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UNIQUENESS AND EXISTENCE OF POSITIVE SOLUTIONS FOR SINGULAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the existence of positive solutions for the singular fractional boundary value problem

$$\begin{gathered} -D^{\alpha}u(t) = Af(t,u(t)) + \sum_{i=1}^{k} B_i I^{\beta_i} g_i(t,u(t)), \quad t \in (0,1), \\ D^{\delta}u(0) = 0, \quad D^{\delta}u(1) = aD^{\frac{\alpha-\delta-1}{2}}(D^{\delta}u(t))\big|_{t=\xi} \end{gathered}$$

where $1 < \alpha \le 2$, $0 < \xi \le 1/2$, $a \in [0,\infty)$, $1 < \alpha - \delta < 2$, $0 < \beta_i < 1$, A, B_i, $1 \le i \le k$, are real constant, D^{α} is the Riemann-Liouville fractional derivative of order α. By using the Banach's fixed point theorem and Leray-Schauder's alternative, the existence of positive solutions is obtained. At last, an example is given for illustration.

1. Introduction

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order, the fractional calculus may be considered an old and yet novel topic.

Recently, fractional differential equations have been of great interest. This is because of both the intensive development of the theory of fractional calculus itself and its applications in various sciences, such as physics, mechanics, chemistry, engineering, etc. For example, for fractional initial value problems, the existence and multiplicity of solutions were discussed in [2, 4, 10, 11], moreover, fractional derivative arises from many physical processes,such as a charge transport in amorphous semiconductors [18], electrochemistry and material science are also described by differential equations of fractional order [5, 6, 7, 14, 15]. Bai and Lü [3] considered the boundary value problem of fractional order differential equation

$$\begin{gathered} D_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{gathered}$$

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where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order $1 < \alpha \leq 2$ and
$f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous.

Hussein [9], considered the following nonlinear m-point boundary value problem of fractional type

$$\begin{gathered} D_{0+}^{\alpha} x(t) + q(t) f(t, x(t)) = 0, \quad \text{a.e. on } [0,1], \alpha \in (n-1, n], n \geq 2, \\ x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \xi_i x(\eta_i), \end{gathered}$$

where $0 < \eta_1 < \cdots < \eta_{m-2} < 1, \xi_i > 0$ with $\sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-1} < 1$, q is a real-valued continuous function and f is a nonlinear Pettis integrable function.

Motivated by the above works, the purpose of this paper is to discuss the following singular fractional boundary value problem:

$$\begin{gathered} -D^{\alpha} u(t) = A f(t, u(t)) + \sum_{i=1}^{k} B_i I^{\beta_i} g_i(t, u(t)), \quad t \in (0,1), \\ D^{\delta} u(0) = 0, \quad D^{\delta} u(1) = a D^{\frac{\alpha-\delta-1}{2}} (D^{\delta} u(t))\big|_{t=\xi}, \end{gathered} \tag{1.1}$$

where $1 < \alpha \leq 2$, $0 < \xi \leq \frac{1}{2}$, $a \in (0, \infty)$, $1 < \alpha - \delta < 2$, $0 < \beta_i < 1$, A, B_i, $1 \leq i \leq k$, are real constant, D^{α} is the Riemann-Liouville fractional derivative of order α.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries. In Sections 3 and 4, we study the existence and uniqueness of solutions for system (1.1) by Banach's fixed point theorem and Leray-Schauder's alternative, respectively. At last, in Section 5, an example is also given to illustrate our theory.

2. Preliminaries

In this section, we present notation and some preliminary lemmas that will be used in the proofs of the main results.

Definition 2.1 ([16, 17])**.** The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2 ([16, 17])**.** The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined as

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where $n = [\alpha] + 1$.

Lemma 2.3 ([12])**.** *The equality $D_{0+}^{\gamma} I_{0+}^{\gamma} f(t) = f(t)$ with $\gamma > 0$ holds for $f \in L^1(0,1)$.*

Lemma 2.4 ([12])**.** *Let $\alpha > 0$. Then the differential equation*

$$D_{0+}^{\alpha} u = 0$$

has a unique solution $u(t) = c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, \dots, n$, *there* $n - 1 < \alpha \le n$.

Lemma 2.5 ([12]). *Let* $\alpha > 0$. *Then the following equality holds for* $u \in L^1(0,1)$, $D^\alpha_{0+}u \in L^1(0,1)$;

$$I^\alpha_{0+}D^\alpha_{0+}u(t) = u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n},$$

$c_i \in \mathbb{R}$, $i = 1, \dots, n$, *there* $n - 1 < \alpha \le n$.

