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INTEGRAL INEQUALITIES WITH TIME DELAY IN TWO INDEPENDENT VARIABLES

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Abstract. In this article, we generalize some retarded integral inequalities in two independent variables to more general situations. These integral inequalities can be applied as tools to the study of certain class of integral and differential equations with time delay.

1. Introduction

Integral inequalities play an important role in the qualitative analysis of the solutions to differential and integral equations. Over the years many retarded inequalities have been discovered (see [1, 8, 10]). The literature on such inequalities and their applications is vast; see [3, 5, 8] and the references given therein.

In his study of boundedness of solutions to linear second order differential equations, Pachpatte [9] established and applied the following useful nonlinear integral inequality.

$$u(t) \leq a + \int_{t_0}^{t} f(s)w(u(s))ds \tag{1.1}$$

where $a > 0$ is a constant. Replacing t by a function $b(t)$ in (1.1), Lipovan [6] investigates the retarded Gronwall-like inequalities

$$u(t) \leq a + \int_{t_0}^{t} f(s)w(u(s))ds + \int_{b(t)}^{b(t)} g(s)w(u(s))ds \tag{1.2}$$

In recent years, Pachpatte [10] discovered some new integral inequalities involving functions in two independent variables. These inequalities are applied to study the boundedness and uniqueness of the solutions of the following terminal value problem for the hyperbolic partial differential equation (1.3) with conditions (1.4),

$$D_1D_2u(x,y) = h(x,y,u(x,y)) + r(x,y), \tag{1.3}$$

$$u(x,\infty) = \sigma_\infty(x), \quad u(\infty,y) = \tau_\infty(y), \quad \phi(\infty,\infty) = k, \tag{1.4}$$

These inequalities have been generalized to more than one variable. Many authors have established Gronwall-like type integral inequalities in two independent

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variables; see for example [3, 11, 12]. Recently, Khan in [2] obtained the inequality

$$\begin{aligned}\phi(x,y) = c + \int_0^x A(s,y)\phi(s,y)ds + \int_0^y B(x,t)\phi^p(x,t)dt \\ + \int_0^x \int_0^y H(s,t)\phi^p(x,y)dtds,\end{aligned} \tag{1.5}$$

and its variants, where $1 > p > 0$ and $c > 0$ are constants and $\phi(x,y) \in C(\mathbb{R}_+^2, \mathbb{R}_+)$.

However, sometimes we need to study such inequalities with a function $c(x,y)$ in place of the constant term c. Our main aim here, motivated by the works of [2, 6, 10], is to establish some new and more general retarded Gronwall-like integral inequalities with two independent variables which are useful in the analysis of certain classes of partial differential equations.

In this article we discuss more general forms of integral inequality

$$\begin{aligned}\phi^q(x,y) \le c(x,y) + \sum_{i=1}^{n_1} \int_{x_0}^x a_i(s,y)\phi^p(s,y)ds + \sum_{j=1}^{n_2} \int_{y_0}^y b_j(x,t)\phi^p(x,t)dt \\ + \sum_{k=1}^{n_3} v_k(x,y) \int_{x_0}^x \int_{y_0}^y d_k(x,y,s,t) g(\phi(s,t))\, ds\, dt,\end{aligned} \tag{1.6}$$

where $c(x,y) \ge 0$ is a function and $q \ge p > 0$ are constants for all $(x,y) \in \Delta$. Our results remain valid if we replace $\phi^p(x,t)$ by $w(u(s,y))$ in (1.6) where $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing function with $w(\phi) > 0$ for $\phi > 0$. Furthermore, we show that the results of [2, 6] can be deduced from our results in some special cases.

Motivated by the hyperbolic partial differential equation (1.3)-(1.4) in [10, Pachpatte], we give the boundedness of the solutions of the initial boundary value problem for hyperbolic partial delay differential equations.

2. Main results

In what follows, we define $I = [x_0, X)$ and $J = [x_0, Y)$ are the given subsets of $\mathbb{R}_+$, and $\Delta = I \times J$, $E = \{(x,y,s,t) \in \Delta^2 : x_0 \le s \le x \le X; y_0 \le t \le y \le Y\}$. We also assume that all improper integrals appeared in the sequel are always convergent, and suppose that

- (H1) All $a_i(x,y)$ $(i = 1,2,\dots,n_1)$; $b_j(x,y)$ $(j = 1,\dots,n_2)$; $c(x,y)$ and $\phi(x,y)$ are nonnegative, continuous functions and nondecreasing in each variables on Δ.
- (H2) All $\alpha : I \to I$, $\beta : J \to J$ are continuously differentiable and nondecreasing such that $\alpha(x) \le x$ on $I, \beta(y) \le y$ on J.
- (H3) All $v_k(x,y)$ $(k = 1,2,\dots,n_3)$ are nonnegative, continuous functions and nondecreasing in each variables on Δ.
- (H4) All $d_k(x,y,s,t) : E \to \mathbb{R}_+$ $(k = 1,2,\dots,n_3)$ are nonnegative, continuous functions and nondecreasing in x and y for each variables (s,t) on Δ.
- (H5) $g : \mathbb{R}_+ \to \mathbb{R}_+$ be nonnegative, continuous, nondecreasing and submultiplicative function with $w(\phi) > 0$ for $\phi > 0$.

The following lemma is useful in our main results.

Lemma 2.1. *Let c, ϕ and $a_i \in C(I, \mathbb{R}_+)$ be nonnegative continuous functions for any $x \in I$and $i = 1,2,\dots,n$ with $c(x)$ is nondecreasing function for $x \in I$ and assume that $\alpha \in C^1(I,J)$, $\beta \in C^1(I,J)$ be nondecreasing with $\alpha(x) \le x$ on I,*

$\beta(y) \le y$ on J. Suppose that $q \ge p > 0$ are constants. If $\phi(x)$ satisfies the inequality

$$\phi^q(x) \le c(x) + \sum_{i=1}^{n} \int_{\alpha(x_0)}^{\alpha(x)} a_i(s)\phi^p(s)ds, \tag{2.1}$$

for $x_0 \le s \le x$, then the following inequalities hold

$$\phi(x) \le \begin{cases} c^{1/p}(x)\exp\Big(\frac{1}{p}\sum_{i=1}^{n}\int_{\alpha(x_0)}^{\alpha(x)} a_i(s)ds\Big), & \text{if } p = q,\\ c^{1/q}(x) + \Big(1 + \frac{q-p}{q}\sum_{i=1}^{n}\int_{\alpha(x_0)}^{\alpha(x)} c^{(p-q)/q}(s)a_i(s).ds\Big)^{\frac{1}{q-p}} & \text{if } p < q, \end{cases} \tag{2.2}$$

for $x \in I$.

