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RADIAL POSITIVE SOLUTIONS FOR A NONPOSITONE PROBLEM IN AN ANNULUS

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Abstract. The main purpose of this article is to prove the existence of radial positive solutions for a nonpositone problem in an annulus when the nonlinearity is superlinear and has more than one zero.

1. Introduction

In this article we study the existence of radial positive solutions for the boundary-value problem

$$\begin{gathered} -\Delta u(x) = \lambda f(u(x)) \quad x \in \Omega, \\ u(x) = 0 \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where $\lambda > 0$, $f : [0, +\infty) \to \mathbb{R}$ is a continuous nonlinear function that has more than one zero, and $\Omega \subset \mathbb{R}^N$ is the annulus: $\Omega = C(0, R, \widehat{R}) = \{x \in \mathbb{R}^N : R < |x| < \widehat{R}\}$ ($N > 2$, $0 < R < \widehat{R}$).

When f is a nondecreasing nonlinearity satisfying $f(0) < 0$ (the nonpositone case) and has only one zero, problem (1.1) has been studied by Arcoya and Zertiti [1] and by Hakimi and Zertiti in a ball when f has more than one zero [5].

We observe that the existence of radial positive solutions of (1.1) is equivalent to the existence of positive solutions of the problem

$$\begin{gathered} -u''(r) - \frac{N-1}{r}u'(r) = \lambda f(u(r)) \quad R < r < \widehat{R} \\ u(R) = u(\widehat{R}) = 0. \end{gathered} \tag{1.2}$$

Our main objective in this article is to prove that the result of existence of radial positive solutions of the problem (1.1) remains valid when f has more than one zero and is not increasing entirely on $[0, +\infty)$; see [1, Theorem 2.4].

Remark 1.1. In this article, we assume (without loss of generality) that f has exactly three zeros.

We assume that the map $f : [0, +\infty) \to \mathbb{R}$ satisfies the following hypotheses:

- (F1) $f \in C^1([0, +\infty), \mathbb{R})$ such that f has three zeros $\beta_1 < \beta_2 < \beta_3$, with $f'(\beta_i) \neq 0$ for all $i \in \{1, 2, 3\}$. Moreover, $f' \geq 0$ on $[\beta_3, +\infty)$.
- (F2) $f(0) < 0$.

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(F3) $\lim_{u\to+\infty} \frac{f(u)}{u} = +\infty$.
(F4) The function $h(u) = NF(u) - \frac{N-2}{2} f(u)u$ is bounded from below in $[0,+\infty)$, where $F(x) = \int_0^x f(r)dr$.

Remark 1.2. We observe that our arguments also work in the case $\Omega = B(O,R)$, improving slightly the results in [5]. In fact in [5], besides imposing that f is increasing, we need (F1), (F2), (F3) and that For some $k \in (0,1)$,

$$\lim_{d\to+\infty} \big(\frac{d}{f(d)}\big)^{N/2}\big(F(kd) - \frac{N-2}{2N}\, df(d)\big) = +\infty.$$

On the other hand, it is clear that our hypothesis (F4) is more general than this assumption.

For a nonexistence result of positive solutions for superlinearities satisfying (F1), (F2) and (F3) see [6]. Also see [3] for existence and nonexistence of positive solutions for a class of superlinear semipositone systems, and [4] for existence and multiplicity results for semipositone problems.

2. Main Result

In this section, we give the main result in this work. More precisely we shall prove the following theorem.

Theorem 2.1. *Assume that the hypotheses* (F1)–(F4) *are satisfied. Then there exists a positive real number λ_* such that if $\lambda < \lambda_*$, problem* (1.1) *has at least one radial positive solution.*

To prove Theorem 2.1, we need the next four technical lemmas. The first lemma assures the existence of a unique solution $u(.,d,\lambda)$ of (1.2) in $[R,+\infty)$ for all $\lambda, d > 0$. The three last lemmas concern the behaviour of the solution of (1.2).