Lemma 2.6 ([3]). *For* $\lambda > -1$ *and* $\alpha > 0$,

$$D^\alpha_{0+}t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}t^{\gamma-\alpha}.$$

Lemma 2.7 ([13]). *Suppose that* $g \in L^1(0,1)$ *and* α, β *be two constant such that* $0 \le \beta \le 1 < \alpha$, *then*

$$D^\beta_{0+}\int_0^t (t-s)^{\alpha-1}g(s)ds = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}g(s)\,ds.$$

Now, we consider (1.1). By the substitution $u(t) = I^\delta y(t) = D^{-\delta}y(t)$, problem (1.1) is turned into

$$\begin{gathered} -D^{\alpha-\delta}y(t) = Af(t, I^\delta y(t)) + \sum_{i=1}^k B_iI^{\beta_i}g_i(t, I^\delta y(t)), \quad t \in (0,1), \\ y(0) = 0, \quad y(1) = aD^{\frac{\alpha-\delta-1}{2}}y(t))\big|_{t=\xi}. \end{gathered} \tag{2.1}$$

Lemma 2.8. *For any* $h \in C[0,1] \cap L(0,1)$, *the unique solution of the boundary value problem*

$$\begin{gathered} -D^{\alpha-\delta}y(t) = h(t), \quad t \in (0,1), \\ y(0) = 0, \quad y(1) = aD^{\frac{\alpha-\delta-1}{2}}y(t))\big|_{t=\xi} \end{gathered} \tag{2.2}$$

is

$$y(t) = -I^{\alpha-\delta}h(t) + \frac{t^{\alpha-\delta-1}\Gamma(\frac{\alpha-\delta+1}{2})}{\Gamma(\frac{\alpha-\delta+1}{2}) - a\Gamma(\alpha-\delta)\xi^{\frac{\alpha-\delta-1}{2}}}\Big(I^{\alpha-\delta}h(1) - aI^{\frac{\alpha-\delta+1}{2}}h(\xi)\Big).$$

Proof. By applying Lemma 2.5, equation (2.2) is equivalent to the integral equation

$$y(t) = -I^{\alpha-\delta}h(t) - c_1t^{\alpha-\delta-1} - c_2t^{\alpha-\delta-2}, \tag{2.3}$$

for some arbitrary constants $c_1, c_2 \in \mathbb{R}$.

By the boundary condition $y(0) = 0$, we conclude that $c_2 = 0$. Then, we have

$$y(1) = -I^{\alpha-\delta}h(1) - c_1,$$

and it follows from lemma (2.6) that

$$\begin{aligned} D^{\frac{\alpha-\delta-1}{2}}y(t) &= -D^{\frac{\alpha-\delta-1}{2}}I^{\alpha-\delta}h(t) - c_1D^{\frac{\alpha-\delta-1}{2}}t^{\alpha-\delta-1} \\ &= -I^{\frac{\alpha-\delta+1}{2}}h(t) - c_1\frac{\Gamma(\alpha-\delta)}{\Gamma(\frac{\alpha-\delta+1}{2})}t^{\frac{\alpha-\delta-1}{2}}. \end{aligned}$$

Therefore,

$$D^{\frac{\alpha-\delta-1}{2}}y(t)\Big|_{t=\xi} = -\int_0^\xi \frac{(\xi-s)^{\frac{\alpha-\delta-1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2})}h(s)ds - c_1\frac{\Gamma(\alpha-\delta)}{\Gamma(\frac{\alpha-\delta+1}{2})}t^{\frac{\alpha-\delta-1}{2}}.$$

So, by the boundary condition $y(1) = aD^{\frac{\alpha-\delta-1}{2}}y(t))\big|_{t=\xi}$, we obtain that

$$
\begin{aligned}
c_1 = \frac{\Gamma(\frac{\alpha-\delta+1}{2})}{\Gamma(\frac{\alpha-\delta+1}{2}) - a\Gamma(\alpha-\delta)\xi^{\frac{\alpha-\delta-1}{2}}}\Big[&-\int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}h(s)ds \\
&+ a\int_0^\xi \frac{(\xi-s)^{\frac{\alpha-\delta-1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2})}h(s)ds\Big].
\end{aligned}
$$

Therefore, the unique solution of equation (2.2) is

$$
\begin{aligned}
y(t) = -I^{\alpha-\delta}h(t) + \frac{t^{\alpha-\delta-1}\Gamma(\frac{\alpha-\delta+1}{2})}{\Gamma(\frac{\alpha-\delta+1}{2}) - a\Gamma(\alpha-\delta)\xi^{\frac{\alpha-\delta-1}{2}}}\Big[&\int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}h(s)ds \\
&- a\int_0^\xi \frac{(\xi-s)^{\frac{\alpha-\delta-1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2})}h(s)ds\Big].
\end{aligned}
$$

The proof is complete. $\square$

Thus, the solution of the problem (1.1) can be written as

$$
\begin{aligned}
u(t) &= I^\delta y(t) \\
&= I^\delta\Big[- I^{\alpha-\delta}h(t) + \frac{t^{\alpha-\delta-1}\Gamma(\frac{\alpha-\delta+1}{2})}{\Gamma(\frac{\alpha-\delta+1}{2}) - a\Gamma(\alpha-\delta)\xi^{\frac{\alpha-\delta-1}{2}}}\Big(I^{\alpha-\delta}h(1) - aI^{\frac{\alpha-\delta+1}{2}}h(\xi)\Big)\Big] \\
&= -I^\alpha h(t) + \frac{\Gamma(\frac{\alpha-\delta+1}{2})}{\Gamma(\frac{\alpha-\delta+1}{2}) - a\Gamma(\alpha-\delta)\xi^{\frac{\alpha-\delta-1}{2}}}\Big(I^{\alpha-\delta}h(1) - aI^{\frac{\alpha-\delta+1}{2}}h(\xi)\Big) \\
&\quad\times \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}s^{\alpha-\delta-1}ds \\
&= -I^\alpha h(t) + \frac{\Gamma(\frac{\alpha-\delta+1}{2})}{\Gamma(\frac{\alpha-\delta+1}{2}) - a\Gamma(\alpha-\delta)\xi^{\frac{\alpha-\delta-1}{2}}}\Big(I^{\alpha-\delta}h(1) - aI^{\frac{\alpha-\delta+1}{2}}h(\xi)\Big) \\
&\quad\times \Big\{\frac{t^{\alpha-1}}{\Gamma(\delta)}\int_0^1 (1-\nu)^{\delta-1}\nu^{\alpha-\delta-1}d\nu\Big\},
\end{aligned}
$$