Now, let us list our main results.

Theorem 2.2. *Suppose* (H1)–(H2) *hold and the constant p satisfies $1 > p > 0$.*

(1) If $\phi(x, y)$ satisfies

$$\phi(x, y) \le c(x, y) + \sum_{i=1}^{n_1} \int_{\alpha(x_0)}^{\alpha(x)} a_i(s, y)\phi(s, y)ds + \sum_{i=1}^{n_2} \int_{\beta(y_0)}^{\beta(y)} b_j(x, t)\phi^p(x, t)dt, \tag{2.3}$$

for all $(x, y) \in \Delta$, then

$$\phi(x, y) \le c(x, y)E_1(x, y)Q_1(x, y), \tag{2.4}$$

for all $(x, y) \in \Delta$. Where

$$E_1(x, y) = \exp\Big(\sum_{i=1}^{n_1} \int_{\alpha(x_0)}^{\alpha(x)} a_i(s, y)ds\Big) \tag{2.5}$$

$$Q_1(x, y) = \Big(1 + (1-p)\sum_{i=1}^{n_2} \int_{\beta(y_0)}^{\beta(y)} b_j(x, t)c^{(p-1)}(x, t)E_1^p(x, t)dt\Big)^{\frac{1}{1-p}}. \tag{2.6}$$

(2) If $\phi(x, y)$ satisfies

$$\phi(x, y) \le c(x, y) + \sum_{i=1}^{n_1} \int_{\alpha(x_0)}^{\alpha(x)} a(s, y)\phi^p(s, y)ds + \sum_{i=1}^{n_2} \int_{\beta(y_0)}^{\beta(y)} b(x, t)\phi(x, t)dt, \tag{2.7}$$

for all $(x, y) \in \Delta$, then

$$\phi(x, y) \le c(x, y)E_2(x, y)Q_2(x, y), \tag{2.8}$$

for all $(x, y) \in \Delta$. Where

$$E_1(x, y) = \exp\Big(\sum_{j=1}^{n_2} \int_{\beta(y_0)}^{\beta(y)} b(x, t)dt\Big) \tag{2.9}$$

$$Q_1(x, y) = \Big(1 + (1-p)\sum_{i=1}^{n_1} \int_{\alpha(x_0)}^{\alpha(x)} a_i(s, y)c^{(p-1)}(s, y)E_2^p(s, y)dt\Big)^{\frac{1}{1-p}}. \tag{2.10}$$

The proof of the theorem will be given in the next section.

Theorem 2.3. *Suppose* (H1)–(H2) *hold and $q \ge p > 0$ are constants. If $\phi(x, y)$ satisfies the inequality*

$$\phi^q(x, y) \le c(x, y) + \sum_{i=1}^{n_1} \int_{\alpha(x_0)}^{\alpha(x)} a_i(s, y)\phi^p(s, y)ds + \sum_{j=1}^{n_2} \int_{\beta(y_0)}^{\beta(y)} b_j(x, t)\phi^p(x, t)dt, \tag{2.11}$$

for all $(x,y)\in\Delta$, then we have:

(1) If $p=q$, then

$$\phi(x,y)\le c^{1/p}(x,y)E_1^{1/p}(x,y)Q_3^{1/p}(x,y), \tag{2.12}$$

for all $(x,y)\in\Delta$, where

$$Q_3(x,y)=\exp\Big(\sum_{j=1}^{n_2}\int_{\beta(y_0)}^{\beta(y)} b_j(x,t)E_1(x,t)dt\Big), \tag{2.13}$$

and E_1 is defined in 2.5.

(2) If $p<q$, then

$$\phi(x,y)\le c^{1/q}(x,y)E_4(x,y)Q_4(x,y), \tag{2.14}$$

for all $(x,y)\in\Delta$, where

$$Q_4(x,y)=\Big[1+\frac{q-p}{q}\sum_{j=1}^{n_2}\int_{\beta(y_0)}^{\beta(y)} z^{\frac{p-q}{q}}(x,t)b_j(x,t)dt\Big]^{\frac{1}{q-p}}, \tag{2.15}$$

$$E_4(x,y)=\Big[1+\frac{q-p}{q}\sum_{i=1}^{n_1}\int_{\alpha(x_0)}^{\alpha(x)} c^{\frac{p-q}{q}}(s,y)a_i(s,y)Q_4^p(s,y)dt\Big]^{\frac{1}{q-p}}. \tag{2.16}$$

Where $z(x,y)\le c(x,y)E_4^q(x,y)$, for all all $(x,y)\in\Delta$.

Remark 2.4. If we take $b(x,y)=0$ and keep y fixed, then Theorem 2.3 reduce exactly to Lemma 2.1.

Remark 2.5. Using similar methods to those in the proof our main result above, if we replace $\phi^p(x,t)$ by $w(u(s,y))$ where $w\in C(\mathbb{R}_+,\mathbb{R}_+)$ is a nondecreasing function with $w(\phi)>0$ for $\phi>0$, an estimate of the inequality (2.11) can be easily obtained; in this case our result above reduces to the main results in [2].

Using Theorems 2.2 and 2.3, we can get some more generalized results as follow.

Theorem 2.6. *Suppose* (H1)–(H5) *hold and $1\ge p>0$ is constant.*

(1) If $\phi(x,y)$ satisfies

$$\begin{aligned}\phi(x,y)\le c(x,y)&+\sum_{i=1}^{n_1}\int_{\alpha(x_0)}^{\alpha(x)} a_i(s,y)\phi(s,y)ds+\sum_{j=1}^{n_2}\int_{\beta(y_0)}^{\beta(y)} b_j(x,t)\phi^p(x,t)dt\\&+\sum_{k=1}^{n_3}v_k(x,y)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} d_k(x,y,s,t)g(\phi(s,t))\,ds\,dt\end{aligned} \tag{2.17}$$

for all $(x,y)\in\Delta$, then

$$\phi(x,y)\le M_1(x,y)E_1(x,y)\tilde{Q}_1(x,y), \tag{2.18}$$

for all $x_0\le x\le x_1$, $y_0\le y\le y_1$. Where

$$\begin{aligned}M_1(x,y)\le G^{-1}\Big[G(c(x,y))&+\sum_{k=1}^{n_3}v_k(x,y)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} d_k(x,y,s,t)\\&\times g(E_1(s,t))g(\tilde{Q}_1(s,t))\,ds\,dt\Big],\end{aligned} \tag{2.19}$$

for all $x_0\le x\le x_1$, $y_0\le y\le y_1$ and

$$G(\phi)=\int_{\phi_0}^{\phi}\frac{\delta t}{g(t)},\quad \phi\ge\phi_0>0. \tag{2.20}$$

