Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 109, pp. 1–17.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS TO A NORMALIZED F-INFINITY LAPLACIAN EQUATION

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Abstract. In this article, for a continuous function F that is twice differentiable at a point x_0, we define the normalized F-infinity Laplacian $\Delta^N_{F;\infty}$ which is a generalization of the usual normalized infinity Laplacian. Then for a bounded domain $\Omega \subset \mathbb{R}^n$, $f \in C(\Omega)$ with $\inf_\Omega f(x) > 0$ and $g \in C(\partial\Omega)$, we obtain existence and uniqueness of viscosity solutions to the Dirichlet boundary-value problem

$$\begin{gathered}\Delta^N_{F;\infty} u = f, \quad \text{in } \Omega,\\ u = g, \quad \text{on } \partial\Omega.\end{gathered}$$

1. Introduction

Let $F : \mathbb{R}^n \to [0, +\infty)$ be a function which satisfies the following conditions:

(a) $F \in C^2(\mathbb{R}^n \setminus \{0\})$, $F(0) = 0$, $F(p) > 0$, for any $p \in \mathbb{R}^n \setminus \{0\}$;
(b) F is positively homogeneous of degree 1: $F(tp) = tF(p)$, for any $t > 0$ and $p \in \mathbb{R}^n$;
(c) $\mathrm{Hess}(F^2)$ is positive definite in $\mathbb{R}^n \setminus \{0\}$.

Let Ω be a bounded domain in $\mathbb{R}^n$. For a $C^2(\Omega)$ function u, we define the F-infinity Laplacian $\Delta_{F;\infty}$ and the normalized F-infