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EXISTENCE OF POSITIVE SOLUTIONS TO A SINGULAR BOUNDARY-VALUE PROBLEM USING VARIATIONAL METHODS

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Abstract. In this article, we study a class of nonlinear singular boundary-value problems. We show the existence of positive weak solutions by using variational methods.

1. Introduction

Variational methods are a powerful tool in the resolution of specific nonlinear boundary-value problems appearing in many areas. During the last decade, they are intensively applied to boundary-value problems for differential equations. They are motivated by the modeling of certain nonlinear problems from biological neural networks, elastic mechanics, to anisotropic problems, and so forth. Recently, many differential equations have been studied via variational methods in many classical works, see [11, 13, 14, 15, 16, 18, 19, 20]. The study of singular differential equation via variational methods was initiated by Agarwal, Perera and O'Regan [1, 2]. Since then there is a trend to study differential equation via variational methods which leads to many meaningful results, see [3, 8, 10] and the references therein.

Agarwal et al [1] studied the singular boundary-value problem via variational methods

$$\begin{gathered} -y''(t) = f(t,y), \quad t \in (0,1), \\ y(0) = y(1) = 0, \end{gathered} \tag{1.1}$$

where $f \in C((0,1) \times (0,\infty), [0,\infty))$ satisfies

$$2\varepsilon \le f(t,y) \le Cy^{-\gamma}, \quad (t,y) \in (0,1) \times (0,\varepsilon). \tag{1.2}$$

for some $\varepsilon, C > 0$ and $\gamma \in (0,1)$, the authors introduced a variational formulation for singular Dirichlet boundary-value problem.

Motivated by the above mentioned work, in this paper we consider the singular boundary-value problem

$$\begin{gathered} -u''(t) = f(t,u) + e(t), \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{gathered} \tag{1.3}$$

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where $f \in C((0,1) \times (0,\infty), [0,\infty))$, $e(t) \in L^1(0,1)$ satisfy

$$2\varepsilon - e(t) \le f(t,u) \le Cu^{-\gamma}, (t,u) \in (0,1) \times (0,\varepsilon). \tag{1.4}$$

for some $\varepsilon, C > 0$ and $\gamma \in (0,1)$.

Singular boundary-value problems have been discussed extensively by the method of upper and lower solutions, index theorems, fixed point theorems, nonlinear alternative principle, etc. See [5, 6, 7, 9, 12] and the references therein. In this paper, we consider the existence of weak solutions for (1.3) and obtain some new existence theorems of solutions by using variational methods. It is worth noting that there are some works concerning on the results of positive solutions for singular problems up to now. For example, Cid et al [8] obtained the existence of infinitely many solutions for a second-order singular problem with initial value condition. Agarwal et al [4] studied the existence and multiplicity of positive solutions of a singular by using the direct method of the calculus of variations, Ekeland's Variational Principle and an idea of Tarantello. However, their results cannot cover our results obtained in this paper, and our results are different from those obtained by classical methods such as fixed point theorems, nonlinear Leary-Schauder alternative principle, the method of upper and lower solutions and so on.

The rest of this article is organized as follows: In Section 2, we give several important definitions and lemmas. The main theorems are formulated and proved in Section 3. In Section 4, some examples are presented to illustrate our results.

2. Preliminaries and Lemmas

We denote by H be the Hilbert space of absolutely continuous functions $u : (0,1) \to \mathbb{R}$ such that $u' \in L^2(0,1)$ and $u(0) = u(1) = 0$. Consider the Hilbert space H with the inner product and norm

$$(u,v) = \int_0^1 u'(t)v'(t)dt, \quad \|u\| = \Big(\int_0^1 (u'(t))^2 dt\Big)^{1/2}. \tag{2.1}$$

Define $f_\varepsilon \in C((0,1) \times \mathbb{R}, [0,\infty))$ by

$$f_\varepsilon(t,u) = f(t, (u - \phi_\varepsilon(t))^+ + \phi_\varepsilon(t)), \tag{2.2}$$

where $u^\pm = \max\{\pm u, 0\}$ and $\phi_\varepsilon(t) = \varepsilon t(1-t)$ is the solution of

$$\begin{gathered} -u''(t) = 2\varepsilon, \quad t \in (0,1), \\ u(0) = u(1) = 0. \end{gathered} \tag{2.3}$$

