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FIXED POINTS FOR α-ψ CONTRACTIVE MAPPINGS WITH AN APPLICATION TO QUADRATIC INTEGRAL EQUATIONS

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Abstract. Recently, Samet et al [24] introduced the concept of α-ψ contractive mappings and studied the existence of fixed points for such mappings. In this article, we prove three fixed point theorems for this class of operators in complete metric spaces. Our results extend the results in [24] and well known fixed point theorems due to Banach, Kannan, Chatterjea, Zamfirescu, Berinde, Suzuki, Ćirić, Nieto, López, and many others. We prove that α-ψ contractions unify large classes of contractive type operators, whose fixed points can be obtained by means of the Picard iteration. Finally, we utilize our results to discuss the existence and uniqueness of solutions to a class of quadratic integral equations.

1. Introduction

Fixed point theory plays an important role in nonlinear analysis. This is because many practical problems in applied science, economics, physics, and engineering can be reformulated as a problem of finding fixed points of nonlinear mappings. The Banach contraction principle [4] is one of the fundamental results in fixed point theory. It guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces and provides a constructive method to approximate those fixed points. During the last few decades, several extensions of this famous principle have been established.

Recently, Samet et al [24] introduced the class of α-ψ contractive mappings and studied the existence of fixed points for such mappings. Let us recall the main results obtained in this work. Let Ψ be the family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(P1) ψ is nondecreasing;
(P2) $\sum_{k=0}^{\infty} \psi^k(t) < \infty$, for all $t > 0$, where ψ^k is the k-th iterate of ψ.

A function $\psi \in \Psi$ is called a (c)-comparison function.

Definition 1.1. Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an α-ψ-contraction if there exist a (c)-comparison function $\psi \in \Psi$

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and a function $\alpha : X \times X \to \mathbb{R}$ such that

$$\alpha(x,y)d(Tx,Ty) \le \psi(d(x,y)), \quad \text{for all } x,y \in X. \tag{1.1}$$

Definition 1.2. Let X be a nonempty set, $T : X \to X$ be a given mapping and $\alpha : X \times X \to \mathbb{R}$. We say that T is α-admissible if

$$x,y \in X,\ \alpha(x,y) \ge 1 \Longrightarrow \alpha(Tx,Ty) \ge 1. \tag{1.2}$$

The results obtained in [24] can be summarized as follows.

Theorem 1.3. *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi$ such that*

(i) *Inequality* (1.1) *holds;*
(ii) *T is α-admissible;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;*
(iv) *T is continuous or*
(v) *for every $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that $x_n \to x \in X$ and $\alpha(x_n, x_{n+1}) \ge 1$ for $n \in \mathbb{N}$, we have $\alpha(x_n, x) \ge 1$ for $n \in \mathbb{N}$.*

Then T has a fixed point. Moreover, if in addition we suppose that for every pair $(u,v) \in X \times X$ there exists $w \in X$ such that $\alpha(u,w) \ge 1$ and $\alpha(v,w) \ge 1$, we have a unique fixed point.

For other results in this direction, we refer the reader to [1, 16, 17, 18].

In this paper, we extend and improve Theorem 1.3 without the condition (1.2). Moreover, we prove that our results unify the most existing fixed point theorems, where the fixed points can be obtained by means of the Picard iteration. Finally, we utilize our results to discuss the existence and uniqueness of solutions to a class of quadratic integral equations.

2. Main results

If $T : X \to X$ is a given mapping, we denote by $\mathrm{Fix}(T)$ the set of its fixed points; that is,

$$\mathrm{Fix}(T) = \{x \in X : x = Tx\}.$$

The following lemma will be useful later.

Lemma 2.1 ([6])**.** *Let $\psi \in \Psi$. Then*

(i) *$\psi(t) < t$, for all $t > 0$;*
(ii) *$\psi(0) = 0$;*
(iii) *ψ is continuous at $t = 0$.*

For a given $\psi \in \Psi$, let Σ_ψ be the set defined by

$$\Sigma_\psi = \{\sigma \in (0,\infty) : \sigma\psi \in \Psi\}.$$

We start with the following proposition.

Proposition 2.2. *Let (X,d) be a metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi$ such that T is an α-ψ contraction. Suppose that there exists $\sigma \in \Sigma_\psi$ and for some positive integer p, there exists a finite sequence $\{\xi_i\}_{i=0}^p \subset X$ such that*

$$\xi_0 = x_0, \quad \xi_p = Tx_0, \quad \alpha(T^n\xi_i, T^n\xi_{i+1}) \ge \sigma^{-1}, \quad n \in \mathbb{N},\ i = 0,\dots,p-1. \tag{2.1}$$

Then $\{T^n x_0\}$ is a Cauchy sequence in (X,d).

Proof. Let $\varphi = \sigma\psi$. By definition of Σ_ψ, we have $\varphi \in \Psi$. Let $\{\xi_i\}_{i=0}^p$ be a finite sequence in X satisfying (2.1). Consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ in X defined by $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. We claim that

$$d(T^r\xi_i, T^r\xi_{i+1}) \le \varphi^r(d(\xi_i, \xi_{i+1})), \quad r \in \mathbb{N},\ i = 0, \dots, p-1. \tag{2.2}$$

Let $i \in \{0, 1, \dots, p-1\}$. From (2.1), we have

$$\sigma^{-1} d(T\xi_i, T\xi_{i+1}) \le \alpha(\xi_i, \xi_{i+1}) d(T\xi_i, T\xi_{i+1}) \le \psi(d(\xi_i, \xi_{i+1})),$$

which implies that

$$d(T\xi_i, T\xi_{i+1}) \le \varphi(d(\xi_i, \xi_{i+1})). \tag{2.3}$$

Again, we have

$$\sigma^{-1} d(T^2\xi_i, T^2\xi_{i+1}) \le \alpha(T\xi_i, T\xi_{i+1}) d(T(T\xi_i), T(T\xi_{i+1})) \le \psi(d(T\xi_i, T\xi_{i+1})),$$

which implies that

$$d(T^2\xi_i, T^2\xi_{i+1}) \le \varphi(d(T\xi_i, T\xi_{i+1})). \tag{2.4}$$

Since φ is a nondecreasing function (from property (Ψ_1)), from (2.3) and (2.4), we obtain that

$$d(T^2\xi_i, T^2\xi_{i+1}) \le \varphi^2(d(\xi_i, \xi_{i+1})).$$

Continuing this process, by induction we obtain (2.2). Now, using the triangle inequality and (2.2), for every $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(T^n x_0, T^{n+1} x_0) \\ &\le d(T^n\xi_0, T^n\xi_1) + d(T^n\xi_1, T^n\xi_2) + \dots + d(T^n\xi_{p-1}, T^n\xi_p) \\ &= \sum_{i=0}^{p-1} d(T^n\xi_i, T^n\xi_{i+1}) \\ &\le \sum_{i=0}^{p-1} \varphi^n(d(\xi_i, \xi_{i+1})). \end{aligned}$$

