remark 2.7, we can get

$$\begin{aligned}
T(r,\frac{H'}{H})&=m(r,\frac{H'}{H})+\overline{N}(r,H)+\overline{N}(r,\frac{1}{H})\\
&\le\sum_{i=1}^{n}\overline{N}(r,f(q_iz))+\sum_{i=1}^{n}\overline{N}(r,\frac{1}{f(q_iz)})+S(r,f)\\
&=\sum_{i=1}^{n}\overline{N}(|q_i|r,f(z))+\sum_{i=1}^{n}\overline{N}(|q_i|r,\frac{1}{f(z)})+S(r,f)\\
&=O(r^{\beta})+S(r,f),
\end{aligned}$$

which shows

$$T(r_n,\frac{H'}{H})=S(r_n,f). \tag{3.6}$$

By substituting P,T which are defined in (3.1) into (3.3), we obtain

$$(\frac{H'}{H}-\frac{A'}{A})(f^{p+t}+Q_{p+t-1})=P'T-PT'=(p-t)\frac{f'}{f}f^{p+t}+\tilde{Q}_{p+t-1}$$

where Q_{p+t-1} and $\tilde{Q}_{p+t-1}$ are differential polynomials in f with coefficients being small functions with degree at most $p+t-1$. Thus from the equation above, we obtain

$$[\frac{H'}{H}-\frac{A'}{A}-(p-t)\frac{f'}{f}]f^{p+t}=\tilde{Q}_{p+t-1}-(\frac{H'}{H}-\frac{A'}{A})Q_{p+t-1}. \tag{3.7}$$

From (3.5) and (3.6), we see $\frac{H'}{H}-\frac{A'}{A}-(p-t)\frac{f'}{f}$ is small function of $f(z)$ (for r_n). Thus if $\frac{H'}{H}-\frac{A'}{A}-(p-t)\frac{f'}{f}\not\equiv 0$, then by applying Lemma 2.5 to (3.7), we obtain

$$(p+t)T(r_n,f)\le(p+t-1)T(r_n,f)+S(r_n,f),$$

which is impossible. Thus

$$\frac{H'}{H} - \frac{A'}{A} - (p-t)\frac{f'}{f} \equiv 0.$$

Then we solve the equation above and get

$$f^{p-t} = k\frac{H}{A} = k\frac{P}{T}, \tag{3.8}$$

where k is a nonzero constant. By (3.8) and Lemma 2.5, we obtain $|p-t| = \max\{p,t\}$, which is impossible since p and t are positive integers. Thus $p = 0$ or $t = 0$, and we distinguish two cases as follows.

Case 1. $t = 0$, then (1.2) becomes

$$F := \prod_{i=1}^{n} f(q_i z) = a_0(z) + a_1(z)f(z) + \cdots + a_p(z)f^p(z).$$

By Lemma 2.6 and the equation above, we obtain

$$T(r,f) \le \overline{N}(r, \frac{1}{F}) + \overline{N}(r,f) + S(r,f) \tag{3.9}$$

or

$$F = a_p(f + \frac{a_{p-1}}{pa_p})^p. \tag{3.10}$$

If Equation (3.9) holds, then

$$T(r,f) \le \sum_{i=1}^{n} \overline{N}(r, \frac{1}{f(q_i z)}) + \overline{N}(r,f) + S(r,f) = O(r^\beta) + S(r,f).$$

Thus by the same discussion above, we see $T(r_n, f) \le S(r_n, f)$, which is impossible.

If Equation (3.10) holds, and $a_{p-1} \not\equiv 0$, then the second main theorem related to small functions implies

$$\begin{aligned} T(r,f) &\le \overline{N}(r, \frac{1}{f}) + \overline{N}(r,f) + \overline{N}(r, \frac{1}{f + \frac{a_{p-1}}{pa_p}}) + \varepsilon T(r,f) + S(r,f) \\ &\le O(r^\beta) + \overline{N}(r, \frac{1}{F}) + \varepsilon T(r,f) + S(r,f) \\ &\le \sum_{i=1}^{n} \overline{N}(r, \frac{1}{f(q_i z)}) + O(r^\beta) + \varepsilon T(r,f) + S(r,f) \\ &= O(r^\beta) + \varepsilon T(r,f) + S(r,f). \end{aligned}$$

That is,

$$T(r_n, f) \le \varepsilon T(r_n, f) + S(r_n, f),$$

which is impossible since can ε be set small enough. Thus $a_{p-1} \equiv 0$, and then Equation (1.2) becomes $\prod_{i=1}^{n} f(q_i z) = a_p f^p$.