where we have used the substitution $s = t\nu$ in the integral of the last term. Using the relation for the Beta function $B(\cdot,\cdot)$,

$$
B(\alpha,\beta) = \int_0^1 (1-u)^{\alpha-1}u^{\beta-1}du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},
$$

one has

$$
u(t) = -I^\alpha h(t) + \frac{t^{\alpha-1}\Gamma(\alpha-\delta)\Gamma(\frac{\alpha-\delta+1}{2})}{\Gamma(\alpha)\Big(\Gamma(\frac{\alpha-\delta+1}{2}) - a\Gamma(\alpha-\delta)\xi^{\frac{\alpha-\delta-1}{2}}\Big)}\Big(I^{\alpha-\delta}h(1) - aI^{\frac{\alpha-\delta+1}{2}}h(\xi)\Big). \tag{2.4}
$$

The solution of the original nonlinear problem (1.1) can be obtained by replacing h with the right hand side of the fractional equation of (1.1) in (2.4).

The basic space used in this paper is the real Banach space $\mathcal{C} = C([0,1],\mathbb{R})$ of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with the norm $\|u\| = \sup_{t\in[0,1]}|u(t)|$.

In relation to problem (1.1), we define an operator $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ as

$$
(\mathcal{T}u)(t) = -A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))ds - \sum_{i=1}^k B_i\int_0^t \frac{(t-s)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)}g_i(s,u(s))ds
$$

$$
\begin{aligned}
&+ t^{\alpha-1}A_0\Big[A\int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}f(s,u(s))ds \\
&+ \sum_{i=1}^k B_i\int_0^1 \frac{(1-s)^{\alpha-\delta+\beta_i-1}}{\Gamma(\alpha-\delta+\beta_i)}g_i(s,u(s))ds \\
&- aA\int_0^\xi \frac{(\xi-s)^{\frac{\alpha-\delta-1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2})}f(s,u(s))ds \\
&- a\sum_{i=1}^k B_i\int_0^\xi \frac{(\xi-s)^{\frac{\alpha-\delta-1}{2}+\beta_i}}{\Gamma(\frac{\alpha-\delta+1}{2}+\beta_i)}g_i(s,u(s))ds\Big],
\end{aligned}
$$

where

$$
A_0 = \frac{\Gamma(\alpha-\delta)\Gamma(\frac{\alpha-\delta+1}{2})}{\Gamma(\alpha)\Big(\Gamma(\frac{\alpha-\delta+1}{2}) - a\Gamma(\alpha-\delta)\xi^{\frac{\alpha-\delta-1}{2}}\Big)}.
$$

It is clear that the existence of a positive solution for the system (1.1) is equivalent to the existence of a nontrivial fixed point of $\mathcal{T}$ on $\mathcal{C}$.

For convenience of the reader, we set

$$
\begin{aligned}
\Delta = \sup_{t\in[0,1]}\Big\{&|A|\Big[\frac{t^\alpha}{\Gamma(\alpha+1)} + |A_0|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+1)} + a\frac{\xi^{\frac{\alpha-\delta+1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2}+1)}\Big)\Big] \\
&+ \sum_{i=1}^k |B_i|\Big[\frac{t^{\alpha+\beta_i}}{\Gamma(\alpha+\beta_i+1)} + |A_0|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+\beta_i+1)} \\
&+ a\frac{\xi^{\frac{\alpha-\delta+1}{2}+\beta_i}}{\Gamma(\frac{\alpha-\delta+1}{2}+\beta_i+1)}\Big)\Big]\Big\}.
\end{aligned} \tag{2.5}
$$

3. Existence results via Banach's fixed point theorem

In this section, by using Banach's fixed point theorem, we will establish find a unique solution of problem (1.1). Now, we state our results.

Theorem 3.1. *Assume that $f, g_i : [0,1]\times\mathbb{R} \to \mathbb{R}, i = 1,\dots,k$, are continuous functions satisfying the condition*

- (A1) $|f(t,u) - f(t,v)| \le L_1|u-v|$, $|g_i(t,u) - g_i(t,v)| \le L_{i+1}|u-v|$, *for* $i = 1,\dots,k$, $t\in[0,1]$, $L_i > 0, (i = 1,\dots,k+1)$, $u, v \in \mathbb{R}$.

Then the boundary-value problem (1.1) *has a unique solution if* $L < \frac{1}{\Delta}$, *where* $L = \max\{L_i : i = 1,\dots,k+1\}$ *and* Δ *is given by* (2.5).