Thus we proved that

$$d(x_n, x_{n+1}) \le \sum_{i=0}^{p-1} \varphi^n(d(\xi_i, \xi_{i+1})), \quad n \in \mathbb{N},$$

which implies that for $n < m$,

$$\begin{aligned} d(x_n, x_m) &\le \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \\ &\le \sum_{j=n}^{m-1} \sum_{i=0}^{p-1} \varphi^j(d(\xi_i, \xi_{i+1})) \\ &= \sum_{i=0}^{p-1} \sum_{j=n}^{m-1} \varphi^j(d(\xi_i, \xi_{i+1})). \end{aligned}$$

On the other hand, from property (P2), we have

$$\sum_{i=0}^{p-1} \sum_{j=n}^{m-1} \varphi^j(d(\xi_i, \xi_{i+1})) \to 0 \quad \text{as } n, m \to \infty.$$

Then we proved that $d(x_n, x_m) \to 0$ as $n, m \to \infty$; that is, $\{x_n\}$ is a Cauchy sequence in the metric space (X, d). □

Our first main theorem is the following fixed point result obtained under the continuity assumption of the mapping T.

Theorem 2.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi$ such that T is an α-ψ contraction. Suppose also that* (2.1) *is satisfied. Then $\{T^n x_0\}$ converges to some $x^* \in X$. Moreover, if T is continuous, then x^* is a fixed point of T.*

Proof. From Proposition 2.2, we know that $\{T^n x_0\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(T^n x_0, x^*) = 0.$$

Since T is continuous, we have also

$$\lim_{n\to\infty} d(T^{n+1} x_0, Tx^*) = 0.$$

By the uniqueness of the limit, we obtain $x^* = Tx^*$. □

The next theorem does not require the continuity assumption of T.

Theorem 2.4. *Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi$ such that T is an α-ψ contraction. Suppose also that* (2.1) *is satisfied. Then $\{T^n x_0\}$ converges to some $x^* \in X$. Moreover, if there exists a subsequence $\{T^{\gamma(n)} x_0\}$ of $\{T^n x_0\}$ such that*

$$\lim_{n\to\infty} \alpha(T^{\gamma(n)} x_0, x^*) = \ell \in (0, \infty),$$

then x^* is a fixed point of T.

Proof. From Proposition 2.2 and the completeness of the metric space (X, d), we know that $\{T^n x_0\}$ converges to some $x^* \in X$. Suppose now that there exists a subsequence $\{T^{\gamma(n)} x_0\}$ of $\{T^n x_0\}$ such that

$$\lim_{n\to\infty} \alpha(T^{\gamma(n)} x_0, x^*) = \ell \in (0, \infty). \tag{2.5}$$

Since T is an α-ψ contraction, we have

$$\alpha(T^{\gamma(n)} x_0, x^*) d(T^{\gamma(n)+1} x_0, Tx^*) \le \psi(d(T^{\gamma(n)} x_0, x^*)), \ n \in \mathbb{N}.$$

Letting $n \to \infty$ in the above inequality, using (2.5), the properties (ii) and (iii) of Lemma 2.1, we obtain

$$\ell\, d(x^*, Tx^*) \le \psi(0) = 0,$$

which implies that x^* is a fixed point of T. □

The next theorem gives us a sufficient condition for the uniqueness of the fixed point.

Theorem 2.5. *Let (X, d) be a metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi$ such that T is an α-ψ contraction. Suppose also that*

(i) $\mathrm{Fix}(T) \neq \emptyset$*;*

(ii) *for every pair* $(x, y) \in \operatorname{Fix}(T) \times \operatorname{Fix}(T)$ *with* $x \neq y$, *if* $\alpha(x, y) < 1$, *then there exists* $\eta \in \Sigma_\psi$ *and for some positive integer* q, *there is a finite sequence* $\{\zeta_i(x, y)\}_{i=0}^q \subset X$ *such that*

$$\zeta_0(x, y) = x, \quad \zeta_q(x, y) = y, \quad \alpha(T^n\zeta_i(x, y), T^n\zeta_{i+1}(x, y)) \geq \eta^{-1},$$

for $n \in \mathbb{N}$ *and* $i = 0, \dots, q-1$.

Then T *has a unique fixed point.*

Proof. Let $\varphi = \eta\psi \in \Psi$. Suppose that $u, v \in X$ are two fixed points of T such that $d(u, v) > 0$. We consider two cases.

Case 1: $\alpha(u, v) \geq 1$. Since T is an α-ψ contraction, we have

$$d(u, v) \leq \alpha(u, v)d(Tu, Tv) \leq \psi(d(u, v)).$$

From the property (i) of Lemma 2.1, we have $\psi(d(u, v)) < d(u, v)$, which yields $d(u, v) < d(u, v)$, leading to a a contradiction.

Case 2: $\alpha(u, v) < 1$. By assumption, there exists a finite sequence $\{\zeta_i(u, v)\}_{i=0}^q$ in X such that

$$\zeta_0(u, v) = u, \quad \zeta_q(u, v) = v, \quad \alpha(T^n\zeta_i(u, v), T^n\zeta_{i+1}(u, v)) \geq \eta^{-1},$$

for $n \in \mathbb{N}$ and $i = 0, \dots, q-1$. As in the proof of Proposition 2.2, we can establish that

$$d(T^r\zeta_i(u, v), T^r\zeta_{i+1}(u, v)) \leq \varphi^r(d(\zeta_i(u, v), \zeta_{i+1}(u, v))), \quad r \in \mathbb{N}, \quad i = 0, \dots, q-1. \tag{2.6}$$

Using the triangle inequality and (2.6), we have

$$\begin{aligned} d(u, v) &= d(T^nu, T^nv) \\ &\leq \sum_{i=0}^{q-1} d(T^n\zeta_i(u, v), T^n\zeta_{i+1}(u, v)) \\ &\leq \sum_{i=0}^{q-1} \varphi^n(d(\zeta_i(u, v), \zeta_{i+1}(u, v))) \to 0 \quad \text{as } n \to \infty \end{aligned}$$

(from (P2)). Then $u = v$, which is contradicts the assumption $d(u, v) > 0$. □

3. Consequences

In this section, we will see that the most existing fixed point results, where the fixed points can be obtained by means of the Picard iteration, are particular cases of our main theorems.

3.1. The class of ψ-contractive mappings.

The class of ψ-contractive mappings is defined as follows.