Remark 2.2. In this article we follow the work of Arcoya and Zertiti [1], and we note that the proofs of Lemmas 2.4 and 2.7 are analogous with those of [1, Lemmas 1.1 and 2.3]. On the other hand, the proofs of the second and third lemmas are different from that of [1, Lemma 2.1 and 2.2]. This is so because our f has more than one zero. So we apply the Shooting method. For this we consider the auxiliary boundary-value problem

$$\begin{gathered} -u''(r) - \frac{N-1}{r} u'(r) = \lambda f(u(r)), \quad r > R \\ u(R) = 0, \quad u'(R) = d, \end{gathered} \tag{2.1}$$

where d is the parameter of Shooting method.

Remark 2.3. For suitable d, problem (2.1) has a solution $u := u(.,d,\lambda)$ such that $u > 0$ on $(R,\widehat{R})$ and $u(\widehat{R}) = 0$. So, such solution u of (2.1) is also a positive solution of (1.2).

In this sequel, we suppose that the nonlinearity $f \in C^1([0,+\infty))$ is always extended to $\mathbb{R}$ by $f|_{(-\infty,0)} \equiv f(0)$.

Lemma 2.4. *Let $\lambda, d > 0$ and $f \in C^1([0,+\infty))$ a function which is bounded from below. Then problem* (2.1) *has a unique solution $u(.,d,\lambda)$ defined in $[R,+\infty)$, In addition, for every $d > 0$ there exist $M = M(d) > 0$ and $\lambda = \lambda(d) > 0$ such that*

$$\max_{r\in[R,\widehat{R}]} |u(r,d,\lambda)| \le M, \quad \forall \lambda \in (0,\lambda(d)).$$

Proof. The proof of the existence is given in two steps. In first, we show the existence and uniqueness of a local solution of (2.1); i.e, the existence a $\varepsilon = \varepsilon(d, \lambda) > 0$ such that (2.1) has a unique solution on $[R, R+\varepsilon]$. In the second step we prove that this unique solution can be extended to $[R, +\infty)$.

Step 1: (Local solution). Consider the problem

$$\begin{gathered} -u''(r) - \frac{N-1}{r}u'(r) = \lambda f(u(r)), \quad r > R_1 \\ u(R_1) = a, \quad u'(R_1) = b, \end{gathered} \tag{2.2}$$

where $R_1 \geq R$. Let u be a solution of (2.2). Multiplying the equation by r^{N-1} and using the initial conditions, we obtain

$$u'(r) = \frac{1}{r^{N-1}}\big\{R_1^{N-1}b - \lambda\int_{R_1}^r s^{N-1}f(u(s))ds\big\}. \tag{2.3}$$

from which u satisfies

$$u(r) = a + \frac{bR_1^{N-1}}{N-2}\Big(\frac{1}{R_1^{N-2}} - \frac{1}{r^{N-2}}\Big) - \lambda\int_{R_1}^r \frac{1}{t^{N-1}}\Big[\int_{R_1}^t s^{N-1}f(u(s))ds\Big]dt. \tag{2.4}$$

Conversely, if u is a continuous function satisfying (2.4), then u is a solution of (2.2).

Hence, to prove the existence and uniqueness of a solution u of (2.2) defined in some interval $[R_1, R_1+\varepsilon]$, it is sufficient to show the existence of a unique fixed point of the operator T defined on X (the Banach space of the real continuous functions on $[R_1, R_1+\varepsilon]$ with the uniform norm),

$$\begin{aligned} T : X = C([R_1, R_1+\varepsilon], \mathbb{R}) &\to X \\ v &\mapsto Tv, \end{aligned}$$

where

$$(Tv)(r) = a + \frac{bR_1^{N-1}}{N-2}\Big(\frac{1}{R_1^{N-2}} - \frac{1}{r^{N-2}}\Big) - \lambda\int_{R_1}^r \frac{1}{t^{N-1}}\Big[\int_{R_1}^t s^{N-1}f(v(s))ds\Big]dt, \tag{2.5}$$

for all $r \in [R_1, R_1+\varepsilon]$ and $v \in X$. To check this, Let $\delta > 0$ such that $\delta > |a|$ and $\overline{B}(0,\delta) = \{u \in X : \|u\| \leq \delta\}$. For all $u, v \in \overline{B}(0,\delta)$, we have