Proof. Assume that $M = \max\{M_i : i = 1,\dots,k+1\}$, where M_i are finite numbers given by $\sup_{t\in[0,1]}|f(t,0)| = M_1$, $\sup_{t\in[0,1]}|g_i(t,0)| = M_{i+1}$. Selecting $r > \frac{\Delta M}{1-L\Delta}$, we show that $\mathcal{T}B_r \subset B_r$, where $B_r = \{u\in\mathcal{C} : \|u\| \le r\}$. Using that $|f(s,u(s))| \le |f(s,u(s)) - f(s,0)| + |f(s,0)| \le L_1 r + M_1$, $|g_i(s,u(s))| \le |g_i(s,u(s)) - g_i(s,0)| + |g_i(s,0)| \le L_{i+1}r + M_{i+1}, i = 1,\dots,k$, for $u\in B_r$, and (2.5) we can show that

$$
\begin{aligned}
&\|(\mathcal{T}u)\| \\
&\le (Lr+M)\sup_{t\in[0,1]}\Big\{|A|\Big[\frac{t^\alpha}{\Gamma(\alpha+1)} + |A_0|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+1)} + a\frac{\xi^{\frac{\alpha-\delta+1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2}+1)}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{k}|B_i|\Big[\frac{t^{\alpha+\beta_i}}{\Gamma(\alpha+\beta_i+1)}+|A_0|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+\beta_i+1)}\\
&+a\frac{\xi^{\frac{\alpha-\delta+1}{2}+\beta_i}}{\Gamma(\frac{\alpha-\delta+1}{2}+\beta_i+1)}\Big)\Big]\Big\}\\
&\le (Lr+M)\Delta<r,
\end{aligned}
$$

which implies that $\mathcal{T}B_r\subset B_r$. Now, for $u,v\in\mathcal{C}$ we obtain

$$
\begin{aligned}
&\|\mathcal{T}u-\mathcal{T}v\|\\
&\le \sup_{t\in[0,1]}\Big\{|A|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))-f(s,v(s))|ds\\
&\quad+\sum_{i=1}^{k}|B_i|\int_0^t\frac{(t-s)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)}|g_i(s,u(s))-g_i(s,v(s))|ds\\
&\quad+|A_0|t^{\alpha-1}\Big[|A|\int_0^1\frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}|f(s,u(s))-f(s,v(s))|ds\\
&\quad+\sum_{i=1}^{k}|B_i|\int_0^1\frac{(1-s)^{\alpha-\delta+\beta_i-1}}{\Gamma(\alpha-\delta+\beta_i)}|g_i(s,u(s))-g_i(s,v(s))|ds\\
&\quad+|A|a\int_0^\xi\frac{(\xi-s)^{\frac{\alpha-\delta-1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2})}|f(s,u(s))-f(s,v(s))|ds\\
&\quad+a\sum_{i=1}^{k}|B_i|\int_0^\xi\frac{(\xi-s)^{\frac{\alpha-\delta-1}{2}+\beta_i-1}}{\Gamma(\frac{\alpha-\delta-1}{2}+\beta_i)}|g_i(s,u(s))-g_i(s,v(s))|ds\Big]\Big\}\\
&\le L\sup_{t\in[0,1]}\Big\{|A|\Big[\frac{t^{\alpha}}{\Gamma(\alpha+1)}+|A_0|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+1)}+a\frac{\xi^{\frac{\alpha-\delta+1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2}+1)}\Big)\Big]\\
&\quad+\sum_{i=1}^{k}|B_i|\Big[\frac{t^{\alpha+\beta_i}}{\Gamma(\alpha+\beta_i+1)}+|A_0|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+\beta_i+1)}\\
&\quad+a\frac{\xi^{\frac{\alpha-\delta+1}{2}+\beta_i}}{\Gamma(\frac{\alpha-\delta+1}{2}+\beta_i+1)}\Big)\Big]\Big\}\|u-v\|\\
&= L\Delta\|u-v\|.
\end{aligned}
$$

By the assumption, $L<1/\Delta$. Therefore, $\mathcal{T}$ is a contraction. Thus, by the contraction mapping principle (Banach's fixed point theorem) the proof is complete. $\square$

Now we present another variant of existence uniqueness result based on the Hölder inequality.

Theorem 3.2. *Suppose that the continuous functions f,g_i satisfy the following conditions:*

- (A2) $|f(t,u(t))-f(t,v(t))|\le m(t)|u-v|$, $|g_i(t,u(t))-g_i(t,v(t))|\le n_i(t)|u-v|$, *for* $t\in[0,1]$, $u,v\in\mathbb{R}$, $m,n_i\in L^{\frac{1}{\gamma}}([0,1],\mathbb{R}^+)$, $i=1,\dots,k$, *and* $\gamma\in(0,\alpha-\delta-2)$.
- (A3) $|A|\|m\|Z_1+\sum_{i=1}^{k}|B_i|\|n_i\|Z_{i+1}<1$, *where*

$$
Z_1=\frac{1}{\Gamma(\alpha)}\Big(\frac{1-\gamma}{\alpha-\gamma}\Big)^{1-\gamma}+\frac{|A_0|}{\Gamma(\alpha-\delta)}\Big(\frac{1-\gamma}{\alpha-\delta-\gamma}\Big)^{1-\gamma}
$$

$$+\frac{a|A_0|}{\Gamma(\frac{\alpha-\delta+1}{2})}\Big(\frac{1-\gamma}{\frac{\alpha-\delta+1}{2}-\gamma}\Big)^{1-\gamma}\xi^{\frac{\alpha-\delta+1}{2}-\gamma},$$

