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ENERGY DECAY FOR ELASTIC WAVE EQUATIONS WITH CRITICAL DAMPING

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Abstract. We show that the total energy decays at the rate $E_u(t) = O(t^{-2})$, as $t \to +\infty$, for solutions to the Cauchy problem of a linear system of elastic wave with a variable damping term. It should be mentioned that the the critical decay satisfies $V(x) \geq C_0(1 + |x|)^{-1}$ for $C_0 > 2b$, where b represents the speed of propagation of the P-wave.

1. Introduction

We consider the Cauchy problem for the linear system of elastic wave equations with a critical potential type of damping $V(x)$ in $\mathbb{R}^2$:

$$\begin{gathered} u_{tt}(t,x) - a^2\Delta u(t,x) - (b^2 - a^2)\nabla(\operatorname{div} u(t,x)) + V(x)u_t(t,x) = 0, \\ (t,x) \in (0,\infty) \times \mathbb{R}^2, \end{gathered} \tag{1.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^2, \tag{1.2}$$

where the vector displacement $u = u(t,x) = (u_1(x,t), u_2(t,x))$ and the coefficients a and b are related to the Lamé coefficients and the satisfy the condition of ellipticity $0 < a^2 \leq b^2$. The initial data u_0 and u_1 are compactly supported from the energy space; that is,

$$\begin{gathered} u_0 \in (H^1(\mathbb{R}^2))^2, \quad u_1 \in (L^2(\mathbb{R}^2))^2, \\ \operatorname{supp} u_i \subset B(R_0) := \{x \in \mathbb{R}^2 : |x| < R_0\}, \quad (i = 0, 1). \end{gathered}$$

The system of elastic waves satisfies the property of finite speed of propagation given by coefficient b, which is the speed of propagation of the longitudinal P-wave. The coefficient a is the speed of propagation of the transverse S-wave (cf. [1]).

The damping coefficient $V(x)$ belongs to $C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and satisfies

(A1) $V(x) \geq \frac{C_0}{1+|x|}$, for all $x \in \mathbb{R}^2$ and for some $C_0 > 0$.

Under these conditions it is standard to prove via semigroups theory (cf. Horbach [5, Theorem 2.1]) that problem (1.1)-(1.2) has a unique solution

$$u \in C([0,+\infty); (H^1(\mathbb{R}^2))^2) \cap C^1([0,+\infty); (L^2(\mathbb{R}^2))^2)$$

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satisfying

$$E_u(t) + \int_0^t \int_{\mathbb{R}^2} V(x)|u_t(s,x)|^2 \, dx \, ds = E_u(0),$$

where the total energy is

$$E_u(t) := \frac{1}{2} \int_{\mathbb{R}^2} \{|u_t(t,x)|^2 + a^2|\nabla u(t,x)|^2 + (b^2 - a^2)(\operatorname{div} u(t,x))^2\} dx. \tag{1.3}$$

Our main results in this article read as follows.

Theorem 1.1. *Let the damping coefficient $V(x)$ satisfy* (A1) *with $C_0 > 2b$. Then the solution $u(t,x)$ to problem* (1.1)-(1.2) *satisfies*

$$E_u(t) = O(t^{-2}) \tag{1.4}$$

as $t \to +\infty$.

Proposition 1.2. *Let $V(x)$ satisfy* (A1) *and C_0 satisfy $0 < C_0 \le 2b$. Then for the solution $u(t,x)$ to* (1.1)-(1.2) *one has the following two possibilities:*

(I) *When $0 < C_0 \le b$ it holds that*

$$E_u(t) = O(t^{-1+\delta}), \quad t \to +\infty \tag{1.5}$$

for any $\delta > 0$ satisfying $1 - \frac{C_0}{b} < \delta < 1$;

(II) *when $b < C_0 \le 2b$ it holds that*

$$E_u(t) = O(t^{-\frac{C_0}{b}+\delta}), \quad \text{as } t \to +\infty \tag{1.6}$$

for any $\delta > 0$.

Remark 1.3. It follows from Proposition 1.2 part (I) that $E_u(t) = O(t^{-\frac{C_0}{b}+\varepsilon})$ for any small $\varepsilon > 0$ which is the same decay rate as that Proposition 1.2 part (II), if one re-set $\delta := 1 - \frac{C_0}{b} + \varepsilon$ with any $\varepsilon \in (0, C_0/b] \subset (0,1]$. This implies that when $0 < C_0 \le 2b$ the obtained decay rate is $E_u(t) = O(t^{-\frac{C_0}{b}+\delta})$ with any small $\delta > 0$.

Remark 1.4. When one compares these results with the one for the scalar wave equations due to [8] the obtained decay rates are (almost) optimal.

