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BIFURCATION FROM INFINITY AND NODAL SOLUTIONS OF QUASILINEAR ELLIPTIC DIFFERENTIAL EQUATIONS

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Abstract. In this article, we establish a unilateral global bifurcation theorem from infinity for a class of N-dimensional p-Laplacian problems. As an application, we study the global behavior of the components of nodal solutions of the problem

$$\begin{gathered}\operatorname{div}(\varphi_p(\nabla u)) + \lambda a(x) f(u) = 0, \quad x \in B, \\ u = 0, \quad x \in \partial B,\end{gathered}$$

where $1 < p < \infty$, $\varphi_p(s) = |s|^{p-2}s$, $B = \{x \in \mathbb{R}^N : |x| < 1\}$, and $a \in C(\bar{B}, [0, \infty))$ is radially symmetric with $a \not\equiv 0$ on any subset of $\bar{B}$, $f \in C(\mathbb{R}, \mathbb{R})$ and there exist two constants $s_2 < 0 < s_1$, such that $f(s_2) = f(s_1) = 0$, and $f(s)s > 0$ for $s \in \mathbb{R} \setminus \{s_2, 0, s_1\}$. Moreover, we give intervals for the parameter λ, where the problem has multiple nodal solutions if $\lim_{s\to 0} f(s)/\varphi_p(s) = f_0 > 0$ and $\lim_{s\to\infty} f(s)/\varphi_p(s) = f_\infty > 0$. We use topological methods and nonlinear analysis techniques to prove our main results.

1. Introduction

In natural sciences, there are various concrete problems involving bifurcation phenomena, for example, Taylor vortices [3], catastrophic shifts in ecosystems [10] and shimmy oscillations of an aircraft nose landing gear [11]. The existence of bifurcation phenomena have called the attention of several mathematicians. Dai et al [4] established a unilateral global bifurcation theorem from infinity for one-dimensional p-Laplacian problem, and studied the global behavior of the components of nodal solutions of nonlinear one-dimensional p-Laplacian eigenvalue problem.

Dai and Ma [5] established a result from trivial solutions line about the continua of radial solutions for the N-dimensional p-Laplacian problem on the unit ball of $\mathbb{R}^N$ with $N \geq 1$ and $1 < p < \infty$. Ambrosetti and Hess [1] studied the global behavior of the components of positive solutions of quasilinear elliptic differential equation under the asymptotically linear growth condition. Ambrosetti et al [2] studied the existence of branches of positive solutions for quasilinear elliptic differential equation under the equidiffusive growth condition, which extend the main result in [1]. However, these references gave no information about the sign-changing solution.

Motivated by the above articles, it is our main purpose to use the results in [5] and in line with the global bifurcation results from infinity by Rabinowitz [9]. We

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shall establish the unilateral global bifurcation result from infinity for the following N-dimensional p-Laplacian problem

$$\begin{gathered} -\operatorname{div}(\varphi_p(\nabla u)) = \lambda a(x)\varphi_p(u) + g(x,u;\lambda), \quad x \in B, \\ u = 0, \quad x \in \partial B, \end{gathered} \tag{1.1}$$

where $1 < p < \infty$, $\varphi_p(s) = |s|^{p-2}s, B$ is the unit ball of $\mathbb{R}^N$, $a \in M(B)$ is a non-negative function with

$$M(B) = \{a \in C(\bar{B}) \text{ is radially symmetric with } a(\cdot) \not\equiv 0 \text{ on any subset of } \bar{B}\},$$

the function $g : B \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition in the first two variables and is radially symmetric with respect x.

It is clear that the radial solutions of (1.1) are the solutions of

$$\begin{gathered} -(r^{N-1}\varphi_p(u'))' = \lambda r^{N-1}a(r)\varphi_p(u) + r^{N-1}g(r,u;\lambda), \quad \text{a.e. } r \in (0,1), \\ u'(0) = u(1) = 0, \end{gathered} \tag{1.2}$$

where $r = |x|$ with $x \in B$, $a \in M(I)$ is a non-negative function with $I = (0,1)$ and

$$M(I) = \{a \in C(\bar{I}) \text{ is radially symmetric with } a(\cdot) \not\equiv 0 \text{ on any subset of } \bar{I}\}.$$

We also assume the perturbation function g satisfies the assumption

$$\lim_{|s|\to\infty} \frac{g(r,s;\lambda)}{|s|^{p-1}} = 0 \tag{1.3}$$

uniformly for a.e. $r \in I$ and λ on bounded sets.

