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PERIODIC SOLUTIONS FOR SECOND-ORDER DIFFERENTIAL INCLUSIONS WITH NONSMOOTH POTENTIALS UNDER WEAK AR-CONDITIONS

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Abstract. In this article, we study a periodic second-order differential inclusions with locally Lipschitz potentials. By means of the least action principle and the minimax principle of nonsmooth type, we prove the existence of two nontrivial periodic solutions under the weak AR-conditions. The method developed in this paper can be applied for studying second-order differential inclusions of periodic type, and for elliptic equations with Neumann boundary condition.

1. Introduction and preliminaries

The AR-condition, which was introduced by Abrosstti-Rabinowitz (see Rabinowitz [15]), is used to investigate the PS-condition (or C-condition) for an energy functional, which probably has critical points (refer to Papageorgiou-Papageorgiou [13], Hu-Papageorgiou [6], and Nikolaos [12]). A nonsmooth function $f : T\times\mathbb{R}^N \to \mathbb{R}$ of Caratheodory type, which growth is constrained by the $r-$th power of $|x|$ ($r > 1$), is said to be satisfying the AR-condition, if

(H0) there exist $\mu > p > 1$ and $M_0 > 0$ such that for a.e. $t \in T$ and all $x \in \mathbb{R}^N$ with $|x| \geq M_0$, the inequality $0 < \mu f(t,x) \leq (u,x)$, or equivalently $\mu f(t,x) \leq -f^0(t,x,-x)$ (see [13]) holds for all $u \in \partial f(t,x)$.

It is well known that (cf. [7], [15, p. 9], or [9, p. 93]), any function $f(t,x)$ satisfying the AR-condition is p-superlinear definitely; i.e., there are $\beta_i > 0$, $i = 1,2$, such that $f(t,x) \geq \beta_1|x|^\mu - \beta_2$ for almost all $t \in T$ and all $x \in \mathbb{R}^N$.

If $r = p$, then the AR-condition can be weakened as (please refer to Kourogenis-Papageorgiou [7])

(H1)
$$\liminf_{|x|\to\infty} \frac{(u,x) - pf(t,x)}{|x|^\alpha} > 0, \quad \alpha \in (0,p)$$
uniformly for all $u \in \partial f(t,x)$, and a.e. $t \in T$,

or even as

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(H2)
$$\liminf_{|x|\to\infty}((u,x)-pf(t,x))=+\infty,$$
uniformly for all $u\in\partial f(t,x)$, and a.e. $t\in T$ (cf Li-Zhou [8]).

In this article, we study the periodic problem
$$\begin{gathered}-(J_p(x'(t)))'+a(t)J_p(x(t))\in\partial f(t,x(t))\quad\text{a.e. on } T=[0,b],\\ x(0)=x(b),\quad x'(0)=x'(b),\quad 1<p<+\infty\end{gathered}\tag{1.1}$$
where J_p denotes the p-Laplacian defined by
$$J_p(x)=\begin{cases}|x|^{p-2}x, & \text{if } x\neq 0,\\ 0, & \text{if } x=0,\end{cases}$$
and the nonsmooth potential $f(t,x)$ satisfying the AR-like condition

(H3) there is a $d_*>0$, such that for a.e. $t\in T$, inequality
$$(u,x)-pf(t,x)\geq d_*\tag{1.2}$$
for all $u\in\partial f(t,x)$ with $|x|\geq M_0$.

Since condition (H3) is weaker than all those mentioned above, it is not used here to verify the C-condition for the energy functional anymore. Instead, an extra condition (see $H(f)_1$ (vii)) which describe the asymptotic property of the potentials is taken into account. In this situation, the weak AR-condition is used only to investigate the asymptotic property of the energy functional in the infinite, and the former role of it to verify the nonsmooth C-condition has been replaced by the additional one.

For the convenience of the reader, we firstly overview briefly the theory of nonsmooth analysis, which is established by Clarke [1].

Let X be a Banach space, on which, we define a real function f, which is said to be locally Lipschitz at x (or equivalently, Lipschitz near x) with the rank L, if there is a neighborhood U of x, such that
$$|f(x_1)-f(x_2)|\leq L\|x_1-x_2\|$$
for all $x_i\in U$, $i=1,2$.

Attached to any locally Lipschitz function f, we can define the Clarke derivative at x along the direction $v\in X$:
$$f^\circ(x,v)=\limsup_{y\to x,\ \lambda\downarrow 0}\frac{f(y+\lambda v)-f(y)}{\lambda}.$$

Let v run over the space X, we get a function $f^\circ(x,\cdot)$, which is subadditive, positively homogeneous and uniformly Lipschitz with the rank L. So there exists a w^*-compact and convex subset $\partial f(x)$ of X^*, for which, $f^\circ(x,\cdot)$ is the corresponding support function; i.e.,
$$\partial f(x)=\{\xi\in X^*:\langle\xi,v\rangle\leq f^\circ(x,v),\ \forall v\in X\},$$
or conversely,
$$f^\circ(x,v)=\sup\{\langle\xi,v\rangle:\xi\in\partial f(x)\}$$
with the supermum being reached for each $v\in X$. In this definition, $\partial f(x)$ is called the Clarke subdifferential of f at the point x.

Here we list two properties of $\partial f(x)$:

(1) $\partial(\lambda f)(x)=\lambda\partial f(x)$, for all $\lambda\in\mathbb{R}$, $x\in X$,

(2) if g is another function locally Lipschitz on X, then for all $x \in X$, the following inclusion holds
$$\partial(f+g)(x) \subseteq \partial f(x) + \partial g(x),$$
and equality holds if one the functions f, g is strictly differentiable.