Case 2. $p = 0$. then Equation (1.2) becomes

$$\tilde{F} := (\prod_{i=1}^{n} f(q_i z))^{-1} = \frac{1}{a_0}(b_0(z) + b_1(z)f(z) + \cdots + b_t(z)f^t(z)).$$

Using the similar method in case 1, we obtain $\prod_{i=1}^{n} f(q_i z) = \frac{a_0(z)}{f^t(z)}$. The proof of Theorem 1.2 is complete. □

Proof of Theorem 1.4. First of all, we suppose that f is a finite order transcendental meromorphic solution of (1.3). Then by Lemma 2.2, Lemma 2.5 and Equation (1.3), we can deduce

$$dT(r,f)+S(r,f)=T(r,\sum_{j=1}^{n}f(z+c_j))\le nT(r,f)+S(r,f),$$

which leads to $n\ge d\ge 2$.

(1) From (1.3), it is easy to see

$$\begin{aligned}P(z,f):=\Big[\sum_{j=1}^{n}f(z+c_j)][b_qf(z)^q+b_{q-1}f(z)^{q-1}+\dots+b_0\Big]\\ -[a_pf(z)^p+a_{p-1}f(z)^{p-1}+\dots+a_0]\equiv 0.\end{aligned}$$

Then we see $P(z,0)=-a_0\not\equiv 0$. It follows from Lemma 2.4 that

$$m(r,\frac{1}{f})=S(r,f)$$

possibly out a set with logarithmic measure, which leads to

$$N(r,\frac{1}{f})=T(r,f)+S(r,f).$$

Thus $\delta(0,f)=0$, i.e., $\lambda(f)=\sigma(f)$.

(2) Let $f(z)=g(z)+z$ and substitute it into (1.3), we see

$$\begin{aligned}\tilde{P}(z,g):=[\sum_{j=1}^{n}g(z+c_j)+\sum_{j=1}^{n}c_j+nz][b_q(g(z)+z)^q+b_{q-1}(g(z)+z)^{q-1}+\dots\\ +b_0]-[a_p(g(z)+z)^p+a_{p-1}(g(z)+z)^{p-1}+\dots+a_0]\equiv 0.\end{aligned}$$

Thus

$$\tilde{P}(z,0)=(nz+\sum_{j=1}^{n}c_j)Q(z)-P(z).$$

If $(nz+\sum_{j=1}^{n}c_j)Q(z)-P(z)\equiv 0$, then $Q(z)|P(z)$. But $Q(z),P(z)$ are relatively prime polynomials, so $Q(z)$ is a nonzero constant and then $P(z)$ is a polynomial with degree 1. This is impossible since $\deg R(z)\ge 2$. Thus $(nz+\sum_{j=1}^{n}c_j)Q(z)-P(z)\not\equiv 0$, that is $\tilde{P}(z,0)\not\equiv 0$. It follows from Lemma 2.4 once again that

$$m(r,\frac{1}{f-z})=m(r,\frac{1}{g})=S(r,f)$$

possibly out a set with logarithmic measure, which leads to

$$N(r,\frac{1}{f-z})=T(r,f)+S(r,f).$$

Thus $\tau(f)=\sigma(f)$.

(3) By applying Lemmas 2.1, 2.3, 2.5 to (1.3), we have

$$\begin{aligned}dT(r,f)+S(r,f)&=T(r,\sum_{j=1}^{n}f(z+c_j))\\ &=m\big(r,\frac{\sum_{j=1}^{n}f(z+c_j)}{f}f\big)+N(r,\sum_{j=1}^{n}f(z+c_j))\end{aligned}$$

$$\begin{aligned}&\le m(r,f)+\sum_{j=1}^{n}N(r,f(z+c_j))+S(r,f)\\&=T(r,f)-N(r,f)+nN(r,f)+S(r,f).\end{aligned}$$

That is,

$$T(r,f)\le(d-1)T(r,f)\le(n-1)N(r,f)+S(r,f),$$

which leads to $\lambda(\frac{1}{f})=\sigma(f)$.