Consider

$$\begin{gathered} -u''(t) = f_\varepsilon(t,u) + e(t), \quad t \in (0,1), \\ u(0) = u(1) = 0. \end{gathered} \tag{2.4}$$

By (2.2) and (1.4), one has

$$2\varepsilon - e(t) \le f_\varepsilon(t,u) \le C\phi_\varepsilon^{-\gamma}, \quad (t,u) \in (0,1) \times (-\infty, \varepsilon), \tag{2.5}$$

$$f_\varepsilon(t,u) = f(t,u), \quad (t,u) \in (0,1) \times [\varepsilon, \infty). \tag{2.6}$$

We observe that if u is a solution of (2.4), then $u \ge \phi_\varepsilon(t)$ and hence also a solution of (1.3). To see this suppose there exists some $t \in (0,1)$ such that

$$u(t) \le \phi_\varepsilon(t). \tag{2.7}$$

By [6, Lemma 2.8.1],

$$u(t) \ge t(1-t)\|u\|_\infty, \ t \in (0,1),$$

where $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$, so (2.7) implies $\|u\|_\infty < \varepsilon$. But one has $-u'' \geq 2\varepsilon = -\phi_\varepsilon''$ by (2.5), so $u \geq \phi_\varepsilon(t)$, contradicting (2.7).

Multiply the first equation of (2.4) by $v \in H$ at both sides, and integrate the equality on the interval (0,1) and combine the boundary condition $u(0) = u(1) = 0$ to obtain

$$\int_0^1 u'(t)v'(t)dt = \int_0^1 f_\varepsilon(t,u)v(t)dt + \int_0^1 e(t)v(t)dt; \tag{2.8}$$

thus, a weak solution of the singular boundary-value problem (2.4) is a function $u \in H$ such that (2.8) holds for any $v \in H$. Let

$$F_\varepsilon(t,u) = \int_\varepsilon^u f_\varepsilon(t,s)ds.$$

For all $u \in H$, noting that $u(0) = u(1) = 0 < \varepsilon$, one has

$$\begin{aligned}
\int_0^1 |F_\varepsilon(t,u)|dt &= \int_{u\geq\varepsilon} |\int_\varepsilon^u f_\varepsilon(t,s)ds|dt + \int_{u<\varepsilon} |\int_\varepsilon^u f_\varepsilon(t,s)ds|dt \\
&\leq \int_{u\geq\varepsilon} |\int_\varepsilon^u f_\varepsilon(t,s)ds|dt + C \max_{t\in[0,1]}(\varepsilon - u(t)) \int_0^1 |\phi_\varepsilon^{-\gamma}(t)|dt \\
&\leq \int_{u\geq\varepsilon} |\int_\varepsilon^u f_\varepsilon(t,s)ds|dt + C(\varepsilon + \|u\|_\infty) \int_0^1 |\phi_\varepsilon^{-\gamma}(t)|dt.
\end{aligned}$$

It is clear that there exists a constant $C_1 > 0$ such that $\int_{u\geq\varepsilon} |\int_\varepsilon^u f_\varepsilon(t,s)ds|dt \leq C_1$, and furthermore, $|\phi_\varepsilon^{-\gamma}| = \phi_\varepsilon^{-\gamma} \in L^1(0,1)$, thus we get $\int_0^1 |F_\varepsilon(t,u)|dt < +\infty$.

We see that the weak solutions of boundary-value problem (2.4) are the critical points of the C^1 functional defined by

$$\varphi(u) = \frac{1}{2}\int_0^1 |u'(t)|^2 dt - \int_0^1 F_\varepsilon(t,u)dt - \int_0^1 e(t)u(t)dt. \tag{2.9}$$

In the following we introduce some necessary definitions and lemmas.

Definition 2.1. Let E be a Banach space and $\varphi : E \to R$, is said to be sequentially weakly lower semi-continuous if $\lim_{k\to+\infty} \inf \varphi(x_k) \geq \varphi(x)$ as $x_k \rightharpoonup x$ in E.

Definition 2.2 ([13, p. 81])**.** Let E be a real reflexive Banach space. For any sequence $u_k \subset E$, if $\varphi(u_k)$ is bounded and $\varphi'(u_k) \to 0$, as $k \to +\infty$ possesses a convergent subsequence, then we say φ satisfies the Palais-Smale condition.