Definition 3.1. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a ψ-contraction if there exists a function $\psi \in \Psi$ such that

$$d(Tx, Ty) \leq \psi(d(x, y)), \quad \text{for all } x, y \in X. \tag{3.1}$$

Theorem 3.2. *Let* (X, d) *be a metric space and* $T : X \to X$ *be a given mapping. Suppose that there exists* $\psi \in \Psi$ *such that* T *is a* ψ*-contraction. Then there exists* $\alpha : X \times X \to \mathbb{R}$ *such that* T *is an* α-ψ *contraction.*

Proof. Consider the function $\alpha : X \times X \to \mathbb{R}$ defined by

$$\alpha(x,y) = 1, \quad \text{for all } x, y \in X. \tag{3.2}$$

Clearly, from (3.1), T is an α-ψ contraction. $\square$

Corollary 3.3 ([6, Theorem 2.8])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a ψ-contraction for some $\psi \in \Psi$. Then T has a unique fixed point.*

Proof. From (i) Lemma 2.1, we have

$$d(Tx,Ty) \leq d(x,y), \quad \text{for all } x, y \in X,$$

which implies that T is a continuous mapping. From Theorem 3.2, T is an α-ψ contraction, where α is defined by (3.2). Clearly, (2.1) is satisfied with $p = 1$ and $\sigma = 1$. By Theorem 2.3, T has a fixed point. The uniqueness follows immediately from (3.2) and Theorem 2.5. $\square$

Note that the Banach contraction principle [4] follows immediately from Corollary 3.3 with $\psi(t) = k\,t$, $t \geq 0$, $k \in (0,1)$.

Observe also that the reverse of Theorem 3.2 is not true in general. As a counter-example, let $T : [0,1] \to [0,1]$ be the mapping defined by

$$Tx = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1/2, \\ 1/2, & \text{otherwise.} \end{cases}$$

Clearly, T is not a ψ-contraction since T is not continuous (w.r.t. the standard metric). However, T is an α-ψ contraction with $\psi(t) = t/2$ and

$$\alpha(x,y) = \begin{cases} 1/4, & \text{if } 0 \leq x, y \leq 1/2, \\ 0, & \text{otherwise.} \end{cases}$$

3.2. **The class of rational contractive mappings.**

3.2.1. *Dass-Gupta contraction.*

Definition 3.4. Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a Dass-Gupta contraction if there exist constants $\lambda, \mu \geq 0$ with $\lambda + \mu < 1$ such that

$$d(Tx,Ty) \leq \mu d(y,Ty)\,\frac{1 + d(x,Tx)}{1 + d(x,y)} + \lambda d(x,y), \quad \text{for all } x, y \in X. \tag{3.3}$$

Theorem 3.5. *Let (X,d) be a metric space and $T : X \to X$ be a given mapping. Suppose that T is a Dass-Gupta contraction. Then there exist $\psi \in \Psi$ and $\alpha : X \times X \to \mathbb{R}$ such that T is an α-ψ contraction.*

Proof. From (3.3), for all $x, y \in X$, we have

$$d(Tx,Ty) - \mu d(y,Ty)\,\frac{1 + d(x,Tx)}{1 + d(x,y)} \leq \lambda d(x,y),$$

which yields

$$\Big(1 - \mu\frac{d(y,Ty)(1 + d(x,Tx))}{(1 + d(x,y))d(Tx,Ty)}\Big)d(Tx,Ty) \leq \lambda d(x,y), \quad x, y \in X,\ Tx \neq Ty. \tag{3.4}$$

Consider the functions $\psi : [0,\infty) \to [0,\infty)$ and $\alpha : X \times X \to \mathbb{R}$ defined by

$$\psi(t) = \lambda\,t, \quad t \geq 0 \tag{3.5}$$

and

$$\alpha(x,y)=\begin{cases}1-\mu\frac{d(y,Ty)(1+d(x,Tx))}{(1+d(x,y))d(Tx,Ty)}, & \text{if } Tx\neq Ty,\\ 0, & \text{otherwise.}\end{cases} \tag{3.6}$$

From (3.4), we have

$$\alpha(x,y)d(Tx,Ty)\leq \psi(d(x,y)),\quad \text{for all } x,y\in X.$$

Then T is an α-ψ contraction. □

Corollary 3.6 (Dass-Gupta [9]). *Let (X,d) be a complete metric space and $T: X\to X$ be a given mapping. Suppose that there exist constants $\lambda,\mu\geq 0$ with $\lambda+\mu<1$ such that (3.3) is satisfied. Then T has a unique fixed point.*

Proof. Let x_0 be an arbitrary point in X. If for some $r\in\mathbb{N}$, $T^rx_0=T^{r+1}x_0$, then T^rx_0 will be a fixed point of T. So we can suppose that $T^rx_0\neq T^{r+1}x_0$, for all $r\in\mathbb{N}$. From (3.6), for all $n\in\mathbb{N}$, we have

$$\begin{aligned}\alpha(T^nx_0,T^{n+1}x_0)&=1-\mu\frac{d(T^{n+1}x_0,T^{n+2}x_0)(1+d(T^nx_0,T^{n+1}x_0))}{(1+d(T^nx_0,T^{n+1}x_0))d(T^{n+1}x_0,T^{n+2}x_0)}\\&=1-\mu>0.\end{aligned}$$

On the other hand, from (3.5), we have

$$(1-\mu)^{-1}\psi(t)=\frac{\lambda}{1-\mu}t,\quad t\geq 0.$$

Since $\lambda+\mu<1$, we have $(1-\mu)^{-1}\psi\in\Psi$; that is, $(1-\mu)^{-1}\in\Sigma_\psi$. Then (2.1) is satisfied with $p=1$ and $\sigma=(1-\mu)^{-1}$. From the first part of Theorem 2.4, the sequence $\{T^nx_0\}$ converges to some $x^*\in X$. Without loss of generality, we can suppose that there exists $N\in\mathbb{N}$ such that

$$T^{n+1}x_0\neq Tx^*,\ n\geq N.$$

Otherwise, x^* will be a fixed point of T. From (3.6), for all $n\geq N$, we have

$$\alpha(T^nx_0,x^*)=1-\mu\frac{d(x^*,Tx^*)(1+d(T^nx_0,T^{n+1}x_0))}{(1+d(T^nx_0,x^*))d(T^{n+1}x_0,Tx^*)}\to 1-\mu\quad \text{as } n\to\infty.$$

From the second part of Theorem 2.4 (with $\ell=1-\mu$), we deduce that x^* is a fixed point of T. For the uniqueness, observe that for every pair $(x,y)\in \mathrm{Fix}(T)\times\mathrm{Fix}(T)$ with $x\neq y$, we have $\alpha(x,y)=1$. By Theorem 2.5, x^* is the unique fixed point of T. □