$$(Tu - Tv)(r) = \lambda\int_{R_1}^r \frac{1}{t^{N-1}}\Big[\int_{R_1}^t s^{N-1}\{f(v(s)) - f(u(s))\}ds\Big]dt,$$

then

$$\begin{aligned} |(Tu - Tv)(r)| &\leq \lambda\int_{R_1}^r \frac{1}{t^{N-1}}[\int_{R_1}^t s^{N-1}\sup_{\zeta\in(0,\delta]}|f'(\zeta)|\,|v(s) - u(s)|ds]dt \\ &\leq \lambda\int_{R_1}^r \frac{1}{t^{N-1}}\Big[\int_{R_1}^t s^{N-1}ds\Big]dt \sup_{\zeta\in(0,\delta]}|f'(\zeta)|\,\|u - v\|. \end{aligned}$$

However,

$$\begin{aligned} \int_{R_1}^r \frac{1}{t^{N-1}}[\int_{R_1}^t s^{N-1}ds]dt &= \int_{R_1}^r \frac{1}{t^{N-1}}[\frac{t^N}{N} - \frac{R_1^N}{N}]dt \\ &\leq \int_{R_1}^r \frac{t}{N}dt - \frac{R_1^N}{N}\int_{R_1}^r \frac{dt}{t^{N-1}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2N}(r^2 - R_1^2)_- \frac{R_1^N}{N}\Big(\frac{1}{(2-N)r^{N-2}} - \frac{1}{(2-N)R_1^{N-2}}\Big)\\
&= \frac{r^2 - R_1^2}{2N} + \frac{1}{N(N-2)}.\frac{R_1^N}{r^{N-2}} - \frac{R_1^2}{N(N-2)}\\
&\le \frac{(R_1+\varepsilon)^2 - R_1^2}{2N}, \quad \text{because } r \in [R_1, R_1+\varepsilon]\\
&= \frac{\varepsilon(2R_1+\varepsilon)}{2N};
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\|Tu - Tv\| &\le \frac{\varepsilon(2R_1+\varepsilon)}{2N}\lambda \sup_{\zeta\in[0,\delta]} |f'(\zeta)|\|u - v\|\\
&\le \frac{\varepsilon(R_1+\varepsilon)}{N}\lambda \sup_{\zeta\in[0,\delta]} |f'(\zeta)|\|u - v\|.
\end{aligned}
$$

Hence

$$
\|Tu - Tv\| \le \frac{\lambda}{N} \sup_{\zeta\in(0,\delta]} |f'(\zeta)|\varepsilon(R_1+\varepsilon)\|u - v\|. \tag{2.6}
$$

Similarly,

$$
\|Tu\| \le |a| + \frac{|b|R_1^{N-1}}{N-2}\Big(\frac{1}{R_1^{N-2}} - \frac{1}{(R_1+\varepsilon)^{N-2}}\Big) + \frac{\lambda}{N} \sup_{\zeta\in[0,\delta]} |f(\zeta)|\varepsilon(R_1+\varepsilon). \tag{2.7}
$$

Now, by (2.6) and (2.7), we can choose $\varepsilon = \varepsilon(\delta) > 0$ (depending on δ) sufficiently small such that T is a contraction from $\overline{B}(0,\delta)$ to $\overline{B}(0,\delta)$. Consequently, T has a fixed point u in $\overline{B}(0,\delta)$. The fixed point u is unique in X for a δ as large as we wanted.