Lemma 2.3 ([17, Theorem 38])**.** *For the functional $F : M \subseteq X \to [-\infty, +\infty]$ with $M \neq \emptyset$, $\min_{u\in M} F(u) = \alpha$ has a solution when the following conditions hold:*

- (i) *X is a real reflexive Banach space;*
- (ii) *M is bounded and weak sequentially closed; i.e., by definition, for each sequence u_n in M such that $u_n \rightharpoonup u$ as $n \to \infty$, we always have $u \in M$;*
- (iii) *F is weak sequentially lower semi-continuous on M.*

Next we have the Mountain pass theorem; [13, Theorem 4.10].

Lemma 2.4. *Let E be a Banach space and $\varphi \in C^1(E,R)$ satisfy Palais-Smale condition. Assume there exist $x_0, x_1 \in E$, and a bounded open neighborhood Ω of x_0 such that $x_1 \notin \overline{\Omega}$ and*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{x\in\partial\Omega} \varphi(x).$$

Then there exists a critical value of φ; that is, there exists $u \in E$ such that $\varphi'(u) = 0$ and $\varphi(u) > \max\{\varphi(x_0), \varphi(x_1)\}$.

Lemma 2.5. *If $u \in H$, then $\|u\|_\infty \le \|u\|$, where $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$.*

Proof. It follows from Hölder's inequality,
$$|u(t)| = |\int_0^t u'(s)ds| \le \int_0^t |u'(s)|ds \le \int_0^1 |u'(t)|dt \le (\int_0^1 |u'(t)|^2 dt)^{1/2} = \|u\|.$$
□

Lemma 2.6. *The functional φ is continuous, continuously differentiable, and weakly lower semi-continuous.*

Proof. By the continuity of f and it is easy to check that functional φ is continuous, continuously differentiable, and $\varphi'(u)$ is defined by
$$\langle \varphi'(u), v\rangle = \int_0^1 u'(t)v'(t)dt - \int_0^1 f_\varepsilon(t,u)v(t)dt - \int_0^1 e(t)v(t)dt. \tag{2.10}$$
To show that φ is weakly lower semi-continuous, let $\{u_n\}$ be a weakly convergent sequence to u in H, then $\|u\| \le \liminf_{n\to\infty} \|u_n\|$, and $\{u_n\}$ converges uniformly to u in $C[0,1]$, so when $n \to \infty$, we have
$$\begin{aligned}
\liminf_{n\to\infty} \varphi(u_n) &= \frac{1}{2}\int_0^1 |u_n'(t)|^2 dt - \int_0^1 F_\varepsilon(t,u_n)dt - \int_0^1 e(t)u_n(t)dt \\
&\ge \frac{1}{2}\int_0^1 |u'(t)|^2 dt - \int_0^1 F_\varepsilon(t,u)dt - \int_0^1 e(t)u(t)dt \\
&= \varphi(u).
\end{aligned}$$
Thus, by Definition 2.1, φ is weakly lower semi-continuous. □

We state the well-known Ambrosetti-Rabinowitz condition as follows: There exist $\mu > 2$ and $r > \varepsilon$ such that
$$0 < \mu F_\varepsilon(t,u) \le f_\varepsilon(t,u)u, \quad u > r, \ \forall t \in (0,1), \tag{2.11}$$

It is well known that the Ambrosetti-Rabinowitz condition is quite natural and convenient not only to ensure the Palais-Smale sequence of the functional φ is bounded but also to guarantee the functional φ has a mountain pass geometry.

Lemma 2.7. *Suppose that Ambrosetti-Rabinowitz condition holds, then the functional φ satisfies Palais-Smale condition.*

Proof. Let $\{u_k\}$ be a sequence in H such that $\{\varphi(u_k)\}$ is bounded and $\varphi'(u_k) \to 0$, as $k \to +\infty$, then we will prove $\{u_k\}$ possesses a convergent subsequence.