For other properties, please refer to [1].

In what follows, we need the nonsmooth C-condition (introduced by Cerami originally) for a locally Lipschitz function f; that is, any sequence $\{x_n\} \in X$ with

(1) $\{\varphi(x_n)\}$ bounded, and
(2) $(1 + \|x_n\|)m(x_n) \to 0$, as $n \to \infty$

contains a strongly convergent subsequence.

In this definition, the notation $m(x)$ denotes $\inf_{\xi \in \partial\varphi(x)} \|\xi\|_*$, which can be attained by some $\xi \in \partial\varphi(x)$ because $\partial\varphi(x)$ is w^*−compact and the norm $\|\cdot\|_*$ is w^*−lsc (cf. [7]).

This article is organized as follows. In Sections 2,3, we pay attention to the periodic differential system of scalar type, where the potential function f satisfying the AR-like condition (H1) or (H3). By truncating the potential function, investigating the asymptotic properties of the corresponding energy functional in the infinite area, and using the critical point theory of nonsmooth type developed by Chang [2] and Kourogenis-Papageorgiou [7], we find two distinct positive solutions for this problem. And in Section 4, we turn to deal with the periodic differential inclusion in $\mathbb{R}^N$ $(N \geq 1)$ under the weaker AR-conditions, and deduce the multiple existence of solutions. Our work space in this paper is
$$W^{1,p}_{\mathrm{per}}(T, \mathbb{R}^N) = \{x \in W^{1,p}(T, \mathbb{R}^N) : x(0) = x(b)\}.$$
For other treatment of positive solutions for the periodic differential systems, please refer to [14] and [3].

2. Properties of the energy functional

Firstly, we give some hypotheses on f and a, which will be used in the following paragraphs.

(HF) $f : T \times \mathbb{R} \to \mathbb{R}$ is a real function satisfying $f(\cdot, 0) \in L^\infty(T, \mathbb{R}_+)$, and $0 \in \partial f(t, 0)$ a.e. on T, and
  (i) for each $x \in \mathbb{R}$, the function $t \to f(t,x)$ is measurable,
  (ii) as the function of x, $f(t,x)$ is locally Lipschitz for a.e. $t \in T$, and
  (iii) for almost all $t \in T$, and all $x \in \mathbb{R}$,
$$|u| \leq a_0(t) + d_0|x|^{p-1} \tag{2.1}$$
  for all $u \in \partial f(t,x)$, where $a_0 \in L^\infty(T, \mathbb{R}_+)$, and $d_0 > 0$,
  (iv) there are constants $M_0, d > 0$ (without loss of generality, we may assume $M_0 \geq 1$ here) and $0 < \alpha < p$, such that
$$ux - pf(t,x) \geq dx^\alpha \tag{2.2}$$
  for a.e. $t \in T$, and all $u \in \partial f(t,x)$ with $x \geq M_0$, or weakly
  (iv') for a.e. $t \in T$,
$$ux - pf(t,x) \geq d \tag{2.3}$$
  for all $u \in \partial f(t,x)$ with $x \geq M_0$,
  (v) $\limsup_{x \to 0^+} x^{-p} f(t,x) \leq 0$, uniformly for a.e $t \in T$,

(vi) there exists $x_0 \geq M_0$ satisfying

$$\frac{7}{6}\int_0^b f(t,x_0)dt > \frac{1}{p}\|a\|_1 x_0^p, \tag{2.4}$$

(vii)

$$x_0^p \sup_{x\geq x_0} \frac{f(t,x)}{x^p} \leq \frac{3}{4}\Big(\frac{d}{p+2} + \frac{1}{b}\int_0^b f(t,x_0)dt\Big) \tag{2.5}$$

uniformly for a.e. $t \in T$.

(HA) $a \in L^1(T,\mathbb{R}_+)$ and $\int_0^b a(t)dt > d_0$.

Remark 2.1. Combining (iii) with the mean value theorem for locally Lipschitz function (please refer to [1, Theorem 2.3.7]), we can deduce the following estimate for $f(t,x)$:

$$|f(t,x)| \leq a_0(t)|x| + \frac{d_0}{p}|x|^p, \tag{2.6}$$

and then for each $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$, such that

$$|f(t,x)| \leq C_\varepsilon + \frac{d_0+\varepsilon}{p}|x|^p, \tag{2.7}$$

for a.e. $t \in T$, and all $x \in \mathbb{R}$.

Remark 2.2. Some properties of f can be derived from (HF) (v) and (vii). The first one is $f(t,0) \leq 0$, and hence $f(t,0) = 0$ a.e. on T, since $f(t,0)$ is nonnegative. And for every $\varepsilon > 0$ and $\mu > p$, there is a $C_{\varepsilon,\mu} > 0$ such that

$$f(t,x) \leq \varepsilon x^\mu + C_{\varepsilon,\mu} \tag{2.8}$$

for a.e. $t \in T$ and all $x \in \mathbb{R}$.

Remark 2.3. Due to hypothesis (HA), we can deduce by contradiction that, for each nonnegative number ε smaller than $\int_0^b a(t)dt$, there is $c_\varepsilon > 0$ (depends on ε) for which, the following inequality

$$\frac{1}{p}\|x'\|_p^p + \frac{1}{p}\int_0^b (a(t)-\varepsilon)|x(t)|^p dt \geq c_\varepsilon \|x\|_{1,p}^p, \tag{2.9}$$

holds on $W^{1,p}_{\rm per}(T)$.