Based on the unilateral global bifurcation results from zero by [5], and the global bifurcation results from infinity, Theorem 2.2, we shall study the existence of radial nodal solutions for the nonlinear eigenvalue problem

$$\begin{gathered} \operatorname{div}(\varphi_p(\nabla u)) + \lambda a(x)f(u) = 0, \quad x \in B, \\ u = 0, \quad x \in \partial B, \end{gathered} \tag{1.4}$$

where a and f satisfy the following assumptions:

- (H1) $a \in C(\bar{B}, [0,\infty))$ with $a \not\equiv 0$ on any subset of $\bar{B}$;
- (H2) there exist $f_0, f_\infty \in (0,\infty)$ such that

$$f_0 = \lim_{s\to 0} \frac{f(s)}{|s|^{p-2}s} \quad \text{and} \quad f_\infty = \lim_{s\to\infty} \frac{f(s)}{|s|^{p-2}s};$$

- (H3) $f \in C(\mathbb{R},\mathbb{R})$, there exist two constants $s_2 < 0 < s_1$, such that $f(s_2) = f(s_1) = f(0) = 0$, and $f(s)s > 0$ for $s \in \mathbb{R} \setminus \{s_2, 0, s_1\}$.

We look for radial nodal solution of (1.4), namely for $u = u(r)$ verifying

$$\begin{gathered} \big(r^{N-1}\varphi_p(u')\big)' + \lambda r^{N-1}a(r)f(u) = 0, \quad \text{a.e. } r \in I, \\ u'(0) = u(1) = 0, \end{gathered} \tag{1.5}$$

where $r = |x|$ with $x \in B$.

The rest of this article is arranged as follows. In Section 2, we establish the unilateral global bifurcation results from infinity of (1.1). In Section 3, we study the global behavior of the components of nodal solutions of problem (1.4).

2. Unilateral global bifurcation from infinity

Let $E := \{u \in C^1(\bar{I}) | u'(0) = u(1) = 0\}$ with the norm $\|u\| = \max_{r\in\bar{I}} |u(r)| + \max_{r\in\bar{I}} |u'(r)|$. Let S_k^+ denote the set of functions in E which have exactly $k-1$ interior nodal zeros in I and are positive near $r = 0$, and set $S_k^- = -S_k^+$ and $S^k = S_k^+ \cup S_k^-$. It is clear that S_k^+ and S_k^- are disjoint and open in E. We also let $\phi_k^\nu = \mathbb{R} \times S_k^\nu$ and $\phi_k = \mathbb{R} \times S_k$ under the product topology, where $\nu \in \{+,-\}$. We use $\mathscr{S}$ to denote the closure of the set of nontrivial solutions of (1.2) in $\mathbb{R} \times E$. We add the points $\{(\lambda,\infty) | \lambda \in \mathbb{R}\}$ to space $\mathbb{R} \times E$.

Lemma 2.1 ([8, Theorem 1.5.3])**.** *Assume* (H1) *holds. Then the problem*

$$\begin{gathered}\big(r^{N-1}\varphi_p(u')\big)' + \lambda r^{N-1} a(r)\varphi_p(u) = 0, \quad a.e.\ r \in I,\\ u'(0) = u(1) = 0\end{gathered} \tag{2.1}$$

has a sequence of simple eigenvalues λ_k with $\lambda_k \to \infty$ as $k \to \infty$, and the corresponding eigenfunctions φ_k have exactly $k-1$ simple zeros, and each $\lambda_k(p)$ depends continuously on p.

Let λ_k denote the k-th eigenvalue of problem (2.1). The main result of this section is the following theorem.

Theorem 2.2. *Let assumption* (1.3) *hold. Then there exists a connected component $\mathcal{D}_k^\nu$ of $\mathscr{S} \cup (\lambda_k \times \{\infty\})$, containing $\lambda_k \times \{\infty\}$. Moreover if $\Lambda \subset \mathbb{R}$ is an interval such that $\Lambda \cap (\cup_{k=1}^\infty \lambda_k) = \lambda_k$ and $\mathcal{U}$ is a neighborhood of $\lambda_k \times \{\infty\}$ whose projection on $\mathbb{R}$ lies in Λ and whose projection on E is bounded away from 0, then either*

(1) *$\mathcal{D}_k^\nu - \mathcal{U}$ is bounded in $\mathbb{R} \times E$ in which case $\mathcal{D}_k^\nu - \mathcal{U}$ meets $\mathscr{R} = \{(\lambda, 0) | \lambda \in \mathbb{R}\}$, or*

(2) *$\mathcal{D}_k^\nu - \mathcal{U}$ is unbounded.*

If (2) occurs and $\mathcal{D}_k^\nu - \mathcal{U}$ has a bounded projection on $\mathbb{R}$, then $\mathcal{D}_k^\nu - \mathcal{U}$ meets $\lambda_j \times \{\infty\}$ for some $j \neq k$.