$$\begin{aligned}Z_{i+1}=&(\frac{1}{\Gamma(\alpha+\beta_i)})\Big(\frac{1-\gamma}{\alpha+\beta_i-\gamma}\Big)^{1-\gamma}+(\frac{|A_0|}{\Gamma(\alpha-\delta+\beta_i)})\Big(\frac{1-\gamma}{\alpha-\delta+\beta_i-\gamma}\Big)^{1-\gamma}\\&+(\frac{a|A_0|}{\Gamma(\frac{\alpha-\delta+1}{2}+\beta_i)})\Big(\frac{1-\gamma}{\frac{\alpha-\delta+1}{2}+\beta_i-\gamma}\Big)^{1-\gamma}\xi^{\frac{\alpha-\delta+1}{2}+\beta_i-\gamma},\quad (i=1,\dots,k),\end{aligned}$$

and $\|\mu\|=(\int_0^1|\mu(s)|^{\frac{1}{\gamma}}ds)^\gamma,\mu=m,n.$

Then, the boundary value problem (1.1) *has a unique solution.*

Proof. For $u,v\in\mathbb{R}$ and $t\in[0,1]$, by Hölder inequality, we have

$$\begin{aligned}&\|\mathcal{T}u-\mathcal{T}v\|\\
&\le\sup_{t\in[0,1]}\Big\{|A|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)|u(s)-v(s)|ds\\
&\quad+\sum_{i=1}^k|B_i|\int_0^t\frac{(t-s)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)}n_i(s)|u(s)-v(s))|ds\\
&\quad+|A_0|\Big[|A|\int_0^1\frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}m(s)|u(s)-v(s)|ds\\
&\quad+\sum_{i=1}^k|B_i|\int_0^1\frac{(1-s)^{\alpha-\delta+\beta_i-1}}{\Gamma(\alpha-\delta+\beta_i)}n_i(s)|u(s)-v(s))|ds\\
&\quad+a|A|\int_0^\xi\frac{(\xi-s)^{\frac{\alpha-\delta-1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2})}m(s)|u(s)-v(s)|ds\\
&\quad+a\sum_{i=1}^k|B_i|\int_0^\xi\frac{(\xi-s)^{\frac{\alpha-\delta-1}{2}+\beta_i}}{\Gamma(\frac{\alpha-\delta+1}{2}+\beta_i)}n_i(s)|u(s)-v(s))|ds\Big]\Big\}\\
&\le\sup_{t\in[0,1]}\Big\{\frac{|A|\|m\|}{\Gamma(\alpha)}\Big(\frac{1-\gamma}{\alpha-\gamma}\Big)^{1-\gamma}t^{\alpha-\gamma}+\sum_{i=1}^k(\frac{|B_i|\|n_i\|}{\Gamma(\alpha+\beta_i)})\Big(\frac{1-\gamma}{\alpha+\beta_i-\gamma}\Big)^{1-\gamma}t^{\alpha+\beta_i-\gamma}\\
&\quad+|A_0|\Big[\frac{|A|\|m\|}{\Gamma(\alpha-\delta)}\Big(\frac{1-\gamma}{\alpha-\delta-\gamma}\Big)^{1-\gamma}+\sum_{i=1}^k(\frac{|B_i|\|n_i\|}{\Gamma(\alpha-\delta+\beta_i)})\Big(\frac{1-\gamma}{\alpha-\delta+\beta_i-\gamma}\Big)^{1-\gamma}\\
&\quad+\frac{a|A|\|m\|}{\Gamma(\frac{\alpha-\delta+1}{2})}\Big(\frac{1-\gamma}{\frac{\alpha-\delta+1}{2}-\gamma}\Big)^{1-\gamma}\xi^{\frac{\alpha-\delta+1}{2}-\gamma}\\
&\quad+a\sum_{i=1}^k(\frac{|B_i|\|n_i\|}{\Gamma(\frac{\alpha-\delta+1}{2}+\beta_i)})\Big(\frac{1-\gamma}{\frac{\alpha-\delta+1}{2}+\beta_i-\gamma}\Big)^{1-\gamma}\xi^{\frac{\alpha-\delta+1}{2}+\beta_i-\gamma}\Big]\Big\}\|u-v\|\\
&\le\Big\{A|\|m\|\Big[\frac{1}{\Gamma(\alpha)}\Big(\frac{1-\gamma}{\alpha-\gamma}\Big)^{1-\gamma}+\frac{|A_0|}{\Gamma(\alpha-\delta)}\Big(\frac{1-\gamma}{\alpha-\delta-\gamma}\Big)^{1-\gamma}\\
&\quad+\frac{a|A_0|}{\Gamma(\frac{\alpha-\delta+1}{2})}\Big(\frac{1-\gamma}{\frac{\alpha-\delta+1}{2}-\gamma}\Big)^{1-\gamma}\xi^{\frac{\alpha-\delta+1}{2}-\gamma}\Big]\\
&\quad+\sum_{i=1}^k|B_i|\|n_i\|\Big[(\frac{1}{\Gamma(\alpha+\beta_i)})\Big(\frac{1-\gamma}{\alpha+\beta_i-\gamma}\Big)^{1-\gamma}\end{aligned}$$

$$
\begin{aligned}
&+ \Big(\frac{|A_0|}{\Gamma(\alpha-\delta+\beta_i)}\Big)\Big(\frac{1-\gamma}{\alpha-\delta+\beta_i-\gamma}\Big)^{1-\gamma} \\
&+ \Big(\frac{a|A_0|}{\Gamma(\frac{\alpha-\delta+1}{2}+\beta_i)}\Big)\Big(\frac{1-\gamma}{\frac{\alpha-\delta+1}{2}+\beta_i-\gamma}\Big)^{1-\gamma}\xi^{\frac{\alpha-\delta+1}{2}+\beta_i-\gamma}\Big]\Big\}\|u-v\| \\
&= [|A|\|m\|Z_1 + \sum_{i=1}^{k}|B_i|\|n_i\|Z_{i+1}]\|u-v\|.
\end{aligned}
$$