Step 2: Let $u(.) = u(., d, \lambda)$ be the unique solution of (2.1) (we take $a = 0$, $b = d$ and $R_1 = R$ in (2.2)), and denote by $[R, R(d,\lambda))$ its maximal domain. We shall prove by contradiction that $R(d,\lambda) = +\infty$. For it, assume $R^* := R(d,\lambda) < +\infty$. u is bounded on $[R, R^*)$. In fact, using (2.4) and that f is bounded from below, we have

$$
\begin{aligned}
\frac{dR}{N-2} &\ge \frac{dR^{N-1}}{N-2}\Big(\frac{1}{R^{N-2}} - \frac{1}{r^{N-2}}\Big)\\
&= u(r) + \lambda \int_R^r \frac{1}{t^{N-1}}\Big[\int_R^t s^{N-1}\, f(u(s))ds\Big]dt\\
&\ge u(r) + \lambda \inf_{\xi\in[0,+\infty)} f(\xi) \int_R^{R^*} \frac{1}{t^{N-1}}\Big[\int_R^t s^{N-1} ds\Big]dt, \quad \forall r \in [R, R^*),
\end{aligned}
$$

then, there exists $K_1 > 0$ such that $u(r) \le K_1$ for all $r \in [R, R^*)$.

On the other hand, using again (2.4), we obtain

$$
\begin{aligned}
u(r) &\ge \frac{dR^{N-1}}{N-2}\Big(\frac{1}{R^{N-2}} - \frac{1}{r^{N-2}}\Big) - \lambda \max_{\xi\in[0,K_1]} f(\xi) \int_R^{R^*} \frac{1}{t^{N-1}}\Big[\int_R^t s^{N-1} ds\Big]dt\\
&\ge -K_2, \quad \forall r \in [R, R^*),
\end{aligned}
$$

for convenient $K_2 > 0$. Hence u is bounded.

By using this and (2.3) and (2.4), we deduce that $\{u(r_n)\}$ and $\{u'(r_n)\}$ are the Cauchy sequence for all sequence $(r_n) \subset [R, R^*)$ converging to R^*. This is equivalent to the existence of the finite limits

$$\lim_{r\to R^{*-}} u(r) = a \quad \text{and} \quad \lim_{r\to R^{*-}} u'(r) = b.$$

Now, consider the problem

$$\begin{gathered} -v''(r) - \frac{N-1}{r} v'(r) = \lambda f(v(r)), \quad R^* < r \\ v(R^*) = a, \quad v'(R^*) = b \end{gathered} \tag{2.8}$$

and by step 1, we deduce the existence of a positive number $\varepsilon > 0$ and a solution v of this problem in $[R^*, R^* + \varepsilon]$. It is easy to see that

$$w(r) = \begin{cases} u(r), & \text{if } R \le r < R^* \\ v(r), & \text{if } R^* \le r \le R^* + \varepsilon, \end{cases}$$

is a solution of (2.1) in $[R, R^* + \varepsilon]$ which is a contradiction, so $R^* = +\infty$.

To prove the second part of the lemma, we consider the operator T defined by (2.5) on $X_0 = C([R, \widehat{R}], \mathbb{R})$ with $R_1 = R$, $a = 0$ and $b = d$. Taking $M = \delta > \frac{2dR}{N-2}$ and

$$\lambda(d) = \min\Big\{\frac{M}{2M_1 \max_{\xi\in[0,M]} |f(\xi)|}, \frac{1}{M_1 \max_{\xi\in[0,M]} |f'(\xi)|}\Big\}$$

with $M_1 = \int_R^{\widetilde{R}} \frac{1}{t^{N-1}} \big[\int_R^t s^{N-1} ds\big] dt$.

By (2.6) and (2.7), we deduce that T is a contraction from $\overline{B}(0, M, X_0)$ into $\overline{B}(0, M, X_0)$, where

$$\overline{B}(0, M, X_0) = \{u \in X_0 : \max_{r\in[R,\widehat{R}]} |u(r)| \le M\}.$$

So, the unique fixed point of T belongs to $\overline{B}(0, M, X_0)$. The lemma is proved. □

Lemma 2.5. *Assume* (F1), (F2) *and let* $d_0 > 0$. *Then there exists* $\lambda_1 = \lambda_1(d_0) > 0$ *such that the unique solution* $u(r, d_0, \lambda)$ *of* (2.1) *satisfies*

$$u(r, d_0, \lambda) > 0, \quad \forall r \in (R, \widehat{R}], \forall \lambda \in (0, \lambda_1).$$

Proof. For $\lambda > 0$, we consider the set

$$\Psi = \{r \in (R, \widehat{R}) : u(.) = u(., d_0, \lambda) \text{is nondecreasing in } (R, r)\}.$$

Since $u'(R) = d_0 > 0$, Ψ is nonempty, and clearly bounded from above. Let $r_1 = \sup \Psi$ (which depends on λ). We have two cases:

Case 1. If $r_1 = \widehat{R}$, the proof is complete.