To begin we mention the motivation and some related results of this research. There are a lot of results concerning the energy decay estimates for the scalar-valued wave equation

$$w_{tt}(t,x) - \Delta w(t,x) + V(x) w_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \tag{1.7}$$

$$w(0,x) = w_0(x), \quad w_t(0,x) = w_1(x), \quad x \in \mathbb{R}^n, \tag{1.8}$$

where the function $V(x)$ typically satisfies

$$V(x) = \frac{C_0}{(1+|x|)^\alpha}, \quad C_0 > 0, \quad \alpha \in [0, +\infty). \tag{1.9}$$

In connection with problem (1.7)-(1.8) it is easy to proove the decreasing property of the total energy

$$E_w(t) := \frac{1}{2} \int_{\mathbb{R}^n} \{|w_t(t,x)|^2 + |\nabla w(t,x)|^2\} dx.$$

So a natural question arises whether $E_w(t)$ decays or not as $t \to +\infty$. About this question Mochizuki [11] first gave an answer that when $\alpha > 1$ (super-critical damping), the solution $w(t,x)$ to (1.7)-(1.8) is asymptotically free, and the corresponding energy satisfies $\lim_{t\to+\infty} E_w(t) > 0$ (non-decay). In this sense, the case $\alpha > 1$ shows a hyperbolic aspect of the equation (1.7). On the other hand, Todorova-Yordanov [14] considered the case when $\alpha \in [0, 1)$ (sub-critical damping), and they derived almost optimal decay estimates of the energy:

$$E_w(t) = O(t^{\delta - \frac{n-\alpha}{2-\alpha} - 1}), \quad \text{as } t \to +\infty$$

for any $\delta > 0$. In connection with this, one can observe that the energy $E_w(t)$ has faster decay rates as $\alpha \to 1$. This sub-critical damping case has a close relation to the so called diffusion phenomenon of the equation (1.7). Recently, Ikehata-Todorova-Yordanov [8] announced a work on the critical damping case $\alpha = 1$, and roughly speaking, they derived the following results:

$$E_w(t) = O(t^{\delta - \min\{C_0, n\}}), \quad \text{as } t \to +\infty,$$

for any $\delta > 0$ (indeed, we can choose $\delta = 0$ in part). There exists a threshold concerning the decay rate from the viewpoint of the dimension n and the damping coefficient C_0.

On the other hand, for the elastic waves it seems that there are not so many results except for the references [2, 3, 4, 5, 9]. That happens, especially for dissipative terms with variable coefficients. The local energy decay property of the equation (1.1) without damping (i.e., $V(x) \equiv 0$) is studied by Kapitonov [9], and the result is an elastic wave version of that derived by Morawetz [13] to the scalar wave equations (1.7) with $V(x) \equiv 0$. For the exterior mixed problem of the elastic wave equation (1.1) with localized damping coefficient $V(x)$ near spatial infinity Charão and Ikehata [3] derived the faster decay estimates of the total energy and L^2-norm of solutions basing on a previous research due to Ikehata [6] about the scalar wave equations. In Charão and Ikehata [4] the equation (1.1) with monotone nonlinearity and critical damping ($\alpha = 1$) has been treated based on a method due to [7], however, the obtained decay rate of the energy seems not to be sharp. Sharp higher order energy decay estimates have been recently studied by Charão-da Luz and Ikehata [2] to the elastic wave equations with "structural damping", but the method in [2] cannot be applied to the x-dependent variable coefficient case like (1.1).

The purpose of this paper is to find sharp decay estimates of the energy to problem (1.1)-(1.2) in the 2-dimensional case, and the strategy for the proof comes from the previous papers due to Ikehata-Inoue [7] and the two dimensional Ikehata-Todorova-Yordanov [8] method. Several basic computations of the energy method have already been prepared by Horbach [5], so basing on this computations due to [5] we will develop a method introduced in [8] to the scalar wave equations. Unlike the scalar wave equation, the elastic wave equation is vector valued and the equation itself has a quite complex form, so the treatment of the elastic wave equation is not so easy as compared with the scalar wave. The advantage is that the proof is very elementary in spite of the complexity of the equation.

The proof of Proposition 1.2 part (I) above is already shown in Horbach [5]; so that we restrict ourselves to prove Theorem 1.1 and Proposition 1.2 part (II) in the next section.

Open problem. Can one have the estimate $E_u(t) = O(t^{-\min\{\frac{C_0}{b}, n\}})$ (as $t \to +\infty$) when $n \geq 3$. This will be an estimate, for a higher dimensional elastic wave version, with the same form as for the scalar wave equations in [8].

1.1. **Notation.** We will use the following symbols:

$$\|u\|^2 := \int_{\mathbb{R}^2} |u(x)|^2 dx = \int_{\mathbb{R}^2} \sum_{i=1}^{2} |u_i(x)|^2 dx,$$

$$\|\nabla u\|^2 := \sum_{i=1}^{2} \|\nabla u_i\|^2 = \sum_{i=1}^{2} \int_{\mathbb{R}^2} |\nabla u_i(x)|^2 dx = \sum_{i,j=1}^{2} \Big\|\frac{\partial u_i}{\partial x_j}\Big\|^2,$$

$$(u,v) = \int_{\mathbb{R}^2} u(x)\cdot v(x) dx = \int_{\mathbb{R}^2} \sum_{i=1}^{2} u_i(x) v_i(x) dx,$$

$$u : \nabla v = u_1 \nabla v_1 + u_2 \nabla v_2 = \Big(\sum_{i=1}^{2} u_i \frac{\partial v_i}{\partial x_1}, \sum_{i=1}^{2} u_i \frac{\partial v_i}{\partial x_2} \Big),$$

$$\operatorname{div}(u : \nabla u) = u \cdot \Delta u + |\nabla u|^2,$$

$$\operatorname{div}(u_t : \nabla u) = u_t \cdot \Delta u + \frac{1}{2}\frac{d}{dt}|\nabla u|^2,$$

$$\operatorname{div}(u \ \operatorname{div} u) = (\operatorname{div} u)^2 + u \cdot \nabla(\operatorname{div} u),$$

$$\operatorname{div}(u_t \ \operatorname{div} u) = \frac{1}{2}\frac{d}{dt}(\operatorname{div} u)^2 + u_t \cdot \nabla(\operatorname{div} u),$$

where $p \cdot q := p_1 q_1 + p_2 q_2$ for $p = (p_1, p_2) \in \mathbb{R}^2$ and $q = (q_1, q_2) \in \mathbb{R}^2$.