By the condition (A3), it follows that $\mathcal{T}$ is a contraction mapping. Hence, by the Banach's fixed point theorem $\mathcal{T}$ has a unique fixed point which is the unique solution of the problem (1.1). Then, the proof is complete. □

4. Existence result via Leray-Schauder's alternative

In this section, by using the Leray-schauder's alternative, we will find at least one solution to problem (1.1). The proof of the main result in this section is based on the Leray-schauder's alternative [1, 8] that we recall here for the reader's convenience.

Lemma 4.1. *(Nonlinear alternative for single valued maps [1, 8]). Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \bar{U} \to C$ is a completely continuous operator. Then, either*

(i) *F has a fixed point in $\bar{U}$, or*
(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

We now state our main result in this section.

Theorem 4.2. *Suppose that $f, g_i : [0,1]\times\mathbb{R} \to \mathbb{R}$, $i = 1,\dots,k$, are continuous functions. Assume that:*

(H1) *There exist functions $p, p_i \in L^1([0,1],\mathbb{R}^+)$, $i = 1,\dots,k$, and nondecreasing functions $\psi, \psi_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1,\dots,k$, such that*
$$|f(t,x)| \le p(t)\psi(\|x\|), \quad |g_i(t,x)| \le p_i(t)\psi_i(\|x\|),$$
for all $(t,x) \in [0,1]\times\mathbb{R}$ and $i = 1,\dots,k$.

(H2) *There exists a constant $M > 0$ such that*
$$\frac{M}{|A|\psi(M)\|p\|_{L^1}\Omega + \sum_{i=1}^{k}|B_i|\Omega_i\psi_i(M)\|p_i\|_{L^1}} > 1$$
where
$$\Omega = \frac{1}{\Gamma(\alpha+1)} + |A_0|\Big(\frac{1}{\Gamma(\alpha-\delta+1)} + a\frac{\xi^{\frac{\alpha-\delta+1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2}+1)}\Big),$$
and
$$\Omega_i = \frac{1}{\Gamma(\alpha+\beta_i+1)} + |A_0|\Big(\frac{1}{\Gamma(\alpha-\delta+\beta_i+1)} + a\frac{\xi^{\frac{\alpha-\delta+1}{2}+\beta_i}}{\Gamma(\frac{\alpha-\delta+1}{2}+\beta_i+1)}\Big),$$
$(i = 1,\dots,k)$.

Then, the boundary-value problem (1.1) has at least one solution on $[0,1]$.

Proof. Consider the operator $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ with

$(\mathcal{T}u)(t)$

$$
\begin{aligned}
&= -A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds - \sum_{i=1}^k B_i \int_0^t \frac{(t-s)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)} g_i(s,u(s))ds \\
&\quad + t^{\alpha-1}A_0\Big[A\int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} f(s,u(s))ds \\
&\quad + \sum_{i=1}^k B_i \int_0^1 \frac{(1-s)^{\alpha-\delta+\beta_i-1}}{\Gamma(\alpha-\delta+\beta_i)} g_i(s,u(s))ds \\
&\quad - aA\int_0^\xi \frac{(\xi-s)^{\frac{\alpha-\delta-1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2})} f(s,u(s))ds - a\sum_{i=1}^k B_i \int_0^\xi \frac{(\xi-s)^{\frac{\alpha-\delta-1}{2}+\beta_i}}{\Gamma(\frac{\alpha-\delta+1}{2}+\beta_i)} g_i(s,u(s))ds\Big].
\end{aligned}
$$

We show that $\mathcal{T}$ maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. for a positive number r, let $B_r = \{u \in C([0,1],\mathbb{R}) : \|u\| \le r\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then, we have