3.2.2. *Jaggi contraction.*

Definition 3.7. Let (X,d) be a metric space and $T:X\to X$ be a given mapping. We say that T is a Jaggi contraction if there exist constants $\lambda,\mu\geq 0$ with $\lambda+\mu<1$ such that

$$d(Tx,Ty)\leq \mu\frac{d(x,Tx)d(y,Ty)}{d(x,y)}+\lambda d(x,y),\quad \text{for all } x,y\in X,\ x\neq y. \tag{3.7}$$

Theorem 3.8. *Let (X,d) be a metric space and $T:X\to X$ be a given mapping. Suppose that T is a Jaggi contraction. Then there exist $\psi\in\Psi$ and $\alpha: X\times X\to\mathbb{R}$ such that T is an α-ψ contraction.*

Proof. From (3.7), for all $x, y \in X$ with $x \neq y$, we have

$$d(Tx, Ty) - \mu \frac{d(x, Tx)d(y, Ty)}{d(x, y)} \leq \lambda d(x, y),$$

which yields

$$\Big(1 - \mu \frac{d(x, Tx)d(y, Ty)}{d(x, y)d(Tx, Ty)}\Big) d(Tx, Ty) \leq \lambda d(x, y), \quad x, y \in X,\ Tx \neq Ty. \tag{3.8}$$

Consider the functions $\psi : [0, \infty) \to [0, \infty)$ and $\alpha : X \times X \to \mathbb{R}$ defined by

$$\psi(t) = \lambda t, \quad t \geq 0 \tag{3.9}$$

and

$$\alpha(x, y) = \begin{cases} 1 - \mu \frac{d(x,Tx)d(y,Ty)}{d(x,y)d(Tx,Ty)}, & \text{if } Tx \neq Ty, \\ 0, & \text{otherwise.} \end{cases} \tag{3.10}$$

From (3.8), we have

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)), \quad \text{for all } x, y \in X.$$

Then T is an α-ψ contraction. □

Corollary 3.9 (Jaggi [14])**.** *Let (X, d) be a complete metric space and $T : X \to X$ be a continuous mapping. Suppose that there exist constants $\lambda, \mu \geq 0$ with $\lambda + \mu < 1$ such that* (3.7) *is satisfied. Then T has a unique fixed point.*

Proof. Let x_0 be an arbitrary point in X. without loss of generality, we can suppose that $T^r x_0 \neq T^{r+1}x_0$, for all $r \in \mathbb{N}$. From (3.10), for all $n \in \mathbb{N}$, we have

$$\alpha(T^n x_0, T^{n+1}x_0) = 1 - \mu \frac{d(T^n x_0, T^{n+1}x_0)d(T^{n+1}x_0, T^{n+2}x_0)}{d(T^n x_0, T^{n+1}x_0)d(T^{n+1}x_0, T^{n+2}x_0)} = 1 - \mu > 0.$$

On the other hand, from (3.9), for all $t \geq 0$, we have

$$(1 - \mu)^{-1}\psi(t) = \frac{\lambda}{1 - \mu} t.$$

Since $\lambda + \mu < 1$, we have $(1 - \mu)^{-1}\psi \in \Psi$; that is, $(1 - \mu)^{-1} \in \Sigma_\psi$. Then (2.1) is satisfied with $p = 1$ and $\sigma = (1 - \mu)^{-1}$. By the first part of Theorem 2.3, $\{T^n x_0\}$ converges to some $x^* \in X$. Since T is continuous, by the second part of Theorem 2.3, x^* is a fixed point of T. Moreover, for every pair $(x, y) \in \mathrm{Fix}(T) \times \mathrm{Fix}(T)$ with $x \neq y$, we have $\alpha(x, y) = 1$. Then by Theorem 2.5, x^* is the unique fixed point of T. □

3.3. **The class of Berinde mappings.** In [5], Berinde introduced the concept of weak contractive mappings and studied the existence of fixed points for such mappings. Moreover, he proved that a large class of contractive type mappings (Kannan's contraction [15], Chatterjee's contraction [7], Zamfirescu contraction [27], Hardy-Rogers contraction [12], and many others) belong to the category of weakly contractive mappings. In this section, we will see that any weak contraction is an α-ψ contraction. Moreover, we will see that Berinde fixed point theorem can be deduced immediately from Theorem 2.4.

Definition 3.10. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a weak contraction if there exists $\lambda \in (0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \lambda\, d(x, y) + L\, d(y, Tx), \quad \text{for all } x, y \in X. \tag{3.11}$$

Theorem 3.11. *Let (X, d) be a metric space and $T : X \to X$ be a given mapping. If T is a weak contraction, then there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi$ such that T is an α-ψ contraction.*

Proof. From (3.11), we have

$$d(Tx, Ty) - L\, d(y, Tx) \leq \lambda d(x, y), \quad \text{for all } x, y \in X,$$

which yields

$$\Big(1 - L\frac{d(y, Tx)}{d(Tx, Ty)}\Big)d(Tx, Ty) \leq \lambda d(x, y), \quad x, y \in X,\ Tx \neq Ty. \tag{3.12}$$

Consider the functions $\psi : [0, \infty) \to [0, \infty)$ and $\alpha : X \times X \to \mathbb{R}$ defined by

$$\psi(t) = \lambda t, \quad t \geq 0,$$

and

$$\alpha(x, y) = \begin{cases} 1 - L\frac{d(y,Tx)}{d(Tx,Ty)}, & \text{if } Tx \neq Ty, \\ 0, & \text{otherwise.} \end{cases} \tag{3.13}$$

From (3.12), we have

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)), \quad \text{for all } x, y \in X.$$

Then T is an α-ψ contraction. □

Corollary 3.12 (Berinde [5])**.** *Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exist constants $\lambda \in (0, 1)$ and $L \geq 0$ such that (3.11) is satisfied. Then T has a fixed point.*

Proof. Let x_0 be an arbitrary point in X. Without loss of generality, we can suppose that $T^r x_0 \neq T^{r+1}x_0$, for all $r \in \mathbb{N}$. From (3.13), for all $n \in \mathbb{N}$, we have

$$\alpha(T^n x_0, T^{n+1}x_0) = 1 - L\frac{d(T^{n+1}x_0, T^{n+1}x_0)}{d(T^{n+1}x_0, T^{n+2}x_0)} = 1.$$

Then (2.1) holds with $\sigma = 1$ and $p = 1$. From the first part of Theorem 2.4, the sequence $\{T^n x_0\}$ converges to some $x^* \in X$. Without loss of generality, we can suppose that there exists some $N \in \mathbb{N}$ such that

$$T^{n+1}x_0 \neq Tx^*, \quad n \geq N.$$

From (3.13), for all $n \geq N$, we have

$$\alpha(T^n x_0, x^*) = 1 - L\frac{d(x^*, T^{n+1}x_0)}{d(T^{n+1}x_0, Tx^*)} \to 1 \quad \text{as } n \to \infty.$$

By the second part of Theorem 2.4 (with $\ell = 1$), we deduce that x^* is a fixed point of T. □

Note that a Berinde mapping need not have a unique fixed point (see [6, Example 2.11]).