Where $E_1(x,y)$ *is defined in 2.9 and*

$$\tilde{Q}_1(x,y) = \Big[1 + (1-p)\sum_{j=1}^{n_2}\int_{\beta(y_0)}^{\beta(y)} b_j(x,t)M_1^{p-1}(x,t)E_1^p(x,t)dt\Big]^{\frac{1}{1-p}}. \tag{2.21}$$

Where G^{-1} *is the inverse function of* G *and the real numbers* $x_1, y_1 \in \mathbb{R}_+$ *are chosen so that* $G(c(x,y)) + \sum_{k=1}^{n_3} v_k(x,y)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} d_k(x,y,s,t)g(E_1(s,t))g(\tilde{Q}_1(s,t))\,ds\,dt$ *is in* $\mathrm{Dom}(G^{-1})$.

(2) *If* $\phi(x,y)$ *satisfies*

$$\begin{aligned}\phi(x,y) \le c(x,y) &+ \sum_{i=1}^{n_1}\int_{\alpha(x_0)}^{\alpha(x)} a_i(s,y)\phi^p(s,y)ds + \sum_{j=1}^{n_2}\int_{\beta(y_0)}^{\beta(y)} b_j(x,t)\phi(x,t)dt \\ &+ \sum_{k=1}^{n_3} v_k(x,y)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} d_k(x,y,s,t)g(\phi(s,t))\,ds\,dt,\end{aligned} \tag{2.22}$$

for all $(x,y) \in \Delta$, *then*

$$\phi(x,y) \le M_2(x,y)E_2(x,y)\tilde{Q}_2(x,y), \tag{2.23}$$

for all $x_0 \le x \le x_2,\ y_0 \le y \le y_2$. *Where*

$$\begin{aligned}M_2(x,y) \le G^{-1}\Big[G(c(x,y)) &+ \sum_{k=1}^{n_3} v_k(x,y)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} d_k(x,y,s,t) \\ &\times g(E_2(s,t))g(\tilde{Q}_2(s,t))\,ds\,dt\Big],\end{aligned} \tag{2.24}$$

for all $x_0 \le x \le x_1,\ y_0 \le y \le y_1$, G *and* E_1 *are defined in 2.20 and 2.9, with*

$$\tilde{Q}_2(x,y) = \Big[1 + (1-p)\sum_{i=1}^{n_1}\int_{\alpha(x_0)}^{\alpha(x)} a_i(s,y)M_2^{p-1}(s,y)E_2^p(s,y)ds\Big]^{\frac{1}{1-p}}. \tag{2.25}$$

Where G^{-1} *is the inverse function of* G *and the real numbers* $x_2, y_2 \in \mathbb{R}_+$ *are chosen so that* $G(c(x,y)) + \sum_{k=1}^{n_3} v_k(x,y)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} d_k(x,y,s,t)g(E_2(s,t))g(\tilde{Q}_2(s,t))\,ds\,dt$ *is in* $\mathrm{Dom}(G^{-1})$.

Remark 2.7. If we take $d_k(x,y,s,t) = 0$ for any $k = 1,2,\dots,n_3$ in the previous Theorem, then Theorem 2.6 reduce to Theorem 2.2.

By choosing suitable functions for g, some interesting new Gronwall-like type inequalities of two variables can be obtained from Theorem 2.6. For example if we take $g(s) = s^r$, the following interesting inequalities are easily obtained.

Corollary 2.8. *Suppose* (H1), (H2), (H4) *hold. Suppose* $1 \ge p > 0$ *and* $0 < r < 1$ *are constants and if* $\phi(x,y)$ *satisfies the inequality*

$$\begin{aligned}\phi(x,y) \le c(x,y) &+ \int_{\alpha(x_0)}^{\alpha(x)} a_1(s,y)\phi(s,y)ds + \int_{\beta(y_0)}^{\beta(y)} b_1(x,t)\phi^p(x,t)dt \\ &+ \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} d_1(x,y,s,t)\phi^r(s,t)\,ds\,dt\end{aligned} \tag{2.26}$$

for all $(x,y) \in \Delta$, *then*

$$\phi(x,y) \le m_1(x,y)e_1(x,y)\tilde{q}_1(x,y), \tag{2.27}$$

for all $(x,y) \in \Delta$. *Where*

$$m_1(x,y) \le \Big[c^{1-r}(x,y) + (1-r)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} d_1(x,y,s,t)e_1^r(s,t)\tilde{q}_1^r(s,t)\,ds\,dt\Big]^{\frac{1}{1-r}}, \tag{2.28}$$

for all $(x,y) \in \Delta$, *and*

$$\tilde{q}_1(x,y) = \Big[1 + (1-p)\int_{\beta(y_0)}^{\beta(y)} b_j(x,t)m_1^{p-1}(x,t)e_1^p(x,t)dt\Big]^{\frac{1}{1-p}}, \tag{2.29}$$

$$e_1(x,y) = \exp\Big(\int_{\alpha(x_0)}^{\alpha(x)} a_1(s,y)ds\Big). \tag{2.30}$$

Remark 2.9. (i) If $r = 0$, by using Theorem 2.6, an estimation of (2.26) can be easily obtained.

(ii) also when $r = 1$, an estimation of the inequality (2.26) can be easily obtained; for space-saving, the details are omitted here.

Remark 2.10. Corollary 2.8 reduces to the main results in [2, Theorem 2.3], when $c(x,y) = c$ (constant), $\alpha(x) = x$, $\beta(y) = y$, $d_1(x,y,s,t) = d(s,t)$, $x_0 = y_0 = 0$ and $r = p$.

We can also get an interesting result as follows.