2. Proof of the main results

First, we multiply the equation (1.1) by $f(t)u_t + g(t)u$, and integrate over $\mathbb{R}^2$ in order to get the following Lemma, where $f(t)$ and $g(t)$ are smooth functions specified later.

Lemma 2.1. *Let $u \in C([0,+\infty); (H^1(\mathbb{R}^2))^2) \cap C^1([0,+\infty); (L^2(\mathbb{R}^2))^2)$ be the solution to* (1.1)-(1.2). *Then*

$$\frac{d}{dt}E(t) + F(t) = 0, \quad t \ge 0, \tag{2.1}$$

where

$$\begin{aligned} E(t) := &\int_{\mathbb{R}^2} \frac{f(t)}{2}\{|u_t|^2 + a^2|\nabla u|^2 + (b^2-a^2)(\operatorname{div} u)^2\} dx \\ &+ g(t)(u,u_t) + \int_{\mathbb{R}^2} \frac{g(t)}{2} V(x)|u|^2 dx - \int_{\mathbb{R}^2} \frac{g_t(t)}{2}|u|^2 dx, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} F(t) := &\frac{1}{2}\int_{\mathbb{R}^2} \{2f(t)V(x) - 2g(t) - f_t(t)\}|u_t|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} \{g_{tt}(t) - g_t(t)V(x)\}|u|^2 dx \\ &+ \frac{a^2}{2}\int_{\mathbb{R}^2} (2g(t) - f_t(t))|\nabla u|^2 dx + \frac{b^2-a^2}{2}\int_{\mathbb{R}^2} (2g(t) - f_t(t))(\operatorname{div} u)^2 dx. \end{aligned} \tag{2.3}$$

Proof. For the moment, one can assume that the corresponding solution $u(t,x)$ is sufficiently smooth and vanishes near infinity to proceed the computations below. The general case follows from density arguments.

We first multiply (1.1) by $f(t)u_t$ to get the equality

$$f(t)(u_t \cdot u_{tt}) - a^2 f(t)(u_t \cdot \Delta u) - (b^2-a^2) f(t)(u_t \cdot \nabla(\operatorname{div} u)) + f(t)V(x)|u_t|^2 = 0,$$

so that

$$\begin{aligned}&\frac{f(t)}{2}\frac{d}{dt}|u_t|^2 - a^2 f(t)\operatorname{div}(u_t : \nabla u) + \frac{a^2 f(t)}{2}\frac{d}{dt}|\nabla u|^2 - (b^2-a^2)f(t)\operatorname{div}(u_t \ \operatorname{div} u)\\ &\quad + \frac{(b^2-a^2)f(t)}{2}\frac{d}{dt}(\operatorname{div} u)^2 + f(t)V(x)|u_t|^2 = 0.\end{aligned} \tag{2.4}$$

Next, we multiply (1.1) by $g(t)u$ to obtain

$$g(t)(u\cdot u_{tt}) - a^2 g(t)(u\cdot\Delta u) - (b^2-a^2)g(t)(u\cdot\nabla(\operatorname{div} u)) + g(t)V(x)(u\cdot u_t) = 0,$$

so that

$$\begin{aligned}&g(t)\frac{d}{dt}(u\cdot u_t) - g(t)|u_t|^2 - a^2 g(t)\operatorname{div}(u:\nabla u) + a^2 g(t)|\nabla u|^2\\ &- (b^2-a^2)g(t)\operatorname{div}(u \ \operatorname{div} u) + (b^2-a^2)g(t)(\operatorname{div} u)^2\\ &+ \frac{g(t)}{2}V(x)\frac{d}{dt}|u|^2 = 0.\end{aligned} \tag{2.5}$$

Thus, by adding (2.4) and (2.5), it follows that

$$\begin{aligned}&\Big\{\frac{f(t)}{2}\frac{d}{dt}|u_t|^2 - a^2 f(t)\operatorname{div}(u_t:\nabla u) + \frac{a^2 f(t)}{2}\frac{d}{dt}|\nabla u|^2\\ &- (b^2-a^2)f(t)\ \operatorname{div}(u_t \ \operatorname{div} u) + \frac{(b^2-a^2)f(t)}{2}\frac{d}{dt}(\operatorname{div} u)^2 + f(t)V(x)|u_t|^2\Big\}\\ &+ \Big\{g(t)\frac{d}{dt}(u\cdot u_t) - g(t)|u_t|^2 - a^2 g(t)\operatorname{div}(u:\nabla u) + a^2 g(t)|\nabla u|^2\\ &- (b^2-a^2)g(t)\ \operatorname{div}(u \ \operatorname{div} u) + (b^2-a^2)g(t)\ (\operatorname{div} u)^2 + \frac{g(t)}{2}V(x)\frac{d}{dt}|u|^2\Big\} = 0.\end{aligned}$$