We first prove that $\{u_k\}$ is bounded. By the Ambrosetti-Rabinowitz condition, one has
$$\begin{aligned}
&\mu\varphi(u_k) - \langle \varphi'(u_k), u_k\rangle \\
&= \frac{\mu}{2}\int_0^1 |u_k'|^2 dt - \mu\int_0^1 F_\varepsilon(t,u_k)dt - \mu\int_0^1 e(t)u_k dt \\
&\quad - \int_0^1 u_k'u_k' dt + \int_0^1 f_\varepsilon(t,u_k)u_k dt + \int_0^1 e(t)u_k dt
\end{aligned}$$

$$
\begin{aligned}
&\geq (\frac{\mu}{2}-1)\|u_k\|^2 - \int_0^1 (\mu F_\varepsilon(t,u_k) - f_\varepsilon(t,u_k)u_k)dt + (1-\mu)\|u_k\|_\infty \|e\|_{L^1} \\
&\geq (\frac{\mu}{2}-1)\|u_k\|^2 - \int_{u_k<\varepsilon} (\mu F_\varepsilon(t,u_k) - f_\varepsilon(t,u_k)u_k)dt \\
&\quad - \int_{\varepsilon\leq u_k\leq r} (\mu F_\varepsilon(t,u_k) - f_\varepsilon(t,u_k)u_k)dt \\
&\quad - \int_{u_k>r} (\mu F_\varepsilon(t,u_k) - f_\varepsilon(t,u_k)u_k)dt + (1-\mu)\|u_k\|_\infty \|e\|_{L^1} \\
&\geq (\frac{\mu}{2}-1)\|u_k\|^2 - C\int_{u_k<\varepsilon} \varphi_\varepsilon(t)^{-\gamma}|u_k|dt - \mu\int_{\varepsilon\leq u_k\leq r} F_\varepsilon(t,u_k)dt \\
&\quad + (1-\mu)\|u_k\|\|e\|_{L^1} \\
&\geq (\frac{\mu}{2}-1)\|u_k\|^2 - C\|u_k^-\|_\infty \int_0^1 \varphi_\varepsilon(t)^{-\gamma}dt + (1-\mu)\|u_k\|\|e\|_{L^1} - C_2,
\end{aligned}
$$

where $C_2 = \mu\int_{\varepsilon\leq u_k\leq r} F_\varepsilon(t,u_k)dt$.

It suffices to show that $\|u_k^-\|_\infty$ is bounded. In fact, by (2.10), one has

$$
\begin{aligned}
&\langle \varphi'(u_k), u_k^-\rangle \\
&= \int_0^1 u_k'(u_k^-)'dt - \int_0^1 f_\varepsilon(t,u_k)u_k^-dt - \int_0^1 e(t)u_k^-dt \\
&= \int_{u_k<0} u_k'(u_k^-)'dt + \int_{u_k\geq 0} u_k'(u_k^-)'dt - \int_0^1 f_\varepsilon(t,u_k)u_k^-dt - \int_0^1 e(t)u_k^-dt \\
&= \int_{u_k<0} u_k'(u_k^-)'dt - \int_0^1 f_\varepsilon(t,u_k)u_k^-dt - \int_0^1 e(t)u_k^-dt \\
&= -\int_0^1 (u_k^-)'(u_k^-)'dt - \int_0^1 f_\varepsilon(t,u_k)u_k^-dt - \int_0^1 e(t)u_k^-dt.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|u_k^-\|^2 &= -\langle \varphi'(u_k), u_k^-\rangle - \int_0^1 f_\varepsilon(t,u_k)u_k^-dt - \int_0^1 e(t)u_k^-dt \\
&\leq -\langle \varphi'(u_k), u_k^-\rangle + \int_0^1 f_\varepsilon(t,u_k)u_k^-dt + \int_0^1 e(t)u_k^-dt \\
&\leq \|\varphi'(u_k)\|\|u_k^-\| + C\|u_k^-\|_\infty \int_{u_k<0} f_\varepsilon(t,u_k)dt + \|u_k^-\|_\infty \|e\|_{L^1} \\
&\leq \|\varphi'(u_k)\|\|u_k^-\| + C\|u_k^-\| \int_0^1 \varphi_\varepsilon(t)^{-\gamma}dt + \|u_k^-\|\|e\|_{L^1}.
\end{aligned}
$$

Therefore,

$$
\|u_k^-\| \leq o(1) + C\int_0^1 \varphi_\varepsilon(t)^{-\gamma}dt + \|e\|_{L^1},
$$

which implies that $\|u_k^-\|$ is bounded. By Lemma 2.5, one has $\|u_k^-\|_\infty$ is also bounded. Thus we proved that $\{u_k\}$ is bounded.