3.4. Ćirić type maps with a non-unique fixed point.

Definition 3.13. Let (X,d) be a metric space and $T:X\to X$ be a given mapping. We say that T is a Ćirić mapping if there exists $\lambda\in(0,1)$ such that for all $x,y\in X$, we have

$$\min\{d(Tx,Ty),d(x,Tx),d(y,Ty)\}-\min\{d(x,Ty),d(y,Tx)\}\le\lambda\,d(x,y). \tag{3.14}$$

Theorem 3.14. *Let (X,d) be a metric space and $T:X\to X$ be a given mapping. Suppose that there exists $\lambda\in(0,1)$ such that (3.14) is satisfied. Then there exist $\alpha:X\times X\to\mathbb{R}$ and $\psi\in\Psi$ such that T is an α-ψ contraction.*

Proof. Consider the functions $\psi:[0,\infty)\to[0,\infty)$ and $\alpha:X\times X\to\mathbb{R}$ defined by

$$\psi(t)=\lambda t,\quad t\ge 0 \tag{3.15}$$

and

$$\alpha(x,y)=\begin{cases}\min\big\{1,\frac{d(x,Tx)}{d(Tx,Ty)},\frac{d(y,Ty)}{d(Tx,Ty)}\big\}-\min\big\{\frac{d(x,Ty)}{d(Tx,Ty)},\frac{d(y,Tx)}{d(Tx,Ty)}\big\}, & \text{if } Tx\ne Ty,\\ 0, & \text{otherwise.}\end{cases} \tag{3.16}$$

From (3.14), we have

$$\alpha(x,y)d(Tx,Ty)\le\psi(d(x,y)),\quad\text{for all } x,y\in X, \tag{3.17}$$

which implies that T is an α-ψ contraction. □

Corollary 3.15 (Ćirić [8])**.** *Let (X,d) be a complete metric space. Suppose that $T:X\to X$ is a continuous Ćirić mapping. Then T has a fixed point.*

Proof. Let $x_0\in X$ be an arbitrary point. Without loss of generality, we can suppose that $T^rx_0\ne T^{r+1}x_0$, for all $r\in\mathbb{N}$. From (3.16), for all $n\in\mathbb{N}$, we have

$$\begin{aligned}\alpha(T^nx_0,T^{n+1}x_0)&=\min\big\{1,\frac{d(T^nx_0,T^{n+1}x_0)}{d(T^{n+1}x_0,T^{n+2}x_0)},\frac{d(T^{n+1}x_0,T^{n+2}x_0)}{d(T^{n+1}x_0,T^{n+2}x_0)}\big\}\\&\quad-\min\big\{\frac{d(T^nx_0,T^{n+2}x_0)}{d(T^{n+1}x_0,T^{n+2}x_0)},\frac{d(T^{n+1}x_0,T^{n+1}x_0)}{d(T^{n+1}x_0,T^{n+2}x_0)}\big\}\\&=\min\big\{1,\frac{d(T^nx_0,T^{n+1}x_0)}{d(T^{n+1}x_0,T^{n+2}x_0)}\big\}.\end{aligned}$$

Suppose that for some $n\in\mathbb{N}$, we have

$$\alpha(T^nx_0,T^{n+1}x_0)=\frac{d(T^nx_0,T^{n+1}x_0)}{d(T^{n+1}x_0,T^{n+2}x_0)}.$$

In this case, from (3.15) and (3.17), we have

$$d(T^nx_0,T^{n+1}x_0)\le\lambda d(T^nx_0,T^{n+1}x_0).$$

This implies (from the assumption $T^rx_0\ne T^{r+1}x_0$, for all $r\in\mathbb{N}$) that $\lambda\ge1$, which is a contradiction. Then

$$\alpha(T^nx_0,T^{n+1}x_0)=1,\quad\text{for all } n\in\mathbb{N}.$$

Then (2.1) is satisfied with $p=1$ and $\sigma=1$. By Theorem 2.4, we deduce that the sequence $\{T^nx_0\}$ converges to a fixed point of T. □

Note that a Ćirić mapping need not have a unique fixed point (see [8]).

3.5. **The class of Suzuki mappings.** We define Suzuki mappings as follows.

Definition 3.16. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a Suzuki mapping if there exists $r \in (0, 1)$ such that

$$(1 + r)^{-1} d(x, Tx) \le d(x, y) \Longrightarrow d(Tx, Ty) \le r\, d(x, y), \quad \text{for all } x, y \in X. \tag{3.18}$$

Theorem 3.17. *Let (X, d) be a metric space and $T : X \to X$ be a given mapping. Suppose that there exists $r \in (0, 1)$ such that (3.18) is satisfied. Then there exist $\alpha : X \times X \to \mathbb{R}$ and $\psi \in \Psi$ such that T is an α-ψ contraction.*

Proof. Consider the functions $\psi : [0, \infty) \to [0, \infty)$ and $\alpha : X \times X \to \mathbb{R}$ defined by

$$\psi(t) = rt, \quad t \ge 0$$

and

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (1 + r)^{-1} d(x, Tx) \le d(x, y), \\ 0, & \text{otherwise.} \end{cases} \tag{3.19}$$

From (3.18), we have

$$\alpha(x, y) d(Tx, Ty) \le \psi(d(x, y)), \quad \text{for all } x, y \in X.$$

Then T is an α-ψ contraction. □

Corollary 3.18 (Suzuki [25])**.** *Let (X, d) be a complete metric space, and suppose that $T : X \to X$ is a Suzuki mapping. Then T has a unique fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary point. For all $n \in \mathbb{N}$, we have

$$(1 + r)^{-1} d(T^n x_0, T(T^n x_0)) \le d(T^n x_0, T^{n+1} x_0), \tag{3.20}$$

which implies that $\alpha(T^n x_0, T^{n+1} x_0) = 1$, for all $n \in \mathbb{N}$, where α is defined by (3.19). Then (2.1) is satisfied with $p = 1$ and $\sigma = 1$. From the first part of Theorem 2.4, the sequence $\{T^n x_0\}$ converges to some $x^* \in X$. From (3.18) and (3.20), we have

$$d(T(T^n x_0), T^2(T^n x_0)) \le r\, d(T^n x_0, T(T^n x_0)), \quad \text{for all } n \in \mathbb{N},$$

which implies from [26, Lemma 2.1] that there exists a subsequence $\{\gamma(n)\}$ of $\{n\}$ such that

$$(1 + r)^{-1} d(T^{\gamma(n)} x_0, T^{\gamma(n)+1} x_0) \le d(T^{\gamma(n)} x_0, x^*), \quad \text{for all } n \in \mathbb{N}.$$