This implies that

$$\begin{aligned}&\frac{f(t)}{2}\frac{d}{dt}\Big(|u_t|^2 + a^2|\nabla u|^2 + (b^2-a^2)(\operatorname{div} u)^2\Big) + g(t)\frac{d}{dt}(u\cdot u_t) + \frac{g(t)}{2}V(x)\frac{d}{dt}|u|^2\\ &- g(t)|u_t|^2 - a^2 g(t)\operatorname{div}(u:\nabla u) + a^2 g(t)|\nabla u|^2 - (b^2-a^2)g(t)\operatorname{div}(u \operatorname{div} u)\\ &+ (b^2-a^2)g(t)(\operatorname{div} u)^2 - a^2 f(t)\operatorname{div}(u_t:\nabla u) - (b^2-a^2)f(t)\operatorname{div}(u_t \operatorname{div} u)\\ &+ f(t)V(x)|u_t|^2 = 0.\end{aligned} \tag{2.6}$$

If we define the density of energy

$$\begin{aligned}e(t,x) := &\frac{f(t)}{2}\Big(|u_t|^2 + a^2|\nabla u|^2 + (b^2-a^2)(\operatorname{div} u)^2\Big)\\ &+ g(t)(u\cdot u_t) + \frac{g(t)}{2}V(x)|u|^2 - \frac{g_t(t)}{2}|u|^2,\end{aligned} \tag{2.7}$$

then we have

$$\begin{aligned}\frac{d}{dt}e(t,x) = &\frac{f_t(t)}{2}\Big(|u_t|^2 + a^2|\nabla u|^2 + (b^2-a^2)(\operatorname{div} u)^2\Big)\\ &+ \frac{f(t)}{2}\frac{d}{dt}\Big(|u_t|^2 + a^2|\nabla u|^2 + (b^2-a^2)(\operatorname{div} u)^2\Big)\\ &+ g(t)\frac{d}{dt}(u\cdot u_t) + \frac{g_t(t)}{2}V(x)|u|^2 + \frac{g(t)}{2}V(x)\frac{d}{dt}|u|^2 - \frac{g_{tt}(t)}{2}|u|^2.\end{aligned}$$

The above identity can be written as

$$
\begin{aligned}
&\frac{f(t)}{2}\frac{d}{dt}\Big(|u_t|^2+a^2|\nabla u|^2+(b^2-a^2)(\operatorname{div}u)^2\Big)+g(t)\frac{d}{dt}(u\cdot u_t)+\frac{g(t)}{2}V(x)\frac{d}{dt}|u|^2\\
&=\frac{d}{dt}e(t,x)-\frac{f_t(t)}{2}\Big(|u_t|^2+a^2|\nabla u|^2+(b^2-a^2)(\operatorname{div}u)^2\Big)\\
&\quad-\frac{g_t(t)}{2}V(x)|u|^2+\frac{g_{tt}(t)}{2}|u|^2.
\end{aligned}
\tag{2.8}
$$

Thus, by (2.6) and (2.8) we have the identity

$$
\begin{aligned}
&\frac{d}{dt}e(t,x)+\Big(f(t)V(x)-g(t)-\frac{f_t(t)}{2}\Big)|u_t|^2+\Big(\frac{g_{tt}(t)}{2}-\frac{g_t(t)}{2}V(x)\Big)|u|^2\\
&+a^2\Big(g(t)-\frac{f_t(t)}{2}\Big)|\nabla u|^2+(b^2-a^2)\Big(g(t)-\frac{f_t(t)}{2}\Big)(\operatorname{div}u)^2-a^2g(t)\operatorname{div}(u:\nabla u)\\
&-(b^2-a^2)g(t)\operatorname{div}(u\operatorname{div}u)-a^2f(t)\ \operatorname{div}(u_t:\nabla u)\\
&-(b^2-a^2)f(t)\operatorname{div}(u_t\operatorname{div}u)=0.
\end{aligned}
\tag{2.9}
$$

We integrate (2.9) over $\mathbb{R}^2$ to obtain the identity

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}e(t,x)dx+\int_{\mathbb{R}^2}\Big(f(t)V(x)-g(t)-\frac{f_t(t)}{2}\Big)|u_t|^2dx\\
&+\int_{\mathbb{R}^2}\Big(\frac{g_{tt}(t)}{2}-\frac{g_t(t)}{2}V(x)\Big)|u|^2dx+a^2\int_{\mathbb{R}^2}\Big(g(t)-\frac{f_t(t)}{2}\Big)|\nabla u|^2dx\\
&+(b^2-a^2)\int_{\mathbb{R}^2}\Big(g(t)-\frac{f_t(t)}{2}\Big)(\operatorname{div}u)^2dx-a^2\int_{\mathbb{R}^2}g(t)\operatorname{div}(u:\nabla u)dx\\
&-(b^2-a^2)\int_{\mathbb{R}^2}g(t)\operatorname{div}(u\operatorname{div}u)dx-a^2\int_{\mathbb{R}^2}f(t)\ \operatorname{div}(u_t:\nabla u)dx\\
&-(b^2-a^2)\int_{\mathbb{R}^2}f(t)\operatorname{div}(u_t\operatorname{div}u)dx=0.
\end{aligned}
$$