Corollary 2.11. *Suppose* (H1), (H2), (H4) *hold. Suppose* $1 \ge p > 0$, $0 < r < 1$ *are a constants and if* $\phi(x,y)$ *satisfies the inequality*

$$\begin{aligned}\phi(x,y) \le c(x,y) &+ \int_{\alpha(x_0)}^{\alpha(x)} a_1(s,y)\phi^p(s,y)ds + \int_{\beta(y_0)}^{\beta(y)} b_1(x,t)\phi(x,t)dt \\ &+ \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} d_1(x,y,s,t)\phi^r(s,t)\,ds\,dt\end{aligned}$$

for all $(x,y) \in \Delta$, *then*

$$\phi(x,y) \le m_2(x,y)e_2(x,y)\tilde{q}_2(x,y),$$

for all $(x,y) \in \Delta$. *Where*

$$m_2(x,y) \le \Big[c^{1-r}(x,y) + (1-r)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} d_1(x,y,s,t)e_2^r(s,t)\tilde{q}_2^r(s,t)\,ds\,dt\Big]^{\frac{1}{1-r}},$$

for all $(x,y) \in \Delta$, *and*

$$\tilde{q}_2(x,y) = \Big[1 + (1-p)\int_{\alpha(x_0)}^{\alpha(x)} a_1(s,y)m_2^{p-1}(s,y)e_2^p(s,y)ds\Big]^{\frac{1}{1-p}},$$

$$e_2(x,y) = \exp\Big(\int_{\beta(y_0)}^{\beta(y)} b_1(x,t)dt\Big).$$

Remark 2.12. Under some suitable conditions, Corollary 2.11 is also a generalization of the main result in [2, Theorem 2.4].

Using Theorem 2.3, we can get some more generalized results as follows.

Theorem 2.13. *Suppose* (H1)–(H5) *hold. Suppose that* $q \geq p > 0$ *are constants. If* $\phi(x,y)$ *satisfies*

$$\phi^q(x,y) \leq c(x,y) + \sum_{i=1}^{n_1} \int_{\alpha(x_0)}^{\alpha(x)} a_i(s,y)\phi^p(s,y)ds + \sum_{j=1}^{n_2} \int_{\beta(y_0)}^{\beta(y)} b_j(x,t)\phi^p(x,t)dt + \sum_{k=1}^{n_3} v_k(x,y) \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} d_k(x,y,s,t)g(\phi(s,t))\,ds\,dt \tag{2.31}$$

for all $(x,y) \in \Delta$, *then the following conclusions are true:*

(1) If $p = q$, *then*

$$\phi(x,y) \leq N_1^{1/p}(x,y)E_1^{1/p}(x,y)Q_3^{1/p}(x,y), \tag{2.32}$$

for all $x_0 \leq x \leq x_3$, $y_0 \leq y \leq y_3$. *Where*

$$N_1(x,y) \leq H^{-1}\Big[H(c(x,y)) + \sum_{k=1}^{n_3} v_k(x,y) \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} d_k(x,y,s,t) \times g(E_1^{1/p}(s,t))g(Q_3^{1/p}(s,t))\,ds\,dt\Big], \tag{2.33}$$

for all $x_0 \leq x \leq x_3$, $y_0 \leq y \leq y_3$*and*

$$H(\phi) = \int_{\phi_0}^{\phi} \frac{dt}{g(t^{1/q})}, \quad \phi \geq \phi_0 > 0. \tag{2.34}$$

Where $E_1(x,y)$ *and* $Q_3(x,y)$ *are defined in in 2.5 and 2.13. Where* H^{-1} *is the inverse function of* H *and the real numbers* x_3, y_3 *are chosen so that* $H(c(x,y)) + \sum_{k=1}^{n_3} v_k(x,y) \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} d_k(x,y,s,t)g(E_1^{1/p}(s,t))g(Q_3^{1/p}(s,t))\,ds\,dt \in \mathrm{Dom}(H^{-1})$.

(2) If $p < q$, *then*

$$\phi(x,y) \leq N_2^{1/q}(x,y)\tilde{E}_4(x,y)\tilde{Q}_4(x,y), \tag{2.35}$$

for all $x_0 \leq x \leq x_4$, $y_0 \leq y \leq y_4$. *Where*

$$N_2(x,y) \leq H^{-1}\Big[H(c(x,y)) + \sum_{k=1}^{n_3} v_k(x,y) \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} d_k(x,y,s,t) \times g(\tilde{E}_4(s,t)\tilde{Q}_4(s,t))\,ds\,dt\Big], \tag{2.36}$$

for all $x_0 \leq x \leq x_4$, $y_0 \leq y \leq y_4$, H *is defined in 2.34, with*

$$\tilde{Q}_4(x,y) = \Big[1 + \frac{(q-p)}{q} \sum_{j=1}^{n_2} \int_{\beta(y_0)}^{\beta(y)} b_j(s,y)\tilde{z}^{(p-q\grave{a}/q}(x,t)dt\Big]^{\frac{1}{q-p}}, \tag{2.37}$$

$$\tilde{E}_4(x,y) = \Big[1 + \frac{(q-p)}{q} \sum_{i=1}^{n_1} \int_{\alpha(x_0)}^{\alpha(x)} a_i(s,y)N_2^{(p-q\grave{a}/q}(s,y)\tilde{Q}_4^p(s,y)ds\Big]^{\frac{1}{q-p}} \tag{2.38}$$

for all $x_0 \leq x \leq x_4$, $y_0 \leq y \leq y_4$, *where* $\tilde{z}(x,y) \leq N_2(x,y)\tilde{E}_4^q(x,y)$. *Here* H^{-1} *is the inverse function of* H *and the real numbers* x_4, y_4 *are chosen so that* $H(c(x,y)) + \sum_{k=1}^{n_3} v_k(x,y) \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} d_k(x,y,s,t)g(\tilde{E}_4(s,t)\tilde{Q}_4(s,t))\,ds\,dt \in \mathrm{Dom}(H^{-1})$.

Remark 2.14. Various choices of c, g, d_k, v_k and p, q, α, β can give many different inequalities. Obviously, our results generalize many results obtained before, for example, let $p = 1$, $\alpha(x) = x$, $\beta(y) = y$, $g(s) = s$, $c(x, y) = c > 0$ (constant), $n_1 = n_2 = n_3 = 1$, $x_0 = y_0 = 0$ and $d_1(x, y, s, t) = d_1(s, t)$, then our Theorem 2.13(1) reduces to [2, Theorem 2.1]. Considering $q = 1$, $\alpha(x) = x$, $\beta(y) = y$, $w(s) = s^p$, $n_1 = n_2 = n_3 = 1$, $x_0 = y_0 = 0$, $d_1(x, y, s, t) = d_1(s, t)$ and $c(x, y) = c \geq 0$ (constant) in Theorem 2.13(2), we obtain [2, Theorem 2.5]. If we take $q = p = 1$, $g(s) = s^r$, $1 > r > 0$, $\alpha(x) = x$, $\beta(y) = y$, $c(s, y) = c \geq 0$ (constant), $n_1 = n_2 = n_3 = 1$, $x_0 = y_0 = 0$, $v_1(x, y) = 1$ and $d_1(x, y, s, t) = d_1(s, t)$, then the inequality established in Theorem 2.13(1) reduces to the [2, Theorem 2.2].