Let us define the truncated function f_1 of f as follows:

$$f_1(t,x) = \begin{cases} f(t,x), & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases} \tag{2.10}$$

Evidently, $f_1(\cdot,x)$ is measurable for each $x \in \mathbb{R}$, and $f_1(t,\cdot)$ is locally Lipschitz for a.e $t \in T$. In view of the chain rules (cf [1, Theorem 2.3.9]), we have

$$\partial f_1(t,x) = \begin{cases} \partial f(t,x), & \text{if } x > 0, \\ co([0,1]\cdot \partial f(t,x)), & \text{if } x = 0, \\ 0, & \text{if } x < 0 \end{cases} \tag{2.11}$$

a.e.on T, where $[0,1]\cdot\partial f(t,x) = \{\lambda y : \lambda \in [0,1], y \in \partial f(t,0)\}$. Thus f_1 has all the properties as f has under hypotheses (HF).

If we define $\phi(x) = \int_0^b f_1(t,x(t))dt$, $x \in L^p(T)$, then we get a locally Lipschitz functional with its subdifferential $\partial\phi(x) \subseteq \int_0^b \partial f_1(t,x(t))dt$, which means for every $\xi \in \partial\phi(x)$, there exists a $L^q(T)$ selection u of $\partial f_1(\cdot,x(\cdot))$ ($\frac{1}{p}+\frac{1}{q}=1$), such that

$$(\xi, y)_{p,q} = \int_0^b u(t)y(t)dt \tag{2.12}$$

for all $y \in L^p(T)$ (see [1, Theorem 2.7.5]).

Since $W^{1,p}_{\rm per}(T)$ is a dense subspace of $L^p(T)$ with the inclusion mapping continuous, if we set $\phi_0 = \phi|_{W^{1,p}_{\rm per}(T)}$, then we obtain another locally Lipschitz functional ϕ_0 defined on $W^{1,p}_{\rm per}(T)$ with $\partial\phi_0(x) = \partial\phi(x) = S^q_{\partial f_1(\cdot,x(\cdot))}$ holding for all $x \in W^{1,p}_{\rm per}(T)$ (see the corollary of [1, Theorem 2.3.10]).

Now, we can consider the energy functional on $W^{1,p}_{\rm per}(T)$ associated with the periodic system (1.1):

$$\Phi(x) = \frac{1}{p}\int_0^b |x'(t)|^p dt + \frac{1}{p}\int_0^b a(t)|x(t)|^p dt - \int_0^b f_1(t,x(t))dt. \tag{2.13}$$

Based on the discussion in [17] and the finite sum rule (see Section 1), we know that Φ is locally Lipschitz and its Clarke subdifferential can be represented by

$$\partial\Phi(x) = \mathcal{J}_p x + a(\cdot)J_p(x(\cdot)) - \partial\phi_0(x), \tag{2.14}$$

where $\mathcal{J}_p : W^{1,p}_{\rm per}(T) \to (W^{1,p}_{\rm per}(T))^*$ is a demicontinuous and $(S)_+$ operator defined by

$$\langle \mathcal{J}_p x, y\rangle = \int_0^b J_p(x'(t))y'(t)dt, \quad \forall\, x,y \in W^{1,p}_{\rm per}(T).$$

Lemma 2.4. *Under hypotheses* (HF) (i)–(iv) *and* (HA)*, the energy functional* Φ *satisfies the nonsmooth C-condition.*

Proof. Suppose $\{\Phi(x_n)\}$ is bounded and $(1+\|x_n\|_{1,p})m(x_n) \to 0$ as $n \to \infty$ for some sequence $\{x_n\}$ of $W^{1,p}_{\rm per}(T)$, we will show that $\{x_n\}$ has a convergent subsequence. For this purpose, we need to introduce two truncated functions for any function x.

$$x^+ = \max\{0,x\}, \quad x^- = \max\{0,-x\}. \tag{2.15}$$

As we know that, both x^+ and x^- are all in $W^{1,p}_{\rm per}(T)$ whenever $x \in W^{1,p}_{\rm per}(T)$ (please refer to Marcus-Mizel [10] or Evans-Gariepy [5], p.130). Moreover, for the derivatives of x^+ and x^-, we have

$$(x^+)'(t) = \begin{cases} x'(t), & \text{if } x > 0,\\ 0, & \text{if } x \le 0,\end{cases}$$

$$(x^-)'(t) = \begin{cases} 0, & \text{if } x \ge 0,\\ -x'(t), & \text{if } x < 0.\end{cases}$$

Consequently, the following relations between x and x^+, x^- hold:

$$\max\{\|x^+\|_{1,p}, \|x^-\|_{1,p}\} \le \|x\|_{1,p}, \quad \|x\|_{1,p} \le \|x^+\|_{1,p} + \|x^-\|_{1,p}, \tag{2.16}$$

$$\int_0^b J_p(x'(t))(x^+)'(t)dt = \int_{\{x>0\}} |x'(t)|^p dt = \int_0^b |(x^+)'(t)|^p dt, \tag{2.17}$$

$$\int_0^b J_p(x'(t))(x^-)'(t)dt = -\int_{\{x<0\}} |x'(t)|^p dt = -\int_0^b |(x^-)'(t)|^p dt. \tag{2.18}$$