Since H is a reflexive Banach space, there exists a subsequence of $\{u_k\}$ (for simplicity denoted again by $\{u_k\}$) such that $\{u_k\}$ weakly converges to some u in

H. Then the sequence $\{u_k\}$ converges uniformly to u in $[0,1]$. Hence,

$$(\varphi'(u_k)-\varphi'(u))(u_k-u)\to 0,$$
$$\int_0^1 (f_\varepsilon(t,u)-f_\varepsilon(t,u_k))(u_k-u)dt\to 0,$$

as $k\to+\infty$. Thus, we have

$$\begin{aligned}(\varphi'(u_k)-\varphi'(u))(u_k-u)&=\varphi'(u_k)(u_k-u)-\varphi'(u)(u_k-u)\\&=\int_0^1 (u_k'-u')^2dt+\int_0^1 (f_\varepsilon(t,u)-f_\varepsilon(t,u_k))(u_k-u)dt\\&=\|u_k-u\|^2+\int_0^1 (f_\varepsilon(t,u)-f_\varepsilon(t,u_k))(u_k-u)dt,\end{aligned}$$

which means $\|u_k-u\|\to 0$, as $k\to+\infty$. That is, $\{u_k\}$ converges strongly to u in H. □

3. Main results

Our main results are the following three theorems.

Theorem 3.1. *Suppose there exists $L>0$ such that*

$$f(t,u)\le L,\quad (t,u)\in(0,1)\times[\varepsilon,\infty). \tag{3.1}$$

Then (1.3) *has at least one positive weak solution.*

Proof. By (2.5) and (3.1), one has

$$F_\varepsilon(t,u)\le\begin{cases}0, & u<\varepsilon,\\ L(u-\varepsilon), & u\ge\varepsilon.\end{cases}$$

For any $u\in H$, one has

$$\begin{aligned}\varphi(u)&=\frac12\int_0^1|u'(t)|^2dt-\int_0^1 F_\varepsilon(t,u)dt-\int_0^1 e(t)u(t)dt\\&\ge\frac12\|u\|^2-\int_{u\ge\varepsilon}F_\varepsilon(t,u)dt-\|u\|_\infty\|e\|_{L^2}\\&\ge\frac12\|u\|^2-L\|u\|-\|u\|\|e\|_{L^2},\end{aligned}$$

which implies that $\liminf_{\|u\|\to\infty}\varphi(u)=+\infty$, thus, φ is coercive. Hence, by [13, Lemma 2.4 and Theorem 1.1], φ has a minimum, which is a critical point of φ, then (1.3) has at least one positive weak solution. □

Analogously we have the following result.

Theorem 3.2. *Suppose there exists $a,b>0$ and $\theta\in(0,1)$ such that*

$$f(t,u)\le au^\theta+b,\quad (t,u)\in(0,1)\times[\varepsilon,\infty). \tag{3.2}$$

Then (1.3) *has at least one positive weak solution.*

Proof. By using the same methods of the above proof of Theorem 3.1 there exists $\eta>0$ such that

$$\varphi(u)\ge\frac12\|u\|^2-a\|u\|^{\theta+1}-\eta\|u\|,$$

which implies that $\liminf_{\|u\|\to\infty}\varphi(u)=+\infty$, thus, φ is coercive. Hence, by [13, Lemma 2.4 and Theorem 1.1], φ has a minimum, which is a critical point of φ, then (1.3) has at least one positive weak solution. □

Theorem 3.3. *Suppose* (2.11) *holds, and there exist $\delta>0$, $\alpha>2$ such that $F_\varepsilon(t,u)\le\delta u^\alpha$, $(t,u)\in(0,1)\times[\varepsilon,\infty)$. Then* (1.3) *has at least two positive weak solutions.*

Proof. Firstly, we will show that there exists $\rho>0$, which will be determined later, such that the functional φ has a local minimum $u_0\in B_\rho=\{u\in H:\|u\|<\rho\}$. By the same methods used in [20] show that $\overline{B}_\rho$ is a bounded and weak sequentially closed. Noting that φ is weak sequentially lower semi-continuous on $\overline{B}_\rho$ and H is a reflexive Banach space. Then by Lemma 2.3 we can know that φ has a local minimum $u_0\in B_\rho$; that is, $\varphi(u_0)=\min_{u\in B_\rho}\varphi(u)$.