Remark 2.15. By replacing $\phi^p(x, t)$ by $w(u(s, y))$ where $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing function with $w(\phi) > 0$ for $\phi > 0$ in the line above (2.31) and using the same arguments in the proof of our theorem 2.13, an estimation of the inequality (2.31) can be easily obtained. In particular, when a (a nonnegative constant) and $d_k(x, y, s, t) = 0$ for all $k = 0, \dots, n_3$, the inequality (2.31) becomes

$$\phi^q(x, y) \leq a + \sum_{i=1}^{n_1} \int_{\alpha(x_0)}^{\alpha(x)} a_i(s, y) w(\phi(s, y)) ds + \sum_{j=1}^{n_2} \int_{\beta(y_0)}^{\beta(y)} b_j(x, t) w(\phi(x, t)) dt \quad (2.39)$$

the general form of (2.31) in the case of two independent variables. Thus our result implies to the main result in [2].

Remark 2.16. By choosing suitable functions for g for example $g(s) = s^r$ with $q \geq r \geq 0$ or when we take $q = r > 0$ (with $p = q$ or $q > p$), using similar arguments in the proof of Theorem 2.13, we can obtain many interesting new retarded integral inequalities, but, for space-saving, the details are omitted here.

Remark 2.17. Using similar method of those in the proof of our main results above, with a suitable conditions, we can obtain some new reversed inequalities of our results.

3. Proof of theorems

Since the proofs resemble each other, we give the details only for Theorem 2.2(1), Theorem 2.3(2), Throem2.6(1) and Theorem 2.13(2). The proofs of the remaining inequalities can be completed by following the proofs of the above-mentioned inequalities. To the best our knowledge, Lemma 2.1 is not found in the literature (in this form). Therefore, we give a proof here.

Proof of Lemma 2.1. If $c(x) > 0$ (i) If $p = q$ holds, letting $z(x) = [\frac{\phi(x)}{c^{1/p}(x)}]^p$, from (2.1) derive that

$$z(x) \leq 1 + \sum_{i=1}^{n} \int_{\alpha(x_0)}^{\alpha(x)} a_i(s) z(s) ds, \quad (3.1)$$

for $x \in I$. define a positive, continuous and nondecreasing function $v(x)$ by the right hand of (3.1), then $z(x) \leq v(x)$ and $v(x_0) = 1$ hold. Since $v(x)$ is positive and by differentiation we obtain

$$v(x) = \sum_{i=1}^{n} \alpha'(x) a_i(\alpha(x)) z(\alpha(x)),$$

$$\frac{v'(x)}{v(x)} \le \sum_{i=1}^{n} \alpha'(x) a_i(\alpha(x)), \quad x \in I. \tag{3.2}$$

By integration of (3.2) from x_0 to x, we have

$$v(x) \le \exp\Big(\sum_{i=1}^{n} \int_{\alpha(x_0)}^{\alpha(x)} a_i(s)ds\Big),$$

hence we obtain

$$[\frac{\phi(x)}{c^{1/p}(x)}]^p = z(x) \le \exp[\sum_{i=1}^{n} \int_{\alpha(x_0)}^{\alpha(x)} a_i(s)ds].$$

This inequality implies the desired inequality (2.2) immediately.

(ii) If $p < q$ holds, letting $y(x) = \frac{\phi(x)}{c^{1/q}(x)}$, from (2.1) we obtain

$$y^q(x) \le 1 + \sum_{i=1}^{n} \int_{\alpha(x_0)}^{\alpha(x)} a_i(s) c^{(p-q)/q}(s) y^p(s) ds, \tag{3.3}$$

Define a positive, continuous and nondecreasing function $h(x)$ by the right hand of (3.3), then $y(x) \le h^{1/q}(x)$ and $h(x_0) = 1$ hold. we carry out the above procedure, we obtain

$$\frac{\phi(x)}{c^{1/q}(x)} = y(x) \le \Big[1 + \frac{q-p}{q} \sum_{i=1}^{n} \int_{\alpha(x_0)}^{\alpha(x)} c^{(p-q)/q}(s) a_i(s) ds\Big]^{\frac{1}{q-p}},$$

This inequality implies the desired inequality (2.2) immediately.

If $c(x) \ge 0$ is nonnegative, we carry out the above procedure in (i) and (ii) with $c(x)+\varepsilon$ instead of $c(x)$, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to obtain (2.2). This completes the proof. □

Proof of Theorem 2.2. (1) We define a function

$$z(x,y) = c(x,y) + \sum_{j=1}^{n_2} \int_{\beta(y_0)}^{\beta(y)} b_j(x,t) \phi^p(x,t) dt, \tag{3.4}$$

by substituting (3.4) in (2.3), we obtain

$$\phi(x,y) \le z(x,y) + \sum_{i=1}^{n_1} \int_{\alpha(x_0)}^{\alpha(x)} a_i(s,y) \phi(s,y) ds, \quad (x,y) \in \Delta. \tag{3.5}$$

Clearly, $z(x,y)$ is a nonnegative, continuous and nondecreasing function in x. Treating y, $y \in I_2$ fixed in (3.5), a suitable application of Lemma 2.1 to (3.5) we obtain

$$\phi(x,y) \le z(x,y) E_1(x,y), \tag{3.6}$$

for $(x,y) \in \Delta$, where $E_1(x,y)$ is defined as in (2.5).

By (3.4) and (3.6), we obtain

$$z(x,y) \le c(x,y) + \sum_{j=1}^{n_2} \int_{\beta(y_0)}^{\beta(y)} b_j(x,t) E_1^p(x,t) z^p(x,t) dt. \tag{3.7}$$

Keeping x fixed in (3.7), an estimation of $z(x,y)$ can be obtained by a suitable application of Lemma 2.1 to (3.7), after that, we obtain

$$z(x,y) \le c(x,y) Q_1(x,y), \tag{3.8}$$

for $(x,y) \in \Delta$, where $Q_1(x,y)$ is defined as in (2.6).

Finally, substituting the last inequality into (3.6), the desired inequality (2.4) follows immediately. □

Proof of Theorem 2.3(2). We define a function

$$z(x,y) = c(x,y) + \sum_{i=1}^{n_1} \int_{\alpha(x_0)}^{\alpha(x)} a_i(s,y)\phi^p(s,y)ds, \tag{3.9}$$

by substituting (3.9) in (2.11), we obtain

$$\phi^q(x,y) \le z(x,y) + \sum_{j=1}^{n_2} \int_{\beta(y_0)}^{\beta(y)} b_j(x,t)\phi^p(x,t)dt. \tag{3.10}$$

Clearly, $z(x,y)$ is a nonnegative, continuous and nondecreasing function in y. Treating x fixed in (3.10), a suitable application of Lemma 2.1 to (3.10) we obtain

$$\phi(x,y) \le z(x,y)^{1/q} Q_4(x,y), \tag{3.11}$$

for $(x,y) \in \Delta$, where $Q_4(x,y)$ is defined as in (2.15).