By applying the Gauss divergence theorem, one notices that

$$
\begin{gathered}
\int_{\mathbb{R}^2}\operatorname{div}(u:\nabla u)dx=0,\quad \int_{\mathbb{R}^2}\operatorname{div}(u\operatorname{div}u)dx=0,\\
\int_{\mathbb{R}^2}\operatorname{div}(u_t:\nabla u)dx=0,\quad \int_{\mathbb{R}^2}\operatorname{div}(u_t\ \operatorname{div}u)dx=0.
\end{gathered}
$$

Therefore, one has arrived at the desired equality.

$$
\frac{d}{dt}E(t)+F(t)=0.
$$

□

Note that when one estimates the functions $E(t)$ and $F(t)$ it is sufficient to consider the spatial integration over the light cone

$$
\Omega(t)=\{x\in\mathbb{R}^2:|x|\le R_0+bt\}
$$

since the finite speed of propagation property can be applied again to the solutions of the corresponding problem (1.1)-(1.2).

Now let us choose the functions $f(t)$ and $g(t)$ in Lemmas 2.2 and 2.3 as follows: When $C_0 > 2b$ we set

$$f(t) = (1+t)^2, \quad g(t) = (1+t), \tag{2.10}$$

when $b < C_0 \le 2b$, for an arbitrarily fixed $\delta > 0$ we choose

$$f(t) = (1+t)^{\frac{C_0}{b}-\delta}, \quad g(t) = \frac{C_0 - b\delta}{2b}(1+t)^{\frac{C_0}{b}-1-\delta}. \tag{2.11}$$

Then one has the following lemmas.

Lemma 2.2. *The smooth functions $f(t)$ and $g(t)$ defined by* (2.10) *and* (2.11) *satisfy the following properties: there exists a large $t_0 > 0$ such that for all $t \ge t_0 \ge 0$,*

(i) $2f(t)V(x) - f_t(t) - 2g(t) \ge 0$, $x \in \Omega(t)$,
(ii) $2g(t) - f_t(t) = 0$.

Proof. We will check only (i), because (ii) is quite easy. First, we consider the case (2.10) to check (i) when $C_0 > 2b$. In fact, for $x \in \Omega(t)$,

$$\begin{aligned}
2f(t)V(x) - f_t(t) - 2g(t) &= 2(1+t)^2V(x) - 2(1+t) - 2(1+t)\\
&= 2(1+t)\{(1+t)V(x) - 2\}\\
&\ge 2(1+t)\Big\{(1+t)\frac{C_0}{1+|x|} - 2\Big\}\\
&\ge 2(1+t)\Big\{(1+t)\frac{C_0}{1+bt+R_0} - 2\Big\}.
\end{aligned}$$

Here, we find that

$$\lim_{t\to\infty}\Big((1+t)\frac{C_0}{1+bt+R_0} - 2\Big) = \frac{C_0}{b} - 2,$$

so that there exists $t_0 > 0$ such that for all $t \in [t_0, +\infty)$,

$$(1+t)\frac{C_0}{1+bt+R_0} - 2 \ge \frac{1}{2}\Big(\frac{C_0}{b} - 2\Big).$$

Thus, we have the inequality

$$2(1+t)\Big\{(1+t)\frac{C_0}{1+bt+R_0} - 2\Big\} \ge (1+t)\Big(\frac{C_0}{b} - 2\Big), \quad 0 \le t_0 \le t.$$

From assumption of (I) of Theorem 1.1, since $\frac{C_0}{b} - 2 > 0$, one can check (i) when $C_0 > 2b$.

On the other hand, when $b < C_0 \le 2b$ it follows from the definition of (2.11) that

$$\begin{aligned}
&2f(t)V(x) - f_t(t) - 2g(t)\\
&= 2(1+t)^{\frac{C_0}{b}-\delta}V(x) - \Big(\frac{C_0}{b} - \delta\Big)(1+t)^{\frac{C_0}{b}-\delta-1} - \Big(\frac{C_0}{b} - \delta\Big)(1+t)^{\frac{C_0}{b}-\delta-1}\\
&= 2(1+t)^{\frac{C_0}{b}-\delta}V(x) - 2\Big(\frac{C_0}{b} - \delta\Big)(1+t)^{\frac{C_0}{b}-\delta-1}\\
&\ge 2(1+t)^{\frac{C_0}{b}-\delta-1}\Big\{(1+t)\frac{C_0}{1+|x|} - \Big(\frac{C_0}{b} - \delta\Big)\Big\}
\end{aligned}$$

$$\geq 2(1+t)^{\frac{C_0}{b}-\delta-1}\Big\{(1+t)\frac{C_0}{1+bt+R_0}-\Big(\frac{C_0}{b}-\delta\Big)\Big\},$$

for all $x \in \Omega(t)$.