Next, we show that $\varphi(u_0)<\inf_{u\in\partial B_\rho}\varphi(u)$. Choose $\rho>0$ such that

$$\frac{1}{2}\rho^2-\delta\rho^\alpha-\rho\|e\|_{L^1}>-2\varepsilon^2. \tag{3.3}$$

For all $u=\rho\omega$, $\omega\in H$ with $\|\omega\|=1$, then $\|u\|=\|\rho\omega\|=\rho\|\omega\|=\rho$, thus $u\in\partial B_\rho$.

By Lemma 2.5 and $F_\varepsilon(t,u)\le\delta u^\alpha$ and $(t,u)\in(0,1)\times[\varepsilon,\infty)$, one has

$$\begin{aligned}
\varphi(u)&=\varphi(\rho\omega)\\
&=\frac{1}{2}\rho^2-\int_0^1F_\varepsilon(t,\rho\omega)dt-\int_0^1e(t)\rho\omega(t)dt\\
&\ge\frac{1}{2}\rho^2-\int_{u\ge\varepsilon}F_\varepsilon(t,\rho\omega)dt-\rho\|\omega\|_\infty\|e\|_{L^1}\\
&\ge\frac{1}{2}\rho^2-\int_{u\ge\varepsilon}F_\varepsilon(t,\rho\omega)dt-\rho\|\omega\|\|e\|_{L^1}\\
&\ge\frac{1}{2}\rho^2-\delta\int_0^1|\rho\omega|^\alpha dt-\rho\|\omega\|\|e\|_{L^1}\\
&\ge\frac{1}{2}\rho^2-\delta\rho^\alpha-\rho\|e\|_{L^1}\\
&>-2\varepsilon^2.
\end{aligned}$$

By (2.5), one has

$$F_\varepsilon(t,u)=\int_\varepsilon^u f(t,s)ds\ge\int_\varepsilon^u(2\varepsilon-e(t))ds=(2\varepsilon-e(t))(u-\varepsilon).$$

Thus, $F_\varepsilon(t,0)\ge-\varepsilon(2\varepsilon-e(t))\ge-2\varepsilon^2$, and we get $\varphi(u)>-2\varepsilon^2\ge\varphi(0)=-F_\varepsilon(t,0)\ge\varphi(u_0)$ for $u\in\partial B_\rho$, which implies $\varphi(u_0)<\inf_{u\in\partial B_\rho}\varphi(u)$.

Secondly, we will show that there exists u_1 with $\|u_1\|>\rho$ such that $\varphi(u_1)<\inf_{u\in\partial B_\rho}\varphi(u)$. By (2.5) and noting that the function $(0,\infty)\ni\xi\to F_\varepsilon(t,\frac{u}{\xi})\xi^\mu$ is nonincreasing when $u\ne0$, see the references [18], one has

$$F_\varepsilon(t,u)\ge F_\varepsilon(t,r)(\frac{u}{r})^\mu,\quad u\ge r.$$

Therefore, we can choose u_1 with $\|u_1\|$ sufficiently large such that $\varphi(u_1)<-2\varepsilon^2$. Thus we have

$$\max\{\varphi(u_0),\varphi(u_1)\}<\inf_{x\in\partial B_\rho}\varphi(x).$$

Lemma 2.7 shows that φ satisfies Palais-Smale condition. Hence, by Lemma 2.4 there exists a critical point $\hat{u}$. Therefore, u_0 and $\hat{u}$ are two critical points of φ, and they are also two positive weak solutions of (1.3). $\square$

4. Examples

Example 4.1. Take $\varepsilon = 1, e(t) = 2\sin t,\ f(t,u) = 2(1 + |\sin\frac{1}{t(1-t)}|)u^{-1/3}$, and consider the equation

$$\begin{gathered} -u''(t) = f(t,u) + e(t), \quad t \in (0,1), \\ u(0) = u(1) = 0\,. \end{gathered}$$

The equation is solvable according to Theorems 3.1 or 3.2.

Example 4.2. *Take $\varepsilon = 1$, $e(t) = 2\sin t$,*

$$f(t,u) = \begin{cases} 2(1 + |\sin\frac{1}{t(1-t)}|)u^{-1/3}, & 0 < u < 1, \\ 2(1 + |\sin\frac{1}{t(1-t)}|)u^2, & u \ge 1, \end{cases}$$

and consider the equation

$$\begin{gathered} -u''(t) = f(t,u) + e(t), \quad t \in (0,1), \\ u(0) = u(1) = 0. \end{gathered}$$

It is easy to verify the conditions of Theorem 3.3 hold, thus this equation has at least two positive weak solutions.

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