From (3.19), we have

$$\alpha(T^{\gamma(n)} x_0, x^*) = 1, \quad \text{for all } n \in \mathbb{N}.$$

By the second part of Theorem 2.4 (with $\ell = 1$), x^* is a fixed point of T. On the other hand, from (3.19), for every pair $(x, y) \in \mathrm{Fix}(T) \times \mathrm{Fix}(T)$ with $x \ne y$, we have $\alpha(x, y) = 1$. By Theorem 2.5, x^* is the unique fixed point of T. □

3.6. **The class of cyclic mappings.** In [23] the following notion was introduced, suggested by the consideration in [19].

Definition 3.19. Let (X, d) be a metric space, m be a positive integer and $T : X \to X$ be an operator. By definition, $X = \cup_{i=1}^m X_i$ is a cyclic representation of X with respect to T if

(i) X_i, $i = 1, \dots, m$ are nonempty sets;
(ii) $T(X_1) \subseteq X_2, \dots, T(X_{m-1}) \subseteq X_m$, $T(X_m) \subset X_1$.

Definition 3.20. Let (X,d) be a metric space, $A_1,\dots,A_m \in P_{cl}(X)$, $Y=\cup_{i=1}^m A_i$, with m a positive integer, and $T:Y\to Y$ be an operator. We say that T is a cyclic ψ-contraction for some $\psi\in\Psi$ if

(i) $\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T;
(ii) for all $i=1,\dots,m$, we have
$$d(Tx,Ty)\le \psi(d(x,y)),\quad \text{for all } x\in A_i,\ y\in A_{i+1},$$
where $A_{m+1}=A_1$.

Here, $P_{cl}(X)$ denotes the collection of nonempty closed subsets of (X,d).

We have the following result.

Theorem 3.21. *Let (X,d) be a metric space, m be a positive integer, $A_1,\dots,A_m\in P_{cl}(X)$, $Y=\cup_{i=1}^m A_i$ and $T:Y\to Y$ be a cyclic ψ-contraction for some $\psi\in\Psi$. Then there exists a function $\alpha:Y\times Y\to\mathbb{R}$ such that*
$$\alpha(x,y)d(Tx,Ty)\le\psi(d(x,y)),\quad \textit{for all } x,y\in Y. \tag{3.21}$$

Proof. Define the function $\alpha:Y\times Y\to\mathbb{R}$ by
$$\alpha(x,y)=\begin{cases}1, & \text{if } (x,y)\in A_i\times A_{i+1} \text{ for some } i=1,\dots,m,\\ 0, & \text{otherwise.}\end{cases} \tag{3.22}$$
From (ii) Definition 3.20, we obtain (3.21). □

Corollary 3.22 (Păcurar and Rus [21])**.** *Let (X,d) be a complete metric space, m be a positive integer, $A_1,\dots,A_m\in P_{cl}(X)$, $Y=\cup_{i=1}^m A_i$, $\psi\in\Psi$ and $T:Y\to Y$ be an operator. Suppose that*

(i) *$\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T;*
(ii) *T is a cyclic ψ-contraction.*

Then T has a unique fixed point $x^*\in\cap_{i=1}^m A_i$.

Proof. Let $x_0\in A_1$ be an arbitrary point. From condition (i) and (3.22), we have
$$\alpha(T^nx_0,T^{n+1}x_0)=1,\quad \text{for all } n\in\mathbb{N}.$$
Then (2.1) is satisfied with $p=1$ and $\sigma=1$. By the first part of Theorem 2.4, the sequence $\{T^nx_0\}$ converges to some $x^*\in Y$. By (i), the sequence $\{T^nx_0\}$ has an infinite number of terms in each A_i, $i=1,\dots,m$, so from each A_i, $i=1,\dots,m$, one can extract a subsequence $\{T^{\gamma_i(n)}x_0\}\subset A_i$ of $\{T^nx_0\}$. Since $\{A_i\}_{i=1}^m\subset P_{cl}(X)$, it follows that $x^*\in\cap_{i=1}^m A_i$. Then by (3.22), for a fixed $j=1\dots,m$, we have $\alpha(T^{\gamma_j(n)}x_0,x^*)=1$, for all $n\in\mathbb{N}$. By the second part of Theorem 2.4 (with $\ell=1$), we deduce that x^* is a fixed point of T. On the other hand, observe that
$$\mathrm{Fix}(T)\times\mathrm{Fix}(T)\subset\cap_{i=1}^m A_i\times\cap_{i=1}^m A_i,$$
which implies from (3.22) that
$$\alpha(x,y)=1,\quad \text{for all } (x,y)\in\mathrm{Fix}(T)\times\mathrm{Fix}(T).$$
By Theorem 2.5, we deduce that x^* is the unique fixed point of T. □

3.7. **Edelstein fixed point theorem.** Another consequence of our main results is the following generalized version of Edelstein fixed point theorem [11].

Corollary 3.23. *Let (X,d) be complete and ε-chainable for some $\varepsilon > 0$; i.e., given $x, y \in X$, there exist a positive integer N and a sequence $\{x_i\}_{i=0}^N \subset X$ such that*

$$x_0 = x, \quad x_N = y, \quad d(x_i, x_{i+1}) < \varepsilon, \quad \text{for } i = 0, \dots, N-1. \tag{3.23}$$

Let $T : X \to X$ be a given mapping such that

$$x, y \in X, \quad d(x,y) < \varepsilon \Longrightarrow d(Tx, Ty) \le \psi(d(x,y)), \tag{3.24}$$

for some $\psi \in \Psi$. Then T has a unique fixed point.