Since the subdifferential $\partial\Phi(x)$ is a w^*-compact subset of $(W^{1,p}_{\rm per}(T))^*$, its minimal norm can be reached by some elements in it. Hence for each $n \in \mathbb{N}^*$, there is $\xi_n \in \partial\Phi(x_n)$, such that

$$\|\xi_n\|_* = m(x_n) \to 0, \tag{2.19}$$

$$\max\{|\langle \xi_n, x_n\rangle|, |\langle \xi_n, x_n^-\rangle|\} \le \|x_n\|_{1,p} m(x_n) \to 0 \tag{2.20}$$

as $n \to \infty$. Therefore we can find constant $M > 0$, sequences of positive numbers $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$, and functions $\{u_n\}$ with $u_n \in S^q_{\partial f_1(\cdot, x_n(\cdot))}$, satisfying

$$-Mp \le \int_0^b |x_n'(t)|^p dt + \int_0^b a(t)|x_n(t)|^p dt - \int_0^b pf_1(t, x_n(t))dt \le Mp, \tag{2.21}$$

$$-\varepsilon_n \le -\int_0^b |x_n'(t)|^p dt - \int_0^b a(t)|x_n(t)|^p dt + \int_0^b u_n(t)x_n(t)dt \le \varepsilon_n, \tag{2.22}$$

$$-\varepsilon_n \le \int_0^b |(x_n^-)'(t)|^p dt + \int_0^b a(t)|x_n^-(t)|^p dt \le \varepsilon_n. \tag{2.23}$$

The above inequality tells us that $\{x_n^-\}$ is bounded in $W^{1,p}_{\rm per}(T)$ by some constant $M_1 > 0$, and inequalities (2.19), (2.20) jointly give

$$-Mp - \varepsilon_n \le \int_0^b (u_n(t)x_n(t) - pf_1(t, x_n(t)))dt \le Mp + \varepsilon_n. \tag{2.24}$$

For the intermediate terms in (2.22), we have

$$\begin{aligned}
&\int_0^b (u_n(t)x_n(t) - pf_1(t, x_n(t)))dt \\
&= \int_{\{x_n > 0\}} (u_n(t)x_n(t) - pf(t, x_n(t)))dt \\
&\le \Big(\int_{\{0 < x_n < M_0\}} + \int_{\{x_n \ge M_0\}}\Big)(u_n(t)x_n(t) - pf(t, x_n(t)))dt,
\end{aligned} \tag{2.25}$$

and

$$\int_{\{0 < x_n < M_0\}} (u_n(t)x_n(t) - pf(t, x_n(t)))dt \le C \tag{2.26}$$

for some $C > 0$. Thus, by the AR-like condition (HF) (iv), we have

$$d\int_{\{x_n \ge M_0\}} x_n^\alpha(t)dt \le Mp + \varepsilon_n + C, \tag{2.27}$$

which yields the boundedness of $\{x_n^+\}$ in $L^\alpha(T)$.

Next, select $p < \mu < p + \alpha$ in (2.4), then there exists $0 < \theta < 1$, such that

$$\frac{1}{\mu} = \frac{1-\theta}{\alpha} + \frac{\theta}{p+\alpha}. \tag{2.28}$$

Thus, using the interpolation inequality, we have

$$\|x_n^+\|_\mu \le \|x_n^+\|_\alpha^{1-\theta} \|x_n^+\|_{p+\alpha}^\theta \le C_1 \|x_n^+\|_{p+\alpha}^\theta. \tag{2.29}$$

This inequality, together with the boundedness of $\Phi\{x_n\}$ and $\{x_n^-\}$ produce the following estimates

$$\begin{aligned}
&\|(x_n^+)'\|_p^p + \int_0^b a(t)|x_n^+(t)|^p dt \\
&\le Mp + C_2 M_1^p + \int_{\{x_n>0\}} (a_1(t) + c_1|x_n'(t)|^\mu)dt \\
&\le Mp + C_2 M_1^p + b\|a_1\|_\infty + c_1\|x_n^+\|_\mu^\mu \\
&\le C_3 + C_4\|x_n^+\|_{p+\alpha}^{\theta\mu} \\
&\le C_3 + C_5\|x_n^+\|_\infty^{\theta\mu} \le C_3 + C_6\|x_n^+\|_{1,p}^{\theta\mu}.
\end{aligned} \tag{2.30}$$

Finally, using (2.7) with $\varepsilon = 0$, we have

$$\|x_n^+\|_{1,p}^p \le C_7 + C_8\|x_n^+\|_{1,p}^{\theta\mu}. \tag{2.31}$$

All the constants $C_i > 0$, $i = 1, 2, \cdots, 8$ in estimates (2.27), (2.28) and (2.29) are independent of n.

It is easily to show that $\theta\mu < p$ (please see [7] for a reference), therefore the consequence $\{x_n^+\}$, and hence $\{x_n\}$ is bounded in $W^{1,p}_{\rm per}(T)$.

On account of the compact embedding of $W^{1,p}_{\rm per}(T)$ into $C(T)$, we deduce that $\{x_n\}$ has a subsequence, which is still denoted by $\{x_n\}$, converging to x weakly in $W^{1,p}_{\rm per}(T)$, and strongly in $C(T)$. Consequently,

$$\int_0^b a(t)(J_p(x_n(t)), x_n(t) - x(t))dt \to 0,$$

$$\int_0^b u_n(t)(x_n(t) - x(t))dt \to 0$$

as $n \to \infty$.

Using $x_n - x$ as a test function in (2.18), we have

$$\limsup_{n\to\infty}\langle \mathcal{J}_p x_n, x_n - x\rangle \le 0, \tag{2.32}$$

which yields $x_n \to x$ in $W^{1,p}_{\rm per}(T)$, since $\mathcal{J}_p$ is of class $(S)_+$. Thus the proof is complete. □

If we replaced (HF) (iv) by (vii), then we can verify the nonsmooth C-condition of Φ more briefly, i.e.