(4) Suppose that b is a deficiency value with respect to f, from result (3), we see $b\neq\infty$. Set $f(z)=g(z)+b$ and substitute it into (1.3), we have

$$nb+\sum_{j=1}^{n}g(z+c_j)=R(z)\circ(g(z)+b)=R(z)\circ(z+b)\circ g(z).$$

That is,

$$\sum_{j=1}^{n}g(z+c_j)=\tilde{R}(z)\circ g(z),$$

where $\tilde{R}(z)=R(z+b)-nb$. If

$$\tilde{a}_0=a_pb^p+a_{p-1}b^{p-1}+\dots+a_0-nb(b_qb^q+b_{q-1}b^{q-1}+\dots+b_0)\neq 0,$$

then from result (1), we see

$$N(r,\frac{1}{f-b})=N(r,\frac{1}{g})=T(r,f)+S(r,f),$$

which implies b is not a deficiency value of f. The proof of Theorem 1.4 is complete. □

Proof of Theorem 1.10. If $n\ge 3$, then by Lemma 2.1, it is easy to see that

$$\begin{gathered}T(r,\frac{\Delta f}{f})=N(r,\frac{\Delta f}{f})+m(r,\frac{\Delta f}{f})\le T(r,f)+S(r,f),\\T(r,\Delta f)=m(r,\Delta f)\le m(r,\frac{\Delta f}{f})+m(r,f)\le T(r,f)+S(r,f).\end{gathered}$$

Thus

$$\begin{aligned}&nT(r,f)+S(r,f)\\&\le(n+s)T(r,f)-sT(r,\frac{\Delta f}{f})\\&\le T(r,f^{n+s}(\frac{\Delta f}{f})^s)\\&=T(r,f^n(\Delta f)^s)\\&\le\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{\Delta f})+\overline{N}(r,\frac{1}{f^n(\Delta f)^s-\alpha(z)})+\varepsilon T(r,f)+S(r,f)\\&\le 2T(r,f)+\overline{N}(r,\frac{1}{f^n(\Delta f)^s-\alpha(z)})+\varepsilon T(r,f)+S(r,f).\end{aligned}$$

That is,

$$(n-2-\varepsilon)T(r,f)\le\overline{N}(r,\frac{1}{f^n(\Delta f)^s-\alpha})+S(r,f).$$

Thus $f^n(\Delta f)^s-\alpha(z)$ has infinitely many zeros since ε can be fixed small enough. Now we just need to consider the case $n=2$. On the contrary, we suppose that

$f^n(\Delta f)^s - \alpha(z)$ has just only finitely many zeros, then we obtain that there exists two polynomials said p and Q such that

$$f^2(\Delta f)^s - \alpha = pe^Q. \tag{3.11}$$

By differentiating (3.11) and eliminating $e^{Q(z)}$, we obtain

$$f[2pf'(\Delta f)^s + sp(\Delta f)^{s-1}(\Delta f)'f - (p' + pQ')(\Delta f)^s f] = p\alpha' - \alpha(p' + pQ'). \tag{3.12}$$

If $p\alpha' - \alpha(p' + pQ') \not\equiv 0$, then from (3.12), we see

$$N(r, \frac{1}{f}) \leq N(r, \frac{1}{p\alpha' - \alpha(p' + pQ')}) = S(r, f).$$

Hence

$$\begin{aligned}
2T(r,f) &\leq T(r, f^2(\Delta f)^s) + S(r,f) \\
&\leq \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{\Delta f}) + \overline{N}(r, \frac{1}{f^2(\Delta f)^s - \alpha(z)}) + \varepsilon T(r,f) + S(r,f) \\
&\leq T(r,f) + \varepsilon T(r,f) + S(r,f),
\end{aligned}$$

which is impossible.

If $p\alpha' - \alpha(p' + pQ') \equiv 0$, we see $pe^Q = k\alpha$, where k is a constant. Thus we substitute it into (3.11), and obtain

$$f^2(\Delta f)^s = (k+1)\alpha.$$

Thus

$$2T(r,f) \leq T(r, f^2(\Delta f)^s) + S(r,f) = T(r, (k+1)\alpha) + S(r,f) = S(r,f),$$

which is a contradiction. The proof of Theorem 1.10 is complete. □

Acknowledgments. This research was supported by the Tianyuan Funds for Mathematics (No. 11326085) and the Fundamental Research Funds for the Central Universities (No. 2011QNA25). The authors would like to thank the anonymous referees for their comments and suggestions.

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Jie Zhang
College of Science, China University of Mining and Technology, Xuzhou 221116, China
E-mail address: zhangjie1981@cumt.edu.cn

Jianjun Zhang (corresponding author)
Mathematics and Information Technology School, Jiangsu Second Normal University, Nanjing 210013, China
E-mail address: zhangjianjun1982@163.com