Here, we find that

$$\lim_{t\to\infty}\Big((1+t)\frac{C_0}{1+bt+R_0}-\Big(\frac{C_0}{b}-\delta\Big)\Big)=\frac{C_0}{b}-\Big(\frac{C_0}{b}-\delta\Big)=\delta>0.$$

So, there exists $t_0 > 0$ such that for all $t \in [t_0, +\infty)$

$$(1+t)\frac{C_0}{1+bt+R_0}-\Big(\frac{C_0}{b}-\delta\Big)\geq\frac{1}{2}\delta.$$

Thus, we have the inequality

$$2(1+t)^{\frac{C_0}{b}-1-\delta}\Big\{(1+t)\frac{C_0}{1+bt+R_0}-\Big(\frac{C_0}{b}-\delta\Big)\Big\}\geq(1+t)^{\frac{C_0}{b}-1-\delta}\delta,\quad 0\leq t_0\leq t,$$

which implies desired estimate. □

Based on Lemmas 2.1 and 2.2 we have the inequality

$$\begin{aligned}\frac{d}{dt}\{f(t)E_u(t)+g(t)(u,u_t)\}&\leq\frac{d}{dt}\Big\{\frac{1}{2}\int_{\mathbb{R}^2}(g_t(t)-g(t)V(x))|u|^2dx\Big\}\\&\quad+\frac{1}{2}\int_{\mathbb{R}^2}(g_t(t)V(x)-g_{tt}(t))|u|^2dx\end{aligned}\tag{2.12}$$

for $t \geq t_0 \geq 0$.

We want to use the following lemma. However, when $C_0 > 2b$ and $f(t)$, $g(t)$ are given by (2.10) the proof of this lemma is easily done. We will prove the lemma only for $b < C_0 \leq 2b$ and $f(t)$, $g(t)$ given by(2.11).

Lemma 2.3. *Assume the functions $f(t)$ and $g(t)$ defined by* (2.10) *and* (2.11) *satisfy the following three properties, for $t \geq t_0 \geq 0$:*

(iii) $-g_{tt}(t) \leq \frac{C_1}{1+bt}$,
(iv) $V(x)g_t(t) \leq C_2V(x)$, *for all* $x \in \mathbb{R}^2$,
(v) $g_t(t) - V(x)g(t) \leq C_3$, *for all* $x \in \mathbb{R}^2$,

where C_i $(i = 1, 2, 3)$ are positive constants.

Proof. We proof only (iii) under the condition $b < C_0 \leq 2b$ with $f(t)$, $g(t)$ given by (2.11). The other cases are easy to proof. Note that

$$\begin{aligned}-g_{tt}(t)&=-\Big(\frac{C_0-\delta b}{2b}\Big)\Big(\frac{C_0}{b}-\delta-1\Big)\Big(\frac{C_0}{b}-\delta-2\Big)(1+t)^{\frac{C_0}{b}-\delta-3}\\&\leq C(1+t)^{\frac{C_0}{b}-\delta-3}\\&\leq C(1+t)^{\frac{2b}{b}-\delta-3}\\&=C(1+t)^{-1-\delta}\\&\leq C(1+t)^{-1}.\end{aligned}$$

Here, we find that

$$\frac{1}{\max\{1,b\}(1+t)}\leq\frac{1}{1+bt},$$

so that

$$\frac{1}{1+t}\leq\frac{\max\{1,b\}}{1+bt}\leq\frac{C}{1+bt}.$$

Thus, we have the following inequality with some $C_1 > 0$:

$$C(1+t)^{-1} \le \frac{C_1}{1+bt},$$

which implies the desired inequality. □

By integrating both sides of (2.12) over $[t_0, t]$, we find that

$$\begin{aligned}
&f(t)E_u(t) + g(t)(u(t,\cdot), u_t(t,\cdot)) - \{f(t_0)E_u(t_0) + g(t_0)(u(t_0,\cdot), u_t(t_0,\cdot))\} \\
&\le \frac{1}{2}\int_{\mathbb{R}^2}(g_t(t) - V(x)g(t))|u(t,x)|^2 dx - \frac{1}{2}\int_{\mathbb{R}^2}(g_t(t_0) - V(x)g(t_0))|u(t_0,x)|^2 dx \\
&\quad + \frac{1}{2}\int_{t_0}^t\int_{\mathbb{R}^2} V(x)g_t(s)|u(s,x)|^2\,dx\,ds + \frac{1}{2}\int_{t_0}^t\int_{\mathbb{R}^2}(-g_{tt}(s))|u(s,x)|^2\,dx\,ds.
\end{aligned}$$

It follows from Lemma 2.3 with large t_0 that

$$\begin{aligned}
&f(t)E_u(t) + g(t)(u(t,\cdot), u_t(t,\cdot)) \\
&\le C_4 + \frac{C_3}{2}\int_{\mathbb{R}^2}|u(t,x)|^2 dx + \frac{C_2}{2}\int_{t_0}^t\int_{\mathbb{R}^2} V(x)|u(s,x)|^2\,dx\,ds \\
&\quad + \frac{C_1}{2C_0}\int_{t_0}^t\int_{\mathbb{R}^2}\frac{C_0}{1+bs}|u(s,x)|^2\,dx\,ds,
\end{aligned} \tag{2.13}$$

where

$$C_4 = f(t_0)E_u(t_0) + g(t_0)(u(t_0,\cdot), u_t(t_0,\cdot)) - \frac{1}{2}\int_{\mathbb{R}^2}(g_t(t_0) - V(x)g(t_0))|u(t_0,x)|^2 dx.$$

We shall rely on the following powerful lemma, motivated by results from Ikehata [6] and Charão-Ikehata [3].