Proof. If $(\lambda, u) \in \mathscr{S}$ with $\|u\| \neq 0$, dividing (1.2) by $\|u\|^2$ and setting $w = u/\|u\|^2$ yield

$$\begin{gathered}-\big(r^{N-1}\varphi_p(w')\big)' = \lambda\big(r^{N-1}a(r)\varphi_p(w)\big) + r^{N-1}\frac{g(r,u;\lambda)}{\|u\|^{2(p-1)}}, \quad \text{a.e. } r \in I,\\ w'(0) = w(1) = 0.\end{gathered} \tag{2.2}$$

Define

$$f(r,w;\lambda) = \begin{cases} \|w\|^{2(p-1)} r^{N-1} g(r, w/\|w\|^2; \lambda), & \text{if } w \neq 0,\\ 0, & \text{if } w = 0,\end{cases}$$

Clearly, (2.2) is equivalent to

$$\begin{gathered}-\big(r^{N-1}\varphi_p(w')\big)' = \lambda\big(r^{N-1}a(r)\varphi_p(w)\big) + f(r,w;\lambda), \quad \text{a.e. } r \in I,\\ w'(0) = w(1) = 0.\end{gathered} \tag{2.3}$$

It is obvious that $(\lambda, 0)$ is always the solution of (2.3). By simple computation, we can show that assumption (1.3) implies

$$f(r,w;\lambda) = o(|w|^{p-1})$$

near $w = 0$, uniformly for all $r \in I$ and on bounded λ intervals.

Now applying [5, Theorem 3.2] to problem (2.3), we have the connected component $\mathcal{C}_k^\nu$ of $\mathscr{S}\cup(\lambda_k\times\{0\})$, containing $\lambda_k\times\{0\}$ is unbounded and lies in $\phi_k^\nu\cup(\lambda_k\times\{0\})$. Under the inversion $w\to w/\|w\|^2=u, \mathcal{C}_k^\nu\to\mathcal{D}_k^\nu$ satisfying problem (1.2). Clearly, $\mathcal{D}_k^\nu$ satisfies the conclusions of this theorem. $\square$

By [6, Lemma 6.4.1] and using the similar argument, we can prove [9, Corollary 1.8] with obvious changes. Also we have the following theorem.

Theorem 2.3. *There exists a neighborhood $\mathcal{N}\subset\mathcal{U}$ of $\lambda_k\times\{\infty\}$ such that $(\lambda,u)\in(\mathcal{D}_k^\nu\cap\mathcal{N})\setminus\{(\lambda_k\times\{\infty\})\}$ implies $(\lambda,u)=(\lambda_k+o(1),\alpha\varphi_k+w)$, where φ_k is the eigenfunction corresponding to λ_k with $\|\varphi_k\|=1,\alpha>0(\alpha<0)$ and $\|w\|=o(|\alpha|)$ at $|\alpha|=\infty$.*

Remark 2.4. Note that Theorem 2.3 implies that $(\mathcal{D}_k^\nu\cap\mathcal{N})\subset(\phi_k^\nu\cup(\lambda_k\times\{\infty\}))$. However, it need not be the case that $\mathcal{D}_k^\nu\subset(\phi_k^\nu\cup(\lambda_k\times\{\infty\}))$ even in the case of $p=2$ (see the example in [9]).

3. Global behavior of the components of nodal solutions

Let $\xi,\eta\in C(\mathbb{R},\mathbb{R})$ be such that

$$f(u)=f_0\varphi_p(u)+\xi(u),\quad f(u)=f_\infty\varphi_p(u)+\eta(u)$$

with

$$\lim_{|u|\to0}\frac{\xi(u)}{\varphi_p(u)}=0,\quad \lim_{|u|\to\infty}\frac{\eta(u)}{\varphi_p(u)}=0.$$

Let us consider

$$\begin{gathered}-\big(r^{N-1}\varphi_p(u')\big)'=\lambda r^{N-1}a(r)f_0\varphi_p(u)+\lambda r^{N-1}a(r)\xi(u),\quad \text{a.e. } r\in I,\\ u'(0)=u(1)=0\end{gathered}\tag{3.1}$$

as a bifurcation problem from the trivial solution $u\equiv0$, and

$$\begin{gathered}-\big(r^{N-1}\varphi_p(u')\big)'=\lambda r^{N-1}a(r)f_\infty\varphi_p(u)+\lambda r^{N-1}a(r)\eta(u),\quad \text{a.e. } r\in I,\\ u'(0)=u(1)=0\end{gathered}\tag{3.2}$$

as a bifurcation problem from infinity.