Case 2. If $r_1 < \widehat{R}$, we shall prove $u(.) = u(., d_0, \lambda) > 0$, for all $r \in (R, \widehat{R}]$ for all λ sufficiently small. In order to show it, assume that $r_1 < \widehat{R}$. Then $u'(r_1) = 0$, and since

$$u'(r) = \frac{1}{r^{N-1}} \Big[R^{N-1} d_0 - \lambda \int_R^r s^{N-1} f(u(s)) ds\Big],$$

then $u(r_1) > \beta_1$. Hence the set $\Gamma = \{r \in [r_1, \widehat{R}] : u(t) \ge \beta_1 \text{ and } u'(t) \le 0, \forall t \in [r_1, r]\}$ is nonempty and bounded from above. Let $r_2 = \sup \Gamma > r_1$. We shall prove that for λ sufficiently small $r_2 = \widehat{R}$. We observe that $u'(r) \le 0$ for all $r \in \Gamma$, then

$u(r) \le u(r_1)$, for all $r \in [R, r_2]$. Therefore, by the mean value theorem, there exists $c \in (r_1, r_2)$ such that

$$u(r_2) = u(r_1) + u'(c)(r_2 - r_1),$$

but

$$u'(c) = -\frac{\lambda}{c^{N-1}} \int_{r_1}^{c} t^{N-1} f(u(t))dt,$$

then

$$u(r_2) > u(r_1) - \frac{\lambda \widehat{R}}{N} \sup_{[\beta_1, u(r_1)]} |f(\zeta)|(\widehat{R} - R).$$

If $M = M(d_0) > 0$ and $\lambda(d_0) > 0$ (defined in Lemma 2.4), then

$$\beta_1 < u(r_1) \le M, \quad \forall \lambda \in (0, \lambda(d_0)).$$

Let $K = K(d_0) > 0$ such that $|f(\zeta)| < K(\zeta - \beta_1)$ for all $\zeta \in (\beta_1, M]$. We deduce that

$$u(r_2) > u(r_1) - \frac{\lambda K \widehat{R}}{N}(\widehat{R} - R)(u(r_1) - \beta_1), \quad \forall \lambda \in (0, \lambda(d_0)),$$

Thus, if $\lambda \in (0, \lambda_1)$ with $\lambda_1 = \min\{\lambda(d_0), \frac{N}{K\widehat{R}(\widehat{R}-R)}\}$ we have $u(r_2) > \beta_1$, which implies that $r_2 = \widehat{R}$. □

Lemma 2.6. *Assume* (F1)–(F3). *Let* $\lambda > 0$. *Then*

(i) $\lim_{d \to +\infty} r_1(d, \lambda) = R$

(ii) $\lim_{d \to +\infty} u(r_1, d, \lambda) = +\infty$

Proof. If (i) is not true, then there exists $\varepsilon > 0$ so that for all n there exists d_n such that

$$|r_1(d_n, \lambda) - R| \ge \varepsilon,$$

from which

$$r_1(d_n, \lambda) \ge R + \varepsilon \quad (\text{because } r_1(d_n, \lambda) \ge R),$$

then there exists $R_0 \in (R, \widehat{R})$ and a sequence $(d_n) \subset (0, +\infty)$ converging to ∞ such that $u_n := u(., d_n, \lambda)$ satisfies

$$u_n(r) > 0, \quad u_n'(r) \ge 0, \quad \forall r \in (R, R_0], \quad \forall n \in \mathbb{N}.$$