Lemma 2.5. *Under Hypotheses* (HF) (i)–(iii), (v), (vii) *together with* (HA)*, the energy functional* Φ *also satisfies the nonsmooth C-condition.*

Proof of this lemma is omitted here, and a similar proof can be find in Section 4.

3. Existence of positive solutions in the scalar case

To derive the existence results for positive solutions of periodic system (1.1), we need the following three lemmas.

Lemma 3.1. *If hypotheses* (HF) (i)–(iii), (v), (HA) *are satisfied, then there exist* $\rho, \beta > 0$*, such that*

$$\inf_{\|x\|_{1,p}=\rho} \Phi(x) \ge \beta. \tag{3.1}$$

Proof. Based on the inequality (2.6), we know that

$$\Phi(x) \geq \frac{1}{p}\|x'\|_p^p + \frac{1}{p}\int_0^b a(t)|x(t)|^p dt - \int_0^b [\frac{\varepsilon}{p}|x(t)|^p + C_{\varepsilon,\mu}|x(t)|^\mu]dt, \tag{3.2}$$

where ε can be selected smaller than $\int_0^b a(t)dt$ so that we can invoking (2.7) to derive the lower estimates for functional Φ; that is,

$$\begin{aligned} \Phi(x) &\geq c_\varepsilon\|x\|_{1,p}^p - C_{\varepsilon,\mu}\|x\|_\mu^\mu \\ &\geq c_\varepsilon\|x\|_{1,p}^p - c_2\|x\|_0^\mu \\ &\geq c_\varepsilon\|x\|_{1,p}^p - c_3\|x\|_{1,p}^\mu, \end{aligned} \tag{3.3}$$

for some constants $c_i > 0$, $i = 2, 3$ independent of $x \in W_{\rm per}^{1,p}(T)$. Consider that $\mu > p$, so if $\rho > 0$ is taken small enough, we then have $\inf_{\|x\|_{1,p}=\rho} \Phi \geq \beta$ for some $\beta > 0$. □

The proof of the next lemma is much similar to that of Lemma 4.5, and here it is omitted.

Lemma 3.2. *If* (HF)$_1$ (i)–(iii), (iv'), (vi), (HA) *hold, then for the fixed number* $x_0 > 0$ *taken in (vi), there is an* $r > 1$ *making the value of* Φ *at* rx_0 *negative.*

Remark 3.3. Lemma 3.2 remains true if (HF) (iv') is replaced by (iv), since the latter is weaker than the former.

Lemma 3.4. *Under conditions* (HF) (i)–(iii), (HA)*, every critical point* x *of* Φ*, which satisfies* $0 \in \partial\Phi(x)$*, lies in the set*

$$C_{\rm per}^1(T)_+ = \{x \in C^1(T) : x(0) = x(b),\ x'(0) = x'(b),\ \text{and } x(t) \geq 0,\ \forall t \in T\}$$

and solves problem (1.1).

Proof. A simple computation shows that every critical point x of Φ lies in $C_{\rm per}^1(T)$ with the equality

$$\mathcal{J}_p x + a(\cdot)J_p(x(\cdot)) - u = 0 \tag{3.4}$$

holding for some $u \in S_{\partial f_1(\cdot,x(\cdot))}^q$ (cf [13] or [14]), which means

$$\langle \mathcal{J}_p x, y\rangle + \int_0^b a(t)J_p(x(t))y(t)dt - \int_0^b u(t)y(t)dt = 0 \tag{3.5}$$

for all $y \in W_{\rm per}^{1,p}(T)$.

Take $y = x^-$ as the test function in (3.5), we have

$$\int_0^b |(x^-)'(t)|^p dt + \int_0^b a(t)|x^-(t)|^p dt = 0, \tag{3.6}$$

hence $\|(x^-)'\|_p = 0$ and $\int_0^b a(t)|x^-(t)|^p dt = 0$, which produce $x^- = 0$ in $W_{\rm per}^{1,p}(T)$ since $\int_0^b a(t)dt > 0$. Therefore, $x(t) = x^+(t) \geq 0$ for all $t \in T$.

By reviewing (3.5) again, we can also find $u(t) = 0$ a.e. on $\{x = 0\}$, this fact combining with $0 \in \partial f(t, 0)$ a.e. on T, leads to the desired results $u \in S_{\partial f(\cdot,x(\cdot))}^q$ and then x is a solution of problem (1.1). □

Theorem 3.5. *Under hypotheses* (HF) (i)–(vi), (HA)*, the periodic problem* (1.1) *has a nontrivial and nonnegative solution in* $C_{\rm per}^1(T)_+$.