By (3.11) and (3.9), we obtain

$$z(x,y) \le c(x,y) + \sum_{i=1}^{n_1} \int_{\alpha(x_0)}^{\alpha(x)} a_i(s,y) Q_4^p(s,y) z^{p/q}(s,y)ds. \tag{3.12}$$

Keeping y fixed in (3.12), an estimation of $z(x,y)$ can be obtained by a suitable application of Lemma 2.1 to (3.12), after that, we obtain

$$z(x,y) \le c(x,y)E_4^q(x,y),$$

for $(x,y) \in \Delta$,where $E_4(x,y)$ is defined as in (2.16).

Finally, substituting the last inequality into (3.11), the desired inequality (2.14) follows immediately.

□

Proof of Theorem 2.6. If $c(x,y) > 0$. Setting

$$M_1(x,y) = c(x,y) + \sum_{k=1}^{n_3} v_k(x,y) \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} d_k(x,y,s,t) g(\phi(s,t))\, ds\, dt, \tag{3.13}$$

inequality (2.17) can be restated as

$$\phi(x,y) \le M_1(x,y) + \sum_{i=1}^{n_1} \int_{\alpha(x_0)}^{\alpha(x)} a_i(s,y)\phi(s,y)ds + \sum_{j=1}^{n_2} \int_{\beta(y_0)}^{\beta(y)} b_j(x,t)\phi^p(x,t)dt, \tag{3.14}$$

Clearly, $M_1(x,y)$ is nonnegative and nondecreasing function in each in x and y . Now a suitable application of the inequality (2.3) in Theorem 2.2 to (3.14), yields

$$\phi(x,y) \le M_1(x,y) E_1(x,y) \widetilde{Q}_1(x,y), \tag{3.15}$$

where $E_1(x,y)$, $\widetilde{Q}_1(x,y)$ are defined in (2.5) and (2.21). From (3.13) and (3.15) and by using the fact that w is submultiplicative, we have

$$\begin{aligned} M_1(x,y) \le c(x,y) + \sum_{k=1}^{n_3} v_k(x,y) \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} d_k(x,y,s,t) \\ \times g\big(E_1(s,t)\widetilde{Q}_1(s,t)\big) g(M_1(s,t))\, ds\, dt, \end{aligned} \tag{3.16}$$

for $(x,y)\in\Delta$. Fixing any numbers $\widetilde{x}_1$ and $\widetilde{y}_1$ with $0<\widetilde{x}_1\le x_1$ and $0<\widetilde{y}_1\le y_1$, from (3.16) we have

$$
\begin{aligned}
M_1(x,y) \le c(\widetilde{x}_1,\widetilde{y}_1) &+ \sum_{k=1}^{n_3} v_k(\widetilde{x}_1,\widetilde{y}_1)\\
&\times \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} d_k(\widetilde{x}_1,\widetilde{y}_1,s,t) g\big(E_1(s,t)\widetilde{Q}_1(s,t)\big) g(M_1(s,t))\,ds\,dt
\end{aligned}
$$

for $x_0\le x\le \widetilde{x}_1, y_0\le y\le \widetilde{y}_1$.

Defining $r_1(x,y)$ as the right-hand side of the last inequality, then $r_1(x_0,y)=r_1(x,y_0)=c(\widetilde{x}_1,\widetilde{y}_1)$, and

$$
M_1(x,y)\le r_1(x,y), \tag{3.17}
$$

with $r_1(x,y)$ is positive and nondecreasing in $y\in[y_0,\widetilde{y}_0]$, and

$$
\begin{aligned}
D_1 r_1(x,y) &= \sum_{k=1}^{n_3} v_k(\widetilde{x}_1,\widetilde{y}_1)\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} d_k(\widetilde{x}_1,\widetilde{y}_1,\alpha(x),t)\\
&\quad\times g\big(E_1(\alpha(x),t)\widetilde{Q}_1(\alpha(x),t)\big) g(M_1(\alpha(x),t))dt,\\
&\le \sum_{k=1}^{n_3} v_k(\widetilde{x}_1,\widetilde{y}_1)\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} d_k(\widetilde{x}_1,\widetilde{y}_1,\alpha(x),t)\\
&\quad\times g\big(E_1(\alpha(x),t)\widetilde{Q}_1(\alpha(x),t)\big) g(r_1(\alpha(x),t))dt.\\
&\le w(r_1(x,y))\sum_{k=1}^{n_3} v_k(\widetilde{x}_1,\widetilde{y}_1)\alpha'(x)\\
&\quad\times \int_{\beta(y_0)}^{\beta(y)} d_k(\widetilde{x}_1,\widetilde{y}_1,\alpha(x),t) g\big(E_1(\alpha(x),t)\widetilde{Q}_1(\alpha(x),t)\big)dt.
\end{aligned} \tag{3.18}
$$

Dividing both sides of (3.18) by $w(r_1(x,y))$, we obtain

$$
\begin{aligned}
\frac{D_1 r_1(x,y)}{g(r_1(x,y))} &\le sum_{k=1}^{n_3} v_k(\widetilde{x}_1,\widetilde{y}_1)\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} d_k(\widetilde{x}_1,\widetilde{y}_1,\alpha(x),t)\\
&\quad\times g\big(E_1(\alpha(x),t)\widetilde{Q}_1(\alpha(x),t)\big)dt,
\end{aligned} \tag{3.19}
$$

from (2.20) and (3.19), we have

$$
\begin{aligned}
D_1 G(r_1(x,y)) &\le \sum_{k=1}^{n_3} v_k(\widetilde{x}_1,\widetilde{y}_1)\alpha'(x)\\
&\quad\times \int_{\beta(y_0)}^{\beta(y)} d_k(\widetilde{x}_1,\widetilde{y}_1,\alpha(x),t) g\left(E_1(\alpha(x),t)\widetilde{Q}_1(\alpha(x),t)\right) dt,.
\end{aligned} \tag{3.20}
$$

Now setting $x=s$ in (3.20) and then integrating with respect to s from x_0 to x, we obtain

$$
\begin{aligned}
G(r_1(x,y)) \le G(r_1(x_0,y)) &+ \sum_{k=1}^{n_3} v_k(\widetilde{x}_1,\widetilde{y}_1)\\
&\times \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} d_k(\widetilde{x}_1,\widetilde{y}_1,s,t) g\big(E_1(s,t)\widetilde{Q}_1(s,t)\big)\,ds\,dt.
\end{aligned}
$$