Proof. It is clear that from (3.24), the mapping T is continuous. Now, consider the function $\alpha : X \times X \to \mathbb{R}$ defined by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } d(x,y) < \varepsilon, \\ 0, & \text{otherwise.} \end{cases} \tag{3.25}$$

From (3.24), we have

$$\alpha(x,y) d(Tx, Ty) \le \psi(d(x,y)), \quad \text{for all } x, y \in X.$$

Let $x_0 \in X$. For $x = x_0$ and $y = Tx_0$, from (3.23) and (3.25), for some positive integer p, there exists a finite sequence $\{\xi_i\}_{i=0}^p \subset X$ such that

$$x_0 = \xi_0, \quad \xi_p = Tx_0, \quad \alpha(\xi_i, \xi_{i+1}) \ge 1, \quad \text{for } i = 0, \dots, p-1.$$

Now, let $i \in \{0, \dots, p-1\}$ be fixed. From (3.25) and (3.24), we have

$$\begin{aligned} \alpha(\xi_i, \xi_{i+1}) \ge 1 &\Longrightarrow d(\xi_i, \xi_{i+1}) < \varepsilon \\ &\Longrightarrow d(T\xi_i, T\xi_{i+1}) \le \psi(d(\xi_i, \xi_{i+1})) \le d(\xi_i, \xi_{i+1}) < \varepsilon \\ &\Longrightarrow \alpha(T\xi_i, T\xi_{i+1}) \ge 1. \end{aligned}$$

Again,

$$\begin{aligned} \alpha(T\xi_i, T\xi_{i+1}) \ge 1 &\Longrightarrow d(T\xi_i, T\xi_{i+1}) < \varepsilon \\ &\Longrightarrow d(T^2\xi_i, T^2\xi_{i+1}) \le \psi(d(T\xi_i, T\xi_{i+1})) \le d(T\xi_i, T\xi_{i+1}) < \varepsilon \\ &\Longrightarrow \alpha(T^2\xi_i, T^2\xi_{i+1}) \ge 1. \end{aligned}$$

By induction, we obtain

$$\alpha(T^n\xi_i, T^{n+1}\xi_{i+1}) \ge 1, \quad \text{for all } n \in \mathbb{N}.$$

Then (2.1) is satisfied with $\sigma = 1$. From Theorem 2.3, the sequence $\{T^n x_0\}$ converges to a fixed point of T. Using a similar argument, we can see that condition (ii) of Theorem 2.5 is satisfied, which implies that T has a unique fixed point. □

3.8. **Contractive mapping theorems in partially ordered sets.** In this section, we use our main results to establish some fixed point theorems in a metric space endowed with a partial order. Let (X,d) be a metric space and $\preceq$ be a partial order on X. We denote

$$\Delta = \{(x,y) \in X \times X : x \preceq y \text{ or } y \preceq x\}.$$

Corollary 3.24. *Let $T : X \to X$ be a given mapping. Suppose that there exists $\psi \in \Psi$ such that*
$$d(Tx,Ty) \leq \psi(d(x,y)), \quad \text{for all } (x,y) \in \Delta. \tag{3.26}$$
Suppose also that

(i) *T is continuous;*
(ii) *for some positive integer p, there exists a finite sequence $\{\xi_i\}_{i=0}^{p} \subset X$ such that*
$$\xi_0 = x_0, \quad \xi_p = Tx_0, \quad (T^n\xi_i, T^n\xi_{i+1}) \in \Delta, \quad n \in \mathbb{N},\ i = 0,\dots,p-1. \tag{3.27}$$

Then $\{T^n x_0\}$ converges to a fixed point of T.

Proof. Consider the function $\alpha : X \times X \to \mathbb{R}$ defined by
$$\alpha(x,y) = \begin{cases} 1, & \text{if } (x,y) \in \Delta, \\ 0, & \text{otherwise.} \end{cases} \tag{3.28}$$
From (3.26), we have
$$\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y)), \quad \text{for all } x,y \in X.$$
Then the result follows from Theorem 2.3 with $\sigma = 1$. □

Corollary 3.25. *Let $T : X \to X$ be a given mapping. Suppose that*

(i) *there exists $\psi \in \Psi$ such that* (3.26) *holds;*
(ii) *Condition* (3.27) *holds.*

Then $\{T^n x_0\}$ converges to some $x^* \in X$. Moreover, if

(iii) *there exist a subsequence $\{T^{\gamma(n)}x_0\}$ of $\{T^n x_0\}$ and $N \in \mathbb{N}$ such that*
$$(T^{\gamma(n)}x_0, x^*) \in \Delta,\ n \geq N,$$

then x^* is a fixed point of T.

Proof. We continue to use the same function α defined by (3.28). From the first part of Theorem 2.4, the sequence $\{T^n x_0\}$ converges to some $x^* \in X$. From (iii) and (3.28), we have
$$\lim_{n\to\infty} \alpha(T^{\gamma(n)}x_0, x^*) = 1.$$
By the second part of Theorem 2.4 (with $\ell = 1$), we deduce that x^* is a fixed point of T. □

Corollary 3.26. *Let $T : X \to X$ be a given mapping. Suppose that*

(i) *there exists $\psi \in \Psi$ such that* (3.26) *holds;*
(ii) $\mathrm{Fix}(T) \neq \emptyset$*;*
(iii) *for every pair $(x,y) \in \mathrm{Fix}(T) \times \mathrm{Fix}(T)$ with $x \neq y$, if $(x,y) \notin \Delta$, there exist a positive integer q and a finite sequence $\{\zeta_i(x,y)\}_{i=0}^{q} \subset X$ such that*
$$\zeta_0(x,y) = x, \quad \zeta_q(x,y) = y, \quad (T^n\zeta_i(x,y), T^n\zeta_{i+1}(x,y)) \in \Delta,$$
for $n \in \mathbb{N}$ and $i = 0,\dots,q-1$.

Then T has a unique fixed point.

The above corollary follows from Theorem 2.5 with $\eta = 1$. Observe that in our results we do not suppose that T is monotone or T preserves order as it is supposed in many papers (see [13, 20, 22] and others).

4. Existence results for a class of nonlinear quadratic integral equations

Quadratic integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. The quadratic integral equations can be very often encountered in many applications (see [2, 3, 10]).

Here, we are concerned with the nonlinear quadratic integral equation

$$x(t) = a(t) + \lambda \int_0^t k_1(t,s) f_1(s, x(s))\, ds \int_0^t k_2(t,s) f_2(s, x(s))\, ds, \quad t \in [0,T],\ T > 0. \tag{4.1}$$

Let $X = C([0,T];\mathbb{R}^N)$ be the set of continuous functions from $[0,T]$ to $\mathbb{R}^N$. We endow X with the metric

$$d(x,y) = \max\big\{|x(t) - y(t)| : t \in [0,T]\big\}, \quad (x,y) \in X \times X.$$

It is well known that (X,d) is a complete metric space. We consider the infinity norm on X defined by

$$\|x\|_\infty = \max\big\{|x(t)| : t \in [0,T]\big\}, \quad x \in X.$$

We endow $\mathbb{R}^N$ with the partial order

$$u = (u_1, u_2, \dots, u_N) \le_{\mathbb{R}^N} v = (v_1, v_2, \dots, v_N) \Longleftrightarrow u_i \le v_i, \quad i = 1, 2, \dots, N.$$

We consider now the following assumptions:

(i) $a : [0,T] \to \mathbb{R}^N$ is continuous;

(ii) $f_i : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ are continuous;

(iii) for almost all $t \in [0,T]$, we have

$$|f_i(t,u) - f_i(t,v)| \le L\,|u - v|, \quad u \le_{\mathbb{R}^N} v,$$

where $L > 0$ is a constant;

(iv) there exists two functions $m_i : [0,T] \to \mathbb{R}$ such that $m_i \in L^1[0,T]$ and

$$|f_i(t,u)| \le m_i(t),\ t \in [0,T],\ u \in \mathbb{R}^N;$$

(v) for all $t \in [0,T]$, we have

$$u, v \in \mathbb{R}^N,\ u \le_{\mathbb{R}^N} v \Longrightarrow f(t,u) \le_{\mathbb{R}^N} f(t,v);$$

(vi) $k_i : [0,T] \times [0,T] \to [0,\infty)$ are continuous, $K_i = \max\{k_i(t,s) : (t,s) \in [0,T] \times [0,T]\}$;

(vii) there exists a constant $K > 0$ such that

$$\int_0^t k_i(t,s) m_i(s)\, ds \le K, \quad t \in [0,T];$$

(viii) there exists $x_0 \in X$ such that

$$x_0(t) \le_{\mathbb{R}^N} a(t) + \lambda \int_0^t k_1(t,s) f_1(s, x_0(s))\, ds \int_0^t k_2(t,s) f_2(s, x_0(s))\, ds, \quad t \in [0,T].$$

For the existence of a unique continuous solution to the quadratic integral equation (4.1) we have the following theorem.

Theorem 4.1. *Suppose conditions* (i)–(viii) are satisfied. If $0 < \lambda < (LKT(K_1 + K_2))^{-1}$, then the quadratic integral equation (4.1) has a unique continuous solution $x^* \in C([0,T];\mathbb{R}^N)$.

Proof. We introduce the mapping T associated with (4.1), defined by

$$Tx(t) = a(t) + \lambda \int_0^t k_1(t,s) f_1(s,x(s))\,ds \int_0^t k_2(t,s) f_2(s,x(s))\,ds,\ x \in X,\ t \in [0,T].$$

We consider several steps for the proof.

Step 1. The operator T maps X into itself. Let $x \in X$, let $t_1, t_2 \in [0,T]$ such that $t_1 < t_2$. After simple manipulation, we obtain

$$\begin{aligned} &|Tx(t_2) - Tx(t_1)| \\ &\le |a(t_2) - a(t_1)| + \lambda K \Big(\int_0^{t_2} |k_2(t_2,s) - k_2(t_1,s)| m_2(s)\,ds + \int_{t_1}^{t_2} k_2(t_1,s) m_2(s)\,ds \Big) \\ &\quad \times \lambda K \Big(\int_0^{t_2} |k_1(t_2,s) - k_1(t_1,s)| m_1(s)\,ds + \int_{t_1}^{t_2} k_1(t_1,s) m_1(s)\,ds \Big). \end{aligned}$$

Using the dominated convergence theorem and the assumptions (i)-(viii), we obtain

$$\lim_{|t_2 - t_1| \to 0} |Tx(t_2) - Tx(t_1)| = 0,$$

which implies the continuity of Tx in $[0,T]$. This proves that $T : X \to X$.

Step 2. T is an α-ψ contraction. Let $\alpha : X \times X \to \mathbb{R}$ be the function defined by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x(t) \le_{\mathbb{R}^N} y(t),\ t \in [0,T], \\ 0, & \text{otherwise.} \end{cases}$$

Consider the function $\psi : [0,\infty) \to [0,\infty)$ defined by

$$\psi(t) = \lambda K L T (K_1 + K_2)\, t,\ t \ge 0.$$

It is easy to show that $\psi \in \Psi$. We shall prove that T is an α-ψ contraction; that is,

$$\alpha(x,y) d(Tx,Ty) \le \psi(d(x,y)), \quad x, y \in X.$$

Let $x, y \in X$. If the condition $x(t) \le_{\mathbb{R}^N} y(t)$ is not satisfied, then the above inequality holds immediately. So we can suppose that $x(t) \le_{\mathbb{R}^N} y(t)$, for all $t \in [0,T]$. In this case, for all $t \in [0,T]$, we have

$$\begin{aligned} &|Tx(t) - Ty(t)| \\ &\le \lambda \int_0^t k_1(t,s) |f_1(s,x(s))|\,ds \int_0^t k_2(t,s) |f_2(s,x(s)) - f_2(s,y(s))|\,ds \\ &\quad + \lambda \int_0^t k_2(t,s) |f_2(s,y(s))|\,ds \int_0^t k_1(t,s) |f_1(s,x(s)) - f_1(s,y(s))|\,ds \\ &\le \lambda K L \Big(\int_0^t k_2(t,s) |x(s) - y(s)|\,ds + \int_0^t k_1(t,s) |x(s) - y(s)|\,ds \Big) \\ &\le \lambda K L T (K_1 + K_2) d(x,y) = \psi(d(x,y)). \end{aligned}$$

Then T is an α-ψ contraction.

Step 3. $\alpha(T^n x_0, T^{n+1} x_0) = 1$, $n \in \mathbb{N}$. From (viii), we have $\alpha(x_0, Tx_0) = 1$. Then our claim holds for $n = 0$. On the other hand, from condition (v), we have

$$\alpha(x,y) = 1 \Longrightarrow \alpha(Tx,Ty) = 1, \quad (x,y) \in X \times X.$$

Then by induction, we obtain easily our claim.

Step 4. Convergence of the Picard sequence $\{T^n x_0\}$. Using Theorem 2.4, we obtain the existence of $x^* \in X$ such that the Picard sequence $\{T^n x_0\}$ converges to x^* with respect to the metric d. Then from the previous step, we obtain immediately

$$\alpha(T^n x_0, x^*) = 1, \quad n \in \mathbb{N}.$$

Step 5. Existence of a solution. Now, we can apply Theorem 2.4 to deduce that x^* is a fixed point of T, that is, $x^* \in X$ is a solution to the integral equation (4.1).

Step 6. Uniqueness of the solution. Let us consider an arbitrary pair $(x, y) \in X \times X$ given by

$$x(t) = (x_1(t), x_2(t), \dots, x_N(t)), \quad y(t) = (y_1(t), y_2(t), \dots, y_N(t)), \quad t \in [0, T].$$

For every $i = 1, 2, \dots, N$, let

$$z_i(t) = \max\{x_i(t), y_i(t)\}, \ t \in [0, T].$$

Clearly we have $\alpha(x, z) = \alpha(y, z) = 1$. The uniqueness follows immediately from Theorem 2.5. □

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