Let $\overline{r} = \frac{R+R_0}{2}$. By the equality

$$u_n(\overline{r}) = \frac{d_n R^{N-1}}{N-2}\Big(\frac{1}{R^{N-2}} - \frac{1}{\overline{r}^{N-2}}\Big) - \lambda \int_R^{\overline{r}} \frac{1}{t^{N-1}} \Big[\int_R^t s^{N-1} f(u_n(s))ds\Big]dt,$$

we observe that $(u_n(\overline{r}))$ is unbounded. Passing to a subsequence of (d_n), if it is necessary, we can suppose $\lim_{n \to +\infty} u_n(\overline{r}) = +\infty$. Now, consider

$$M_n = \inf\Big\{\frac{f(u_n(r))}{u_n(r)} : r \in (\overline{r}, R_0)\Big\}.$$

By (F3), $\lim_{n \to +\infty} M_n = +\infty$. Let $n_0 \in \mathbb{N}$ such that $\lambda M_{n_0} > \mu_3$ where μ_3 is the third eigenvalue of $-[\frac{d^2}{d^2 r} + \frac{N-1}{r}\frac{d}{dr}]$ in $(\overline{r}, R_0)$ with Dirichlet boundary conditions.

We take a nonzero eigenfunction ϕ_3 associated to μ_3; i.e.,

$$\phi_3''(r) + \frac{N-1}{r}\phi_3'(r) + \mu_3 \phi_3(r) = 0, \quad \overline{r} < r < R_0$$

$$\phi_3(\overline{r}) = 0 = \phi_3(R_0).$$

Since ϕ_3 has two zeros in $(\overline{r}, R_0)$, we deduce from the Sturm comparison Theorem [7] that u_{n_0} has at least one zero in $(\overline{r}, R_0)$. Which is a contradiction (because $u_n(r) > 0$ for all $r \in (R, R_0]$ and all $n \in \mathbb{N}$).

(ii) Let r_1 be the same number as in the proof of lemma 2.5. we have $u'(r_1) = 0$. However,

$$u'(r_1) = \frac{1}{r_1^{N-1}}\Big[dR^{N-1} - \lambda \int_R^{r_1} t^{N-1} f(u(t))dt\Big],$$

then

$$dR^{N-1} = \lambda \int_R^{r_1} t^{N-1} f(u(t))dt.$$

Hence

$$\lim_{d\to+\infty} u(r_1, d, \lambda) = +\infty.$$

□

Lemma 2.7. *Assume* (F1)–(F4) *and let γ_1 be a positive number. Then there exists a $\lambda_2 > 0$ such that:*

(a) *For all $\lambda \in (0, \lambda_2)$ the unique solution $u(r, d, \lambda)$ of* (2.1) *satisfies*

$$u^2(r, d, \lambda) + u'^2(r, d, \lambda) > 0, \quad \forall r \in [R, \widehat{R}],\ \forall d \ge \gamma_1.$$

(b) *For all $\lambda \in (0, \lambda_2)$, there exists $d > \gamma_1$ such that $u(r, d, \lambda) < 0$ for some $r \in (R, \widehat{R}]$.*

Proof. (a) Let $\lambda, d > 0$ and $u(.) = u(., d, \lambda)$ the unique solution of (2.1). We define the auxiliary function H on $[R, +\infty)$ by setting

$$H(r) = r\frac{u'^2(r)}{2} + \lambda r F(u(r)) + \frac{N-2}{2} u(r)u'(r), \quad \forall r \in [R, +\infty).$$

We can prove, as in [2, 5] the next identity of Pohozaev-type:

$$r^{N-1}H(r) = t^{N-1}H(t) + \lambda \int_t^r s^{N-1}[NF(u(s)) - \frac{N-2}{2} f(u(s))u(s)]ds, \quad \forall t \in [R, r].$$