Noting $G(r_1(x_0, y)) = G(c(\widetilde{x}_1, \widetilde{y}_1))$, we have

$$\begin{aligned} G(r_1(x, y)) \le G(c(\widetilde{x}_1, \widetilde{y}_1)) + \sum_{k=1}^{n_3} v_k(\widetilde{x}_1, \widetilde{y}_1) \\ \times \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} d_k(\widetilde{x}_1, \widetilde{y}_1, s, t) g\big(E_1(s,t)\widetilde{Q}_1(s,t)\big)\, ds\, dt. \end{aligned}$$

Taking $x = \widetilde{x}_1$, $y = \widetilde{y}_1$ in (3.17) and the last inequality, we obtain

$$M_1(\widetilde{x}_1, \widetilde{y}_1) \le r_1(\widetilde{x}_1, \widetilde{y}_1), \tag{3.21}$$

and

$$\begin{aligned} G(r_1(\widetilde{x}_1, \widetilde{y}_1)) \le G(c(\widetilde{x}_1, \widetilde{y}_1)) + \sum_{k=1}^{n_3} v_k(\widetilde{x}_1, \widetilde{y}_1) \\ \times \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} d_k(\widetilde{x}_1, \widetilde{y}_1, s, t) w\left(E_1(s,t)\widetilde{Q}_1(s,t)\right)\, ds\, dt. \end{aligned} \tag{3.22}$$

Since $0 < \widetilde{x}_1 \le x_1$ and $0 < \widetilde{y}_1 \le y_1$ are arbitrary, from (3.21) and (3.22), we have

$$M_1(x, y) \le r_1(x, y), \tag{3.23}$$

and

$$\begin{aligned} r_1(x, y) \le G^{-1}\Big[G(c(x, y)) + \sum_{k=1}^{n_3} v_k(x, y) \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} d_k(x, y, s, t) \\ g\big(E_1(s,t)\widetilde{Q}_1(s,t)\big)\, ds\, dt\Big]. \end{aligned} \tag{3.24}$$

for all $x_0 < x \le x_1, y_0 < y \le y_1$. Hence by (3.23) and (3.24), we obtain

$$\begin{aligned} M_1(x, y) \le G^{-1}\Big[G(c(x, y)) + \sum_{k=1}^{n_3} v_k(x, y) \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} d_k(x, y, s, t) \\ \times g\big(E_1(s,t)\widetilde{Q}_1(s,t)\big)\, ds\, dt\Big], \end{aligned} \tag{3.25}$$

for all $x_0 < x \le x_1$, $y_0 < y \le y_1$. By (2.17), (3.25) holds also when $x = x_{0,} y = y_0$.

Finally, substituting the last inequality into (3.15), the desired inequality (2.18) follows immediately.

If $c(x, y) \ge 0$ is nonnegative, we carry out the above procedure in the proof of Theorem 2.6(1) with $c(x, y) + \varepsilon$ instead of $c(x, y)$, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to obtain (2.18) This completes the proof.

The proof of Theorem 2.6(2) is similar to the argument in the proof of Theorem 2.6(1) with suitable modifications. We omit the details here. □

Proof of Theorem 2.13. Setting

$$N_2(x, y) = c(x, y) + \sum_{k=1}^{n_3} v_k(x, y) \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} d_k(x, y, s, t) g(\phi(s, t))\, ds\, dt, \tag{3.26}$$

inequality (2.31) can be restated as

$$\phi^q(x,y) \le N_2(x,y) + \sum_{i=1}^{n_1} \int_{\alpha(x_0)}^{\alpha(x)} a_i(s,y)\phi^p(s,y)ds + \sum_{j=1}^{n_2} \int_{\beta(y_0)}^{\beta(y)} b_j(x,t)\phi^p(x,t)dt, \tag{3.27}$$

Clearly, $N_2(x,y)$ is nonnegative and nondecreasing function in each in x and y. Now a suitable application of inequality (2.14) in Theorem 2.3, to (3.27), yields

$$\phi(x,y) \le N_2^{1/q}(x,y)\widetilde{E}_4(x,y)\widetilde{Q}_4(x,y), \tag{3.28}$$

where $\widetilde{E}_4(x,y)$, $\widetilde{Q}_4(x,y)$ are defined in (2.38) and (2.37). From (3.26) and (3.28) and by using the fact that w is submultiplicative, we have

$$\begin{aligned} N_2(x,y) \le c(x,y) + \sum_{k=1}^{n_3} v_k(x,y) \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} d_k(x,y,s,t) \\ \times g\big(\widetilde{E}_4(s,t)\widetilde{Q}_4(s,t)\big) g(N_1^{1/q}(s,t))\, ds\, dt, \end{aligned} \tag{3.29}$$

for $(x,y) \in \Delta$.

By following the same steps from (3.16)-(3.25) in (3.29), we obtain

$$\begin{aligned} N_2(x,y) \le H^{-1}\Big[H(c(x,y)) + \sum_{k=1}^{n_3} v_k(x,y) \int_{x_0}^{x} \int_{y_0}^{y} d_k(x,y,s,t) w\big(\widetilde{E}_4(s,t)\big) \\ \times w(\widetilde{Q}_4(s,t))\, ds\, dt\Big], \end{aligned}$$

for all $x_0 \le x \le x_{41}, y_0 \le y \le y_4$. Finally, substituting the last inequality in (3.28), the desired inequality (2.35) follows immediately. □

4. An application

In this section we present an application of the inequality (2.31) given in Theorem 2.13 to study the boundedness of the solutions of the initial boundary value problem for hyperbolic partial delay differential equations of the form

$$\begin{aligned} D_1D_2\phi^p(x,y) &= h\big(x,y,\phi(x-\alpha(x), y-\beta(y))\big) + D_1 g_1\big(x,y,\phi(x-\alpha(x),y)\big) \\ &\quad + D_2 g_2\big(x,y,\phi(x,y-\beta(y))\big), \\ \phi(x,y_0) &= \sigma_1(x), \quad \phi(x_0,y) = \sigma_2(y), \quad \phi(x_0,y_0) = k, \end{aligned} \tag{4.1}$$

for all $(x,y) \in \Delta$.