Applying [5, Theorem 3.2] to (3.1), we have that for each integer $k\ge1$, there exists a continuum $\mathcal{C}_{k,0}^\nu$, of solutions of (1.5) joining $(\lambda_k/f_0,0)$ to infinity, and $(\mathcal{C}_{k,0}^\nu\backslash\{(\lambda_k/f_0,0)\})\subseteq\phi_k^\nu$. Applying Theorem 2.2 to (3.2), we can show that for each integer $k\ge1$, there exists a continuum $\mathcal{D}_{k,\infty}^\nu$ of solutions of (1.5) meeting $(\lambda_k/f_\infty,\infty)$. Moreover, Theorem 2.3 imply that

$$(\mathcal{D}_{k,\infty}^\nu\backslash\{(\lambda_k/f_\infty,\infty)\})\subseteq\phi_k^\nu.$$

Next, we shall show that these two components are disjoint under the assumption (H3). Hence the essential role is played by the fact of whether f possesses zeros in $\mathbb{R}\backslash\{0\}$.

Theorem 3.1. *Let* (H1)-(H3) *hold. Then*

(i) *for $(\lambda,u)\in(\mathcal{C}_{k,0}^+\cup\mathcal{C}_{k,0}^-)$, we have that $s_2<u(r)<s_1$ for all $r\in\bar I$;*
(ii) *for $(\lambda,u)\in(\mathcal{D}_{k,\infty}^+\cup\mathcal{D}_{k,\infty}^-)$, we have that either $\max_{r\in\bar I}u(r)>s_1$ or $\min_{r\in\bar I}u(r)<s_2$.*

Proof. Suppose on the contrary that there exists $(\lambda, u) \in (\mathcal{C}_{k,0}^+ \cup \mathcal{C}_{k,0}^- \cup \mathcal{D}_{k,\infty}^+ \cup \mathcal{D}_{k,\infty}^-)$ such that either $\max\{u(r)|r \in \bar{I}\} = s_1$ or $\min\{u(r)|r \in \bar{I}\} = s_2$. Let $0 < \tau_1 < \cdots < \tau_k = 1$ denote the zeros of u. We only treat the case of $\max\{u(r)|r \in \bar{I}\} = s_1$ because the proof for the case of $\min\{u(r)|r \in \bar{I}\} = s_2$ can be given similarly. In this case, there exists $j \in \{1, \cdots, k,\}$ such that $\max\{u(r)|r \in \bar{I}\} = s_1$ and $0 \le u(r) \le s_1$ for all $r \in [\tau_j, \tau_{j+1}]$.

We claim that there exists $0 < m < \infty$ such that $f(s) \le m\varphi_p(s_1 - s)$ for any $s \in [0, s_1]$.

Clearly, the claim is true for the case of $s = 0$ or $s = s_1$ by (H3). Suppose on the contrary that there exists $s_0 \in (0, s_1)$ such that

$$f(s_0) > m\varphi_p(s_1 - s_0)$$

for any $m > 0$. It follows that $m < f(s_0)/\varphi_p(s_1 - s_0)$. This contradicts the arbitrariness of m.

Now, let us consider the problem

$$\begin{gathered} -(r^{N-1}\varphi_p((s_1 - u)'))' + \lambda r^{N-1} m a(r)\varphi_p(s_1 - u) \\ = \lambda r^{N-1} m a(r)\varphi_p(s_1 - u) - \lambda r^{N-1} a(r) f(u), \quad r \in (\tau_j, \tau_{j+1}), \\ s_1 - u(\tau_j) > 0, \quad s_1 - u(\tau_{j+1}) > 0. \end{gathered}$$

It is obvious that $f(s) \le m\varphi_p(s_1 - s)$ for any $s \in [0, s_1]$ implies

$$\begin{gathered} -(r^{N-1}\varphi_p((s_1 - u)'))' + \lambda r^{N-1} m a(r)\varphi_p(s_1 - u) \ge 0, \quad r \in (\tau_j, \tau_{j+1}), \\ s_1 - u(\tau_j) > 0, \quad s_1 - u(\tau_{j+1}) > 0. \end{gathered}$$

The strong maximum principle of [7] implies that $s_1 > u(r)$ in $[\tau_j, \tau_{j+1}]$. This is a contradiction. □

Remark 3.2. If $N = 1$, then Theorems 2.2, 2.3 and 3.1 correspond to the main results in [4].

In [2], they needed $f \in C^1(\mathbb{R}^+, \mathbb{R})$, while in this article, we need just $f \in C(\mathbb{R}, \mathbb{R})$. Furthermore, they studied the existence of branches of positive solutions, while we have the existence of branches of sign-changing solutions. So we have extended the results in [2, 4].

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