Taking $t = R$, in this identity we obtain

$$r^{N-1}H(r) = \frac{R^N d^2}{2} + \lambda \int_R^r s^{N-1}\Big[NF(u(s)) - \frac{N-2}{2} f(u(s))u(s)\Big]ds,$$

hence

$$r^{N-1}H(r) \ge \frac{R^N d^2}{2} + \lambda m\Big(\frac{r^N}{N} - \frac{R^N}{N}\Big), \tag{2.9}$$

where m is a strictly negative real such that $NF(u) - \frac{N-2}{2} f(u)u \ge m$ for all $u \in \mathbb{R}$, so

$$r^{N-1}H(r) \ge \frac{R^N \gamma_1^2}{2} + \lambda m\Big(\frac{\widehat{R}^N}{N} - \frac{R^N}{N}\Big), \quad \forall r \in [R, \widehat{R}],\ \forall d \ge \gamma_1.$$

We note that m exists by (f_4). Hence there exists $\lambda_2 > 0$ such that

$$H(r) > 0, \quad \forall r \in [R, \widehat{R}],\ \forall d \ge \gamma_1,\ \forall \lambda \in (0, \lambda_2). \tag{2.10}$$

Therefore,

$$u^2(r, d, \lambda) + u'^2(r, d, \lambda) > 0, \quad \forall r \in [R, \widehat{R}], \quad \forall d \ge \gamma_1,\ \forall \lambda \in (0, \lambda_2).$$

(b) We argue by contradiction: fix $\lambda \in (0, \lambda_2)$ and suppose that

$$u(r, d, \lambda) \ge 0, \quad \forall r \in [R, \widehat{R}],\ \forall d \ge \gamma_1.$$

Choose $\varrho > 0$ such that there exists a solution of $\omega'' + \frac{N-1}{r}\omega' + \varrho\omega = 0$, where

$$\omega(0) = 1, \quad \omega'(0) = 0, \quad \frac{\widehat{R} - R}{4} \text{ is the first zero of } \omega.$$

We note (see [8]) that $\omega(r) \geq 0$ and $\omega'(r) < 0$, for all $r \in (0, \frac{\widehat{R}-R}{4}]$.

By (F3), there exists $d_0 = d_0(\lambda) > \gamma_1$ such that

$$\frac{f(u)}{u} \geq \frac{\varrho}{\lambda}, \quad \forall u \geq d_0. \tag{2.11}$$

On the other hand, let $r_1 = r_1(d, \lambda)$ and $r_2 = r_2(d, \lambda)$ be the same numbers as in the proof of Lemma 2.5. By Lemma 2.6, we can assume that

$$r_1 = r_1(d, \lambda) < R + \frac{\widehat{R} - R}{4} < \widehat{R} \quad \text{and} \quad u(r_1, d, \lambda) > d_0, \quad \forall d \geq d_0,$$

the definitions of r_1 and r_2 imply

$$u'(r, d, \lambda) \leq 0, \quad \forall r \in [r_1, \widehat{R}], \quad \forall d \geq d_0. \tag{2.12}$$

Define $v(r) = u(r_1)\omega(r - r_1)$, hence $v''(r) + \frac{N-1}{r-r_1}v'(r) + \varrho v(r) = 0$, for all $r \in (r_1, r_1 + \frac{\widehat{R}-R}{4})$ with $u(r_1) = v(r_1)$, $v'(r_1) = 0$, $v(r_1 + \frac{\widehat{R}-R}{4}) = 0$, $v(r) > 0$ and $v'(r) \leq 0$, for all $r \in (r_1, r_1 + \frac{\widehat{R}-R}{4})$, thus

$$v''(r) + \frac{N-1}{r}v'(r) + \varrho v(r) \geq 0, \quad \forall r \in (r_1, r_1 + \frac{\widehat{R} - R}{4}),$$

if $u(r) \geq d_0$, for all $r \in (r_1, r_1 + \frac{\widehat{R}-R}{4})$, hence by (2.11) and the Sturm comparison theorem (see [7]), u have a zero in $(r_1, r_1 + \frac{\widehat{R}-R}{4})$. Which is a contradiction. Hence, there exists $r^* \in (r_1, r_1 + \frac{\widehat{R}-R}{4})$ such that $u(r^*, d, \lambda) = d_0$.