$$
\begin{aligned}
&|(\mathcal{T}u)(t)| \\
&\le |A|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)\psi(\|u\|)ds + \sum_{i=1}^k |B_i| \int_0^t \frac{(t-s)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)} p_i(s)\psi_i(\|u\|)ds \\
&\quad + |A_0|t^{\alpha-1}\Big[|A|\int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} p(s)\psi(\|u\|)ds \\
&\quad + \sum_{i=1}^k |B_i| \int_0^1 \frac{(1-s)^{\alpha-\delta+\beta_i-1}}{\Gamma(\alpha-\delta+\beta_i)} p_i(s)\psi_i(\|u\|)ds \\
&\quad + |A|a\int_0^\xi \frac{(\xi-s)^{\frac{\alpha-\delta-1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2})} p(s)\psi(\|u\|)ds \\
&\quad + a\sum_{i=1}^k |B_i| \int_0^\xi \frac{(\xi-s)^{\frac{\alpha-\delta-1}{2}+\beta_i-1}}{\Gamma(\frac{\alpha-\delta-1}{2}+\beta_i)} p_i(s)\psi_i(\|u\|)ds\Big] \\
&\le |A|\psi(r)\|p\|_{L^1}\Big[\frac{t^\alpha}{\Gamma(\alpha+1)} + |A_0|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+1)} + a\frac{\xi^{\frac{\alpha-\delta+1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2}+1)}\Big)\Big] \\
&\quad + \sum_{i=1}^k |B_i|\psi_i(r)\|p_i\|_{L^1}\Big[\frac{t^{\alpha+\beta_i}}{\Gamma(\alpha+\beta_i+1)} + |A_0|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+\beta_i+1)} \\
&\quad + a\frac{\xi^{\frac{\alpha-\delta+1}{2}+\beta_i}}{\Gamma(\frac{\alpha-\delta+1}{2}+\beta_i+1)}\Big)\Big].
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\|\mathcal{T}u\| \\
&\le |A|\psi(r)\|p\|_{L^1}\Big[\frac{1}{\Gamma(\alpha+1)} + |A_0|\Big(\frac{1}{\Gamma(\alpha-\delta+1)} + a\frac{\xi^{\frac{\alpha-\delta+1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2}+1)}\Big)\Big] \\
&\quad + \sum_{i=1}^k |B_i|\psi_i(r)\|p_i\|_{L^1}\Big[\frac{1}{\Gamma(\alpha+\beta_i+1)} + |A_0|\Big(\frac{1}{\Gamma(\alpha-\delta+\beta_i+1)} \\
&\quad + a\frac{\xi^{\frac{\alpha-\delta+1}{2}+\beta_i}}{\Gamma(\frac{\alpha-\delta+1}{2}+\beta_i+1)}\Big)\Big]
\end{aligned}
$$

$$= |A|\psi(r)\|p\|_{L^1}\Omega + \sum_{i=1}^{k} |B_i|\psi_i(r)\|p_i\|_{L^1}\Omega_i,$$

let $K = |A|\psi(r)\|p\|_{L^1}\Omega + \sum_{i=1}^{k} |B_i|\psi_i(r)\|p_i\|_{L^1}\Omega_i$, therefore, we conclude that $\|\mathcal{T}u\| \leq K$. Thus, $\mathcal{T}$ maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$.

Next, we show that $\mathcal{T}$ maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $u \in B_r$; thus, we have

$$\begin{aligned}
&|(\mathcal{T}u)(t_2) - (\mathcal{T}u)(t_1)| \\
&\leq \frac{|A|}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]|f(s,u(s))|ds \\
&\quad + \frac{|A|}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}|f(s,u(s))|ds \\
&\quad + \sum_{i=1}^{k} \frac{|B_i|}{\Gamma(\alpha+\beta_i)} \int_0^{t_1} [(t_2 - s)^{\alpha+\beta_i-1} - (t_1 - s)^{\alpha+\beta_i-1}]|g_i(s,u(s))|ds \\
&\quad + \sum_{i=1}^{k} \frac{|B_i|}{\Gamma(\alpha+\beta_i)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha+\beta_i-1}|g_i(s,u(s))|ds \\
&\quad + [(t_2)^{\alpha-1} - (t_1)^{\alpha-1}]|A_0|\Big[|A| \int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}|f(s,u(s))|ds \\
&\quad + \sum_{i=1}^{k} |B_i| \int_0^1 \frac{(1-s)^{\alpha-\delta+\beta_i-1}}{\Gamma(\alpha-\delta+\beta_i)}|g_i(s,u(s))|ds \\
&\quad + a|A| \int_0^{\xi} \frac{(\xi - s)^{\frac{\alpha-\delta-1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2})}|f(s,u(s))|ds \\
&\quad + a\sum_{i=1}^{k} |B_i| \int_0^{\xi} \frac{(\xi - s)^{\frac{\alpha-\delta-1}{2}+\beta_i}}{\Gamma(\frac{\alpha-\delta+1}{2}+\beta_i)}|g_i(s,u(s))|ds\Big] \\
&\leq \frac{|A|}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]p(s)\psi(r)ds \\
&\quad + \frac{|A|}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}p(s)\psi(r)ds \\
&\quad + \sum_{i=1}^{k} \frac{|B_i|}{\Gamma(\alpha+\beta_i)} \int_0^{t_1} [(t_2 - s)^{\alpha+\beta_i-1} - (t_1 - s)^{\alpha+\beta_i-1}]p_i(s)\psi_i(r)ds \\
&\quad + \sum_{i=1}^{k} \frac{|B_i|}{\Gamma(\alpha+\beta_i)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha+\beta_i-1}p_i(s)\psi_i(r)ds \\
&\quad + [(t_2)^{\alpha-1} - (t_1)^{\alpha-1}]|A_0|\Big[|A| \int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}p(s)\psi(r)ds \\
&\quad + \sum_{i=1}^{k} |B_i| \int_0^1 \frac{(1-s)^{\alpha-\delta+\beta_i-1}}{\Gamma(\alpha-\delta+\beta_i)}p_i(s)\psi_i(r)ds
\end{aligned}$$

$$+ a|A|\int_0^{\xi}\frac{(\xi-s)^{\frac{\alpha-\delta-1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2})}p(s)\psi(r)ds + a\sum_{i=1}^{k}|B_i|\int_0^{\xi}\frac{(\xi-s)^{\frac{\alpha-\delta-1}{2}+\beta_i}}{\Gamma(\frac{\alpha-\delta+1}{2}+\beta_i)}p_i(s)\psi_i(r)ds\Big]$$

Obviously, the right-hand side of the above inequality tends to zero independently of $u \in B_r$ as $t_2 - t_1 \to 0$. Therefore, $\mathcal{T} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous by application of the Arzelá-Ascoli theorem.