Where $h, g_1, g_2 \in C(\Delta \times \mathbb{R}, \mathbb{R})$ and $\sigma_1, \sigma_2 \in C(\mathbb{R}_+, \mathbb{R}_+), k, p > 0$ are constants, $\alpha \in C^1(I,\mathbb{R}), \beta \in C^1(J,\mathbb{R})$ nondecreasing functions such that $\alpha(x) \le x$ on I, $\beta(y) \le y$ on J, and $\alpha'(x) < 1$, $\beta'(y) < 1$ with $\alpha(x_0) = 0$ and $\beta(y_0) = 0$.

Theorem 4.1. *Suppose that*

$$|h(x,y,\phi)| \le d_1(x,y)\phi^r, \tag{4.2}$$

$$|g_1(x,y,\phi)| \le b_1(x,y)\phi^p, \tag{4.3}$$

$$|g_2(x,y,\phi)| \le a_1(x,y)\phi^p, \tag{4.4}$$

$$|c_1(x) + c_2(y) - k| \le c, \quad c \ge 0 \ \textit{(constant)}. \tag{4.5}$$

Where $1 > p > r > 0$, $b_1(x,y), a_1(x,y)$ *are as in Theorem 2.13 and* $d_1(x,y) \in C(\Delta, \mathbb{R}_+)$ *be nondecreasing function, with*

$$c_1(x) = \sigma_1(x) - \int_{x_0}^{x} g_2(s, x_0, \sigma_1(s))ds,$$

$$c_2(y) = \sigma_2(y) - \int_{y_0}^{y} g_1(y_0, t, \sigma_2(t))dt.$$

If $\phi(x,y)$ *is any solution of* (4.1), *then*

$$\begin{aligned}\phi(x,y) \le \big(\tilde{e}(x,y)\tilde{q}(x,y)\big)^{1/p}\Big[&c^{(p-r)/p} \\ &+ \frac{p-r}{p}\int_{\psi(x_0)}^{\psi(x)}\int_{\Omega(y_0)}^{\Omega(y)} \bar{d}_1(s,t)\big(\tilde{e}(x,y)\tilde{q}(x,y)\big)^{r/p}\,ds\,dt\Big]^{\frac{1}{p-r}},\end{aligned} \tag{4.6}$$

for $(x,y) \in \Delta$, *in which* $\psi(x) = x - \alpha(x)$ *on* I *and* $\Omega(y) = y - \beta(y)$ *on* J, *and*

$$\tilde{e}(x,y) = \exp\Big(\int_{\psi(x_0)}^{\psi(x)} \bar{a}_1(s,y)ds\Big), \tag{4.7}$$

$$\tilde{q}(x,y) = \exp\Big(\int_{\Omega(y_0)}^{\Omega(y)} \bar{b}_1(x,t)\tilde{e}(x,t)dt\Big). \tag{4.8}$$

For all $(x,y) \in \Delta$, *where*

$$\bar{a}_1(\delta,t) = \xi_1 \cdot a_1(\delta + \alpha(s), t), \tag{4.9}$$

$$\bar{b}_1(s,\tau) = \xi_2 \cdot b_1(s, \tau + \beta(t)), \tag{4.10}$$

$$\bar{d}_1(\delta,\tau) = \xi_1\xi_2 \cdot d_1(\delta + \alpha(s), \tau + \beta(t)), \tag{4.11}$$

for all $s, \delta \in I$ *and* $t, \tau \in J$; *and*

$$\xi_1 = \underset{x\in I}{Max}\frac{1}{1-\alpha'(x)}, \quad \xi_2 = \underset{y\in I}{Max}\frac{1}{1-\beta'(y)}$$

Proof. It is easy to see that, the solution $\phi(x,y)$ of problem (4.1) satisfies the equivalent integral equation

$$\begin{aligned}\phi^p(x,y) = \sigma_1(x) + \sigma_2(y) - k + \int_0^x\int_0^y h(s,t,\phi(s,t))\,ds\,dt + \int_0^y g_1(x,t,\phi(x,t))dt \\ + \int_0^x g_2(s,y,\phi(s,y)ds - \int_0^x g_2(s,0,\sigma_1(s))ds - \int_0^y g_1(0,t,\sigma_2(t))dt.\end{aligned} \tag{4.12}$$

From (4.12) and with (4.2)-(4.12), we have

$$\begin{aligned}\phi^p(x,y) \le c + \int_0^x a_1(s,y)\phi^p(s-\alpha(s),y)ds + \int_0^y b_1(x,t)\phi^p(x,t-\beta(t))dt \\ + \int_0^x\int_0^y d_1(s,t)\phi^r(s-\alpha(s),t-\beta(t))\,ds\,dt,\end{aligned} \tag{4.13}$$

Using (4.13) and making a change of variables, we have

$$\begin{aligned}\phi^p(x,y) \le c + \int_{\psi(x_0)}^{\psi(x)} \bar{a}_1(s,y)\phi^p(s,y)ds + \int_{\Omega(y_0)}^{\Omega(y)} \bar{b}_1(x,t)\phi^p(x,t)dt \\ + \int_{\psi(x_0)}^{\psi(x)}\int_{\Omega(y_0)}^{\Omega(y)} \bar{d}_1(s,t)\phi^r(s,t)\,ds\,dt,\end{aligned} \tag{4.14}$$

for $(x, y) \in \Delta$, with $p > r > 0$. Where $\psi(x) = x - \alpha(x)$ on I and $\Omega(y) = y - \beta(y)$ on J, and $\bar{a}_1, \bar{b}_1$ and $\bar{d}_1$ are defined in (4.9), (4.9) and (4.10)

Now, a suitable application of Theorem 2.13(1) to (4.14), with $g(s) = s^r$, $c(x, y) = c$, $v_1(x, y) = 1$, $d_1(x, y, s, t) = \bar{d}_1(s, t)$, $n_1 = n_2 = n_3 = 1$, yields

$$\begin{aligned}\phi(x, y) \le \big(\tilde{e}(x, y)\tilde{q}(x, y)\big)^{1/p}\Big[&c^{(p-r)/p}\\&+ \frac{p-r}{p}\int_{\psi(x_0)}^{\psi(x)}\int_{\Omega(y_0)}^{\Omega(y)} \bar{d}_1(s, t)\big(\tilde{e}(x, y)\tilde{q}(x, y)\big)^{r/p}\, ds\, dt\Big]^{\frac{1}{p-r}}\end{aligned}$$

for all $(x, y) \in \Delta$, where $\tilde{e}(x, y)$ and $\tilde{q}(x, y)$ are defined in (4.7) and (4.8). $\square$

We also note that the inequalities established in Theorems 2.13 and 2.13 and the applications given in Theorems 2.13 can be extended very easily to functions involving many independent variables.

Finally, we note that under some suitable conditions, the uniqueness and continuous dependence of the solutions of (4.1), can also be discussed using our results.

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