Lemma 2.4. *Let $u \in C([0,+\infty); (H^1(\mathbb{R}^2))^2) \cap C^1([0,+\infty); (L^2(\mathbb{R}^2))^2)$ be the solution to* (1.1)-(1.2). *Then*

$$\|u(t,\cdot)\|^2 + \int_0^t\int_{\mathbb{R}^2} V(x)|u(s,x)|^2\,dx\,ds \le \|u_0\|^2 + C\|d(\cdot)(V(\cdot)|u_0| + |u_1|)\|^2$$

for all $t \ge 0$, where $C > 0$ is a constant and $d(x) := \{1 + \log(1+|x|)\}(1+|x|)$.

Proof. First, we define an auxiliary function

$$\chi(t,x) = \int_0^t u(s,x)ds$$

that satisfies

$$\chi_{tt} - a^2\Delta\chi - (b^2 - a^2)\nabla(\operatorname{div}\chi) + V(x)\chi_t = V(x)u_0 + u_1, \tag{2.14}$$

$$\chi(0,x) = 0, \quad \chi_t(0,x) = u_0(x) \quad x \in \mathbb{R}^2. \tag{2.15}$$

Multiplying (2.14) by χ_t and integrating over $[0,t] \times \mathbb{R}^2$ we obtain

$$\begin{aligned}
&\frac{1}{2}\|\chi_t\|^2 + \frac{a^2}{2}\|\nabla\chi\|^2 + \frac{(b^2-a^2)}{2}\int_{\mathbb{R}^2}(\operatorname{div}\chi)^2 dx + \int_0^t\int_{\mathbb{R}^2} V(x)|\chi_t|^2\,dx\,ds \\
&= \frac{1}{2}\|u_0\|^2 + \int_{\mathbb{R}^2}(V(x)u_0 + u_1)\cdot\chi(t,x)dx.
\end{aligned} \tag{2.16}$$

The next step is to use the two dimensional Hardy-Sobolev inequality [10],

$$\int_{\mathbb{R}^2}\frac{|v(x)|^2}{d(x)^2}dx \le C\int_{\mathbb{R}^2}|\nabla v(x)|^2 dx, \quad v \in (H^1(\mathbb{R}^2))^2, \tag{2.17}$$

where

$$d(x) := \{1 + \log(1 + |x|)\}(1 + |x|).$$

The last term of (2.16) can be estimated by using (2.17) and the Schwarz inequality as follows.

$$\begin{aligned}
&\int_{\mathbb{R}^2} (V(x)u_0 + u_1)\chi(t,x)dx \\
&\le \int_{\mathbb{R}^2} (V(x)|u_0| + |u_1|)|\chi(t,x)|dx \\
&= \int_{\mathbb{R}^2} d(x)(V(x)|u_0| + |u_1|)\frac{|\chi(t,x)|}{d(x)}dx \\
&\le \Big\{\int_{\mathbb{R}^2} d(x)^2(V(x)|u_0| + |u_1|)^2 dx\Big\}^{1/2}\Big\{\int_{\mathbb{R}^2} \frac{|\chi(t,x)|^2}{d(x)^2}dx\Big\}^{1/2} \\
&\le \frac{1}{2\varepsilon}\|d(\cdot)(V(\cdot)|u_0| + |u_1|)\|^2 + \frac{\varepsilon}{2}\Big\|\frac{\chi(t,\cdot)}{d(\cdot)}\Big\|^2 \\
&\le \frac{1}{2\varepsilon}\|d(\cdot)(V(\cdot)|u_0| + |u_1|)\|^2 + \frac{\varepsilon C}{2}\|\nabla\chi\|^2,
\end{aligned}$$

where $\varepsilon > 0$ is an arbitrary real number and C is the constant in the Hardy-Sobolev inequality.

Combining the above estimate with (2.16), we conclude that

$$\begin{aligned}
&\frac{1}{2}\|\chi_t\|^2 + \frac{a^2}{2}(1 - \frac{\varepsilon C}{a^2})\|\nabla\chi\|^2 + \frac{(b^2 - a^2)}{2}\int_{\mathbb{R}^2} (\operatorname{div}\chi)^2 dx + \int_0^t \int_{\mathbb{R}^2} V(x)|\chi_t|^2\,dx\,ds \\
&\le \frac{1}{2}\|u_0\|^2 + C\|d(\cdot)(V(\cdot)|u_0| + |u_1|)\|^2.
\end{aligned} \tag{2.18}$$

Now, fixing $\varepsilon > 0$ such that $1 - \frac{\varepsilon C}{a^2} > 0$, we obtain

$$\|\chi_t\|^2 + \int_0^t \int_{\mathbb{R}^2} V(x)|\chi_t|^2\,dx\,ds \le \|u_0\|^2 + C\|d(\cdot)(V(\cdot)|u_0| + |u_1|)\|^2, \tag{2.19}$$

which implies the desired statement of Lemma 2.4, with $\chi_t = u$:

$$\|u\|^2 + \int_0^t \int_{\mathbb{R}^2} V(x)|u|^2\,dx\,ds \;\le\; \|u_0\|^2 + C\|d(\cdot)(V(\cdot)|u_0| + |u_1|)\|^2. \tag{2.20}$$

□

As consequence of Lemma 2.4, since

$$V(x) \ge \frac{C_0}{1 + |x|} \ge \frac{C_0}{1 + R_0 + bt} \ge \frac{1}{(1 + R_0)}\frac{C_0}{1 + bt}, \quad x \in \Omega(t)$$

we have

$$\frac{1}{(1 + R_0)}\int_0^t \int_{\mathbb{R}^2} \frac{C_0}{1 + bs}|u(s,x)|^2\,dx\,ds \le \|u_0\|^2 + C\|d(\cdot)(V(\cdot)|u_0| + |u_1|)\|^2, \tag{2.21}$$

where $C > 0$ is a constant.