Now, we can conclude the result by using the Leray-Schauder's nonlinear alternative theorem. Consider the equations $x = \lambda\mathcal{T}x$ for $\lambda \in (0,1)$ and assume that u be a solution. Then, using the computations in proving that $\mathcal{T}$ is bounded, we have

$$\begin{aligned}
\|u\| &= \|\lambda(\mathcal{T}u)\| \\
&\le |A|\psi(\|u\|)\|p\|_{L^1}\Big[\frac{1}{\Gamma(\alpha+1)} + |A_0|\Big(\frac{1}{\Gamma(\alpha-\delta+1)} + a\frac{\xi^{\frac{\alpha-\delta+1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2}+1)}\Big)\Big] \\
&\quad + \sum_{i=1}^{k}|B_i|\psi_i(\|u\|)\|p_i\|_{L^1}\Big[\frac{1}{\Gamma(\alpha+\beta_i+1)} + |A_0|\Big(\frac{1}{\Gamma(\alpha-\delta+\beta_i+1)} \\
&\quad + a\frac{\xi^{\frac{\alpha-\delta+1}{2}+\beta_i}}{\Gamma(\frac{\alpha-\delta+1}{2}+\beta_i+1)}\Big)\Big];
\end{aligned}$$

therefore,

$$\frac{\|u\|}{|A|\psi(\|u\|)\|p\|_{L^1}\Omega + \sum_{i=1}^{k}|B_i|\Omega_i\psi_i(\|u\|)\|p_i\|_{L^1}} \le 1.$$

In view of (H2), there exists M such that $\|u\| \ne M$. Let us set

$$U = \{x \in C([0,1],\mathbb{R}) : \|x\| < M\}.$$

It is obvious that the operator $\mathcal{T} : \bar{U} \to C([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of U there is no $u \in \partial U$ such that $u = \lambda\mathcal{T}(u)$ for some $\lambda \in (0,1)$. Therefore, by the Leray-Schauder's nonlinear alternative theorem (Lemma 4.1), we conclude that $\mathcal{T}$ has a fixed point $u \in \bar{\mathrm{U}}$ which is a solution of the problem (1.1). Thus, the proof is completed. □

5. Application

Example 5.1. Consider the singular boundary value problem

$$\begin{gathered}
-D^{3/2}u(t) = Af(t,u(t)) + \sum_{i=1}^{3}B_iI^{\beta_i}g_i(t,u(t)), \quad t \in (0,1), \\
D^{1/4}u(0) = 0, \quad D^{1/4}u(1) = aD^{1/8}(D^{1/4}u(t))\big|_{t=1/2}
\end{gathered} \tag{5.1}$$

Here, $A = B_i = 1$, $(i = 1,2,3)$, $\beta_1 = 1/2$, $\beta_2 = 1/3$, $\beta_3 = 2/3$, $a = 2$, $f(t,u) = \frac{9}{125}\sqrt{1-t}(1+u)$, $g_i(t,u) = \frac{3}{29}\tan^{-1}u + \cos(e^t)$. With the given data, we obtain:

$$A_0 = \frac{\Gamma(\alpha-\delta)\Gamma(\frac{\alpha-\delta+1}{2})}{\Gamma(\alpha)\Big(\Gamma(\frac{\alpha-\delta+1}{2}) - a\Gamma(\alpha-\delta)\xi^{\frac{\alpha-\delta-1}{2}}\Big)} = -1.3365,$$

$$\Delta = \sup_{t\in[0,1]}\Big\{|A|\Big[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + |A_0|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+1)} + a\frac{\xi^{\frac{\alpha-\delta+1}{2}}}{\Gamma(\frac{\alpha-\delta+1}{2}+1)}\Big)\Big]$$

$$+\sum_{i=1}^{k}|B_i|\Big[\frac{t^{\alpha+\beta_i}}{\Gamma(\alpha+\beta_i+1)}+|A_0|t^{\alpha-1}\Big(\frac{1}{\Gamma(\alpha-\delta+\beta_i+1)}$$
$$+a\frac{\xi^{\frac{\alpha-\delta+1}{2}+\beta_i}}{\Gamma(\frac{\alpha-\delta+1}{2}+\beta_i+1)}\Big)\Big]\Big\}=8.9046,$$

and $L_1=9/125$, $L_2=L_3=L_4=3/29$ as $|f(t,u)-f(t,v)|\le\frac{9}{125}|u-v|$, $|g_i(t,u)-g_i(t,v)|\le\frac{3}{29}|u-v|$. Obviously, $L=\max\{L_i:i=1,\dots,4\}=3/29$ and $L<\frac{1}{\Delta}$. Hence, all the assumptions of Theorem 3.1 are satisfied. Thus, by the conclusion of Theorem 3.1, problem (5.1) has a unique solution.

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