Proof. Lemmas 2.4, 3.1, 3.2 together with the "Mountain Pass Lemma" of nonsmooth type [7, 2] yield a critical point x of the energy functional Φ with $\Phi(x) = c$, where

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)),$$
$$\Gamma = \{\gamma \in C([0,1], W^{1,p}_{\rm per}(T)) : \gamma(0) = 0, \gamma(1) = rx_0\}.$$

Obviously, $c \geq \beta > 0$, which infers $x \neq 0$. Finally, invoking Lemma 3.4, we have $x \in C^1_{\rm per}(T)_+$ and solves problem (1.1). □

Theorem 3.6. *Putting hypotheses* (HF) (i)–(iii), (iv'), (vii), (HA) *together, the periodic problem* (1.1) *has a nontrivial and nonnegative solution which minimizes the energy functional on the whole place* $W^{1,p}_{\rm per}(T)$*.*

Proof. For the minimizer, we appeal to the lower estimates of the energy functional Φ in view of (2.5) and (2.7); i.e,

$$\begin{aligned}\Phi(x) &\geq p^{-1}\|x'\|_p^p + p^{-1}\int_0^b a(t)|x(t)|^p dt - \varepsilon p^{-1}\|x\|_p^p - C_{\varepsilon,\mu} b \\ &\geq c_\varepsilon \|x\|^p_{1,p} - C_{\varepsilon,\mu} b\end{aligned}$$

if $0 < \varepsilon < \int_0^b a(t)dt$. Considering that $W^{1,p}_{\rm per}(T)$ is reflexive, and Φ is weakly semicontinuous, we can use the least action principle of nonsmooth type, to find another critical point of Φ, which realizes the whole minimum of Φ on $W^{1,p}_{\rm per}(T)$, and solves problem (1.1) simultaneously. □

Theorem 3.7. *If hypotheses* (HF) (i)–(iii), (iv'), (v)-(vii), (HA) *are all satisfied, the periodic problem* (1.1) *has at least two nontrivial and nonnegative solutions in* $C^1_{\rm per}(T)_+$.

Proof. Using Theorems 3.5 and 3.6, we can find two solutions of (1.1) denoted by x_1 and x_2 respectively. By reviewing the proofs of the two theorems, we can also find $\Phi(x_1) = c$ and $\Phi(x_2) = \inf_{x\in W^{1,p}_{\rm per}(T)}$. On the other hand, Lemmata 3.1 and 3.2 tell us that, $\inf_{x\in W^{1,p}_{\rm per}(T)} \Phi(x) < 0 < c$, so the two solutions x_1 and x_2 are different and nontrivial, which completes the theorem. □

Remark 3.8. According to the strong maximum principle (cf. Vazquez [16]), if condition (2.1) in $H(f)(iii)$ is replaced by

$$|u| \leq \hat{c}_0|x|^{p-1} \quad (\hat{c}_0 > 0) \quad \forall u \in \partial f(t,x), \tag{3.7}$$

then every solution of problem (1.1) can be positive everywhere (see also [6]).

Remark 3.9. If $a(t) \equiv 0$, and H(f)(vii) is replaced by

(vii') there is a function $\theta \in L^\infty(T)$ with $\theta(t) \leq 0$ and $\int_0^b \theta(t)dt < 0$, satisfying

$$\limsup_{x\to+\infty} \frac{pf(t,x)}{x^p} \leq \theta(t) \tag{3.8}$$

uniformly for a.e. $t \in T$,

then the same conclusions as in Theorem 3.6 and 3.7 can be reached.

4. Existence results in the common case

In this section we assume that $N \geq 1$, and the function $f : T \times \mathbb{R}^N \to \mathbb{R}$ satisfies

(Hf1) (i) for each $x \in \mathbb{R}^N$, $f(\cdot, x)$ is measurable, and $f(\cdot, 0) \in L^\infty(T, \mathbb{R}_+)$,

(ii) for a.e. $t \in T$, $f(t, \cdot)$ is locally Lipschitz, and $0 \in \partial f(t, 0)$,

(iii) for almost all $t \in T$, all $x \in \mathbb{R}^N$ and all $u \in \partial f(t, x)$, the following inequality
$$|u| \leq a_0(t) + d_0|x|^{p-1} \tag{4.1}$$
holds, where $a_0 \in L^\infty(T, \mathbb{R}_+)$, and $d_0 > 0$,

(iv) there are constants $M_0 \geq 1$, $d > 0$, such that
$$(u, x) - pf(t, x) \geq d|x|^\alpha \tag{4.2}$$
for a.e. $t \in T$, and all $u \in \partial f(t, x)$ with $|x| \geq M_0$, or weakly

(iv') for a.e. $t \in T$,
$$(u, x) - pf(t, x) \geq d \tag{4.3}$$
for all $u \in \partial f(t, x)$ with $|x| \geq M_0$,

(v) $\limsup_{|x|\to 0} |x|^{-p} f(t, x) \leq 0$, uniformly for a.e $t \in T$, and

(vi) there exists $x_0 \in \mathbb{R}^N$ with $|x_0| > M_0$ satisfying
$$\frac{7}{6} \int_0^b f(t, x_0)dt > \frac{1}{p}\|a\|_1|x_0|^p, \tag{4.4}$$

(vii)
$$|x_0|^p \sup_{|x|\geq|x|_0} \frac{f(t, x)}{|x|^p} \leq \frac{3}{4}\Big(\frac{d}{p+2} + \frac{1}{b}\int_0^b f(t, x_0)dt\Big) \tag{4.5}$$
uniformly for a.e. $t \in T$.

Remark 4.1. Similar to Section 2, we can derive some inequalities from (Hf1) (iii) (v) and (vii): There exists $a_1 \in L^\infty(T, \mathbb{R}_+)$, and $c_1 > 0$, such that
$$|f(t, x)| \leq a_1(t) + c_1|x|^\mu \tag{4.6}$$
for a.e. $t \in T$, and all $x \in \mathbb{R}^N$, and
$$f(t, x) \leq \frac{\varepsilon}{p}|x|^p + C_{\varepsilon,\mu}, \tag{4.7}$$
$$f(t, x) \leq \frac{\varepsilon}{p}|x|^p + C_{\varepsilon,\mu}|x|^\mu \tag{4.8}$$
for a.e. $t \in T$ and all $x \in \mathbb{R}^N$, where $\mu \geq p$ and $C_{\varepsilon,\mu} > 0$ depend on ε and μ.