After these preparations let us prove only Theorem 1.1. The proof of Proposition 1.2 part (II) is quite similar, using (2.11) in stead of (2.10).

Proof of Theorem 1.1. It follows from Lemma 2.4, (2.13) and (2.21) that

$$f(t)E_u(t) + g(t)(u(t,\cdot), u_t(t,\cdot)) \le C_{R_0}, \tag{2.22}$$

where the constant $C_{R_0} > 0$ depends on the L^2-norm of the initial data and R_0. By using the Schwarz inequality, the definition of the total energy, and Lemma 2.4 again we obtain

$$f(t)E_u(t) \le g(t)\|u(t,\cdot)\| \, \|u_t(t,\cdot)\| + C_{R_0} \le Cg(t)\sqrt{E_u(t)} + C_{R_0}, \quad t \ge t_0,$$

with some constant $C > 0$ depending on R_0 and the initial data. Therefore, if we set $X(t) = \sqrt{E_u(t)}$ for $t \in [t_0, +\infty)$, then we get

$$f(t)X^2(t) - Cg(t)X(t) - C_{R_0} \le 0. \tag{2.23}$$

By solving the quadratic inequality (2.23) for $X(t)$ (cf. [15]) we have

$$\sqrt{E_u(t)} \le \frac{Cg(t) + \sqrt{C^2g^2(t) + 4C_{R_0}f(t)}}{2f(t)}.$$

This inequality leads to

$$E_u(t) \le C\Big(\frac{g(t)}{f(t)}\Big)^2 + C\Big(\frac{1}{f(t)}\Big), \quad t \ge t_0,$$

which implies the desired estimate of the Theorem 1.1. Remember that $f(t) = (1+t)^2$ and $g(t) = (1+t)$ in the present case. □

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References

[1] R. C. Charão, G. P. Menzala; *On Huygens'Principle and perturbed elastic waves*, Diff. Integral Eqs. 5 (1992), 631-646.

[2] R. C. Charão, C. R. da Luz, R. Ikehata; *Optimal decay rates for the system of elastic waves in $\mathbb{R}^n$ with structural damping*, J. Evolution Eqs. 14 (2014), 197-210.

[3] R. C. Charão, R. Ikehata; *Decay of solutions for a semilinear system of elastic waves in an exterior domain with damping near infinity*, Nonlinear Anal. 67 (2007), 398-429.

[4] R. C. Charão, R. Ikehata; *Energy decay rates of elastic waves in unbounded domain with potential type of damping*, J. Math. Anal. Appl. 380 (2011), 46-56.

[5] J. L. Horbach; *Taxas de decaimento para a energia associada a um sistema semilinear de ondas elásticas em $\mathbb{R}^n$ com potencial do tipo dissipativo*, Master Thesis, Department of Mathematics, Federal University of Santa Catarina, Brazil, 2013 (in Portuguese).

[6] R. Ikehata; *Fast decay of solutions for linear wave equations with dissipation localized near infinity in an exterior domain*, J. Differential. Eqs. 188 (2003), 390-405.

[7] R. Ikehata, Y. Inoue; *Total energy decay for semilinear wave equations with a critical potential type of damping*, Nonlinear Anal. 69 (2008), 1369-1401.

[8] R. Ikehata, G. Todorova, B. Yordanov; *Optimal decay rate of the energy for wave equations with critical potential*, J. Math. Soc. Japan 65 (2013), 183-236.

[9] B. V. Kapitonov; *Decrease in the solution of the exterior boundary value problem for a system of elastic theory*, Differential Eqs. 22 (1986), 332-337.

[10] A. Matsumura, N. Yamagata; *Global weak solutions of the Navier-Stokes equations for multidimensional compressible flow subject to large external potential forces*, Osaka J. Math. 38 (2001), 399-418.

[11] K. Mochizuki; *Scattering theory for wave equations with dissipative term*, Publ. RIMS. Kyoto Univ. 12(2) (1976), 383-390.

[12] K. Mochizuki, H. Nakazawa; *Energy decay and asymptotic behavior of solutions to the wave equations with linear dissipation*, Publ. RIMS. Kyoto Univ. 32 (1996), 401-414.

[13] C. S. Morawetz; *The decay of solutions of the exterior initial boundary-value problem for the wave equation*, Comm. Pure Appl. Math. 14 (1961) 561-568.

[14] G. Todorova, B. Yordanov; *Weighted L^2-estimates for dissipative wave equations with variable coefficients*, J. Differential Eqs. 246(12) (2009), 4497-4518.

[15] H. Uesaka; *The total energy decay of solutions for the wave equation with a dissipative term*, J. Math. Kyoto Univ. 20 (1980), 57-65.

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