Now, consider the energy function

$$E(r, d, \lambda) = \frac{u'^2(r, d, \lambda)}{2} + \lambda F(u(r, d, \lambda)), \quad \forall r \geq R.$$

By (2.9), (2.12) and the equality $H(r) = rE(r) + \frac{N-2}{2}u(r)u'(r)$, we obtain

$$\begin{aligned} r^N E(r, d, \lambda) &\geq r^{N-1} H(r, d, \lambda) \\ &\geq \frac{R^N d^2}{2} + \lambda m\Big(\frac{\widehat{R}^N}{N} - \frac{R^N}{N}\Big), \quad \forall r \in [r_1, \widehat{R}], \end{aligned}$$

hence, there exists $d_1 = d_1(\lambda) \geq d_0$ such that

$$E(r, d, \lambda) \geq \lambda F(d_0) + \frac{2}{(\widehat{R} - R)^2} d_0^2, \quad \forall r \in [r_1, \widehat{R}], \quad \forall d \geq d_1.$$

However,

$$E'(r) = -\frac{N-1}{r}u'(r)^2 \leq 0, \quad \forall r \in [R, \widehat{R}],$$

hence

$$E(r^*) \geq E(r), \quad \forall r \in [r^*, \widehat{R}],$$

thus

$$\frac{u'(r)^2}{2} \geq \frac{2d_0^2}{(\widehat{R} - R)^2}, \quad \forall r \in [r^*, \widehat{R}], \ \forall d \geq d_1,$$

and by (2.12), we deduce

$$u'(r) \leq -\frac{2d_0}{\widehat{R}-R}\ , \quad \forall r \in [r^*, \widehat{R}],\ \forall d \geq d_1.$$

The mean value theorem implies that there exists a $c \in (r^*, r^* + \frac{\widehat{R}-R}{2})$ such that

$$u\Big(r^* + \frac{\widehat{R}-R}{2}\Big) - u(r^*) = \frac{\widehat{R}-R}{2} u'(c).$$

Hence

$$u\Big(r^* + \frac{\widehat{R}-R}{2}\Big) \leq 0.$$

Which is a contradiction (because $u'(r^* + \frac{\widehat{R}-R}{2}) < 0$). $\square$

Proof of theorem 2.1. Let $d_0 > 0$. By Lemmas 2.5 and 2.7, there exists $\lambda_* > 0$ such that, if $\lambda \in (0, \lambda_*)$ then

(i) $u(r, d_0, \lambda) > 0$ for all $r \in (R, \widehat{R}]$
(ii) $u'(r, d, \lambda)^2 + u(r, d, \lambda)^2 > 0$ for all $r \in [R, \widehat{R}]$ and all $d \geq d_0$,
(iii) there exist $d_1 > d_0$ and $r \in (R, \widehat{R}]$ such that $u(r, d_1, \lambda) < 0$.

Define $\Gamma = \{d \geq d_0 \ \mid u(r, \overline{d}, \lambda) > 0,\ \forall r \in (R, \widehat{R}),\ \forall \overline{d} \in [d_0, d]\}$. By (i), $d_0 \in \Gamma$ then Γ is nonempty. In addition, by (iii) Γ is bounded from above by d_1. Take $d^* = \sup \Gamma$. it is clear that

$$u(r, d^*, \lambda) \geq 0, \quad \forall r \in [R, \widehat{R}].$$

Since $d^* < d_1$, we deduce (using (ii)) that

$$u(r, d^*, \lambda) > 0, \quad \forall r \in (R, \widehat{R}). \tag{2.13}$$

$u(., d^*, \lambda)$ will be a solution searching, if we prove $u(\widehat{R}, d^*, \lambda) = 0$. Assume that $u(\widehat{R}, d^*, \lambda) > 0$. Then by (2.13) and the fact that $u'(R, d^*, \lambda) = d^* > 0$, we have that

$$u(r, d, \lambda) > 0, \quad \forall r \in (R, \widehat{R}], \forall d \in [d^*, d^* + \delta],$$

where δ is sufficiently small. Hence $d^* + \delta \in \Gamma$, which is a contradiction. Therefore, $u(\widehat{R}, d^*, \lambda) = 0$. $\square$

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