Associated with the periodic system (1.1) with $N \geq 1$, the energy functional on $W^{1,p}_{\rm per}(T)$ is defined by
$$\Phi(x) = \frac{1}{p}\int_0^b |x'(t)|^p dt + \frac{1}{p}\int_0^b a(t)|x(t)|^p dt - \int_0^b f(t, x(t))dt. \tag{4.9}$$
Its Clarke subdifferential can be represented by
$$\partial\Phi(x) = \mathcal{J}_p x + a(\cdot)J_p(x(\cdot)) - \partial\phi_0(x), \tag{4.10}$$
where $\phi_0(x) = \int_0^b f(t, x(t))dt$ for all $x \in W^{1,p}_{\rm per}(T, \mathbb{R}^N)$. Following the steps as in Sections 2 and 3, we have the following results.

Lemma 4.2. *Under conditions* (Hf1) (i)–(iv), (HA)*, the energy functional* Φ *satisfies the nonsmooth* C*-condition.*

Lemma 4.3. *Under conditions* (Hf1) (i)–(iii), (v), (vii), (HA)*, the energy functional* Φ *satisfies the nonsmooth* C*-condition.*

Proof. Suppose $\{\Phi(x_n)\}$ is bounded and $(1+\|x_n\|_{1,p})m(x_n) \to 0$ as $n \to \infty$, we need only to show the boundedness of $\{x_n\}$ in $W^{1,p}_{\rm per}(T, \mathbb{R}^N)$. Similar to Lemma 2.4, there is a constant $M > 0$, and functions $u_n \in S^q_{\partial f(\cdot, x_n(\cdot))}$, $\xi_n = \mathcal{J}_p x_n + a(\cdot)J_p(x_n(\cdot)) - u_n$ with $\|\xi_n\|_* = m(x_n)$, such that

$$-M \le p^{-1} \int_0^b |x_n'(t)|^p dt + p^{-1} \int_0^b a(t)|x_n(t)|^p dt - \int_0^b f(t, x_n(t))dt \le M, \quad (4.11)$$

$$|\langle \xi_n, x_n\rangle| \le (1 + \|x_n\|_{1,p})m(x_n) \to 0 \quad (4.12)$$

Inequality (4.9) together with (4.5) and (2.7), lead to the estimate

$$\begin{aligned} c_0\|x_n\|^p_{1,p} &\le p^{-1}\|x_n'\|^p_p + p^{-1} \int_0^b a(t)|x_n(t)|^p dt \\ &\le M + \int_0^b f(t, x_n(t))dt \\ &\le M + C_{\varepsilon,\mu} b + \varepsilon p^{-1}\|x_n\|^p_p. \end{aligned}$$

Select $\varepsilon > 0$ smaller than $c_0 p$ in (4.11), we can deduce the boundedness of $\{x_n\}$ in $W^{1,p}_{\rm per}(T, \mathbb{R}^N)$. $\square$

Lemma 4.4. *There exist* $\rho, \beta > 0$*, such that*

$$\inf_{\|x\|_{1,p}=\rho} \Phi(x) \ge \beta. \quad (4.13)$$

provided (Hf1) (i)–(iii), (vi), (HA) *hold.*

Lemma 4.5. *If* (Hf1) (i)–(iii), (iv'), (vi), (HA) *hold, then for the fixed point* x_0 *taken in* (*vi*)*, there is an* $r \ge 1$ *making the value of* Φ *at* rx_0 *negative.*

Proof. Take any $\nu \ge p + 2$, and consider the function

$$\lambda(r) = r^{-\nu} f(t, rx_0), \quad r \ge 1. \quad (4.14)$$

It is easy to check that λ is locally Lipschitz on $[1, +\infty)$, and according to the chain rule of multiple functions, we have

$$\lambda'(r) = -\nu r^{-\nu-1} f(t, rx_0) + r^{-\nu}(u, x_0) \quad (4.15)$$

for some $u \in \partial f(t, rx_0)$ and a.e. $r > 1$, thus invoking (iv'), we obtain

$$\lambda'(r) \ge -(\nu - p)r^{-\nu-1} f(t, rx_0) + dr^{-\nu-1} = -(\nu - p)r^{-1}\lambda(r) + dr^{-\nu-1} \quad (4.16)$$

a.e. on $(1, +\infty)$. Notice that

$$r^{-1}\lambda(r) = r^{-2}r^{-\nu+1} f(t, rx_0) \le d_1 r^{-2}, \quad (4.17)$$

where

$$d_1 = \frac{3}{4}(\frac{d}{p+2} + \frac{1}{b}\int_0^b f(t, x_0)dt) \ge \sup_{r\ge1} r^{-\nu+1} f(t, rx_0)$$

is derived from (4.5). Thus we obtain

$$\lambda'(r) \ge -d_1(\nu - p)r^{-2} + dr^{-\nu-1}. \quad (4.18)$$

For each $r > 1$, integrating both sides of the inequality (4.18) on $[1, r]$, we have

$$\lambda(r) - \lambda(1) \ge -d_1(\nu - p)(1 - r^{-1}) + \frac{d}{\nu}(1 - r^{-\nu}) \quad (4.19)$$

which lead to the estimate

$$f(t, rx_0) \geq r^\nu f(t, x_0) - d_1(\nu - p)(r^\nu - r^{\nu-1}) + \frac{d}{\nu}(r^\nu - 1). \tag{4.20}$$

Consequently, for the energy functional Φ, we have

$$\begin{aligned}
\Phi(rx_0) &= \frac{|x_0|^p r^p}{p} \int_0^b a(t)dt - \int_0^b f(t, rx_0)dt \\
&\leq \frac{\|a\|_1 |x_0|^p}{p} r^p - r^\nu \int_0^b f(t, x_0)dt + bd_1(\nu - p)(r^\nu - r^{\nu-1}) - \frac{bd}{\nu}(r^\nu - 1) \\
&= (bd_1(\nu - p) - B - A) r^{\nu-2} [r^2 - \frac{bd_1(\nu - p)}{bd_1(\nu - p) - B - A} r \\
&\quad + \frac{K}{bd_1(\nu - p) - B - A} r^{-\nu+p+2} + \frac{B}{bd_1(\nu - p) - B - A}],
\end{aligned} \tag{4.21}$$

where $A = \int_0^b f(t, x_0)dt$, $B = bd/\nu$ and $K = p^{-1}\|a\|_1 |x_0|^p$.

If $K < A$, i.e. $p^{-1}\|a\|_1 |x_0|^p < \int_0^b f(t, x_0)dt$, then $\Phi(x_0) < 0$, and the desired result follows.

If $A \leq K < 7A/6$, then take $\nu = p + 2$, $r = 1 + s\delta$, $k = K/(A + B)$ and $\delta = (B + A)/(bd_1(\nu - p) - B - A)$, and the part in the square brackets in the last line of (4.21) can be represented by:

$$\begin{aligned}
g(s) &= (1 + s\delta)^2 - (1 + \delta)(1 + s\delta) + k\delta + \frac{B\delta}{A + B} \\
&= (k - (1 - s)(1 + s\delta) + \frac{B}{A + B})\delta, \quad s \in [0, 1].
\end{aligned}$$

Thanks to (4.17), we have $\delta = 2$, and $g(s) = 2(k - (1 - s)(1 + 2s) + B/(A + B))$ which reaches its minimum at $1/4$. On the other hand, from (4.5), we can deduce that $B \geq A/3$, thus, on account of (4.4), we finally have

$$\begin{aligned}
g(\frac{1}{4}) &= 2(k - \frac{1}{8} - \frac{A}{A + B}) \\
&= 2(K - A - \frac{A + B}{8})/(A + B) \\
&\leq 2(K - \frac{7}{6}A)/(A + B) < 0,
\end{aligned}$$

which leads to $\Phi((3/2)x_0) < 0$ and the proof has been completed. □

Remark 4.6. In this proof, the desired inequality $f(t, rx_0) \geq r^\nu f(t, x_0) - Cr^p$ for some $\nu > p$ and $C > 0$ could not be obtained, since it contradicts hypothesis (HF) (vii). In this sense, hypotheses (HF) (vii), (H0) and $\int_0^b f(t, x_0)dt > 0$ are not compatible.

Lemma 4.7. *Under conditions* (Hf1) (i)–(iii), (HA)*, every critical point x of Φ lies in the set*

$$C^1_{\rm per}(T, \mathbb{R}^N) = \{x \in C^1(T, \mathbb{R}^N) : x(0) = x(b),\ x'(0) = x'(b)\}$$

and solves problem (1.1)*.*

Combining all the results, we have the following statement.

Theorem 4.8. *If hypotheses* (Hf1) (i)–(iv), (HA) *are satisfied, then the periodic problem* (1.1) *has a nontrivial solution in* $C^1_{\rm per}(T,\mathbb{R}^N)$.

Remark 4.9. This theorem is a revised result of [13, Theorem 1].

Theorem 4.10. *Under hypotheses* (Hf1) (i)–(iii), (iv'), (vii), (HA)*, the energy functional has a minimizer in* $W^{1,p}_{\rm per}(T,\mathbb{R}^N)$ *which solves problem* (1.1).

Theorem 4.11. *Putting all the hypotheses* (Hf1) (i)–(iii), (iv'), (v)–(vii), (HA) *together, the periodic problem* (1.1) *has at least two nontrivial solutions in* $C^1_{\rm per}(T,\mathbb{R}^N)$.

Corollary 4.12. *If hypotheses* (Hf1) (i)–(iii), (v), (vii), (HA) *hold, and if in addition, we assume $A > K$ (see the proof of Lemma 4.5), then the periodic problem* (1.1) *has at least two nontrivial solutions in* $C^1_{\rm per}(T,\mathbb{R}^N)$.

Corollary 4.13. *If hypotheses* (Hf1) (i)–(iii), (iv'), (v), (vii), (HA) *hold, then there exists a $0 < \varepsilon_0 < 7A/6K$, such that for each $\varepsilon \in (0,\varepsilon_0)$, the periodic problem*

$$
\begin{gathered}
-(J_p(x'(t)))' + \varepsilon a(t)J_p(x(t)) \in \partial f(t,x(t)) \quad \text{a.e. on } T,\\
x(0) = x(b), \quad x'(0) = x'(b), \quad 1 < p < +\infty
\end{gathered}
$$

has at least two nontrivial solutions in $C^1_{\rm per}(T,\mathbb{R}^N)$.

There are some interesting topics in the study of differential inclusions with nonsmooth potentials by means of nonsmooth analysis and critical point theory, such as multiple solutions of elliptic inclusions with p-Laplacian, and infinite many solutions for the hemivariational inequalities etc. For the related discussions, please refer to [11] and [4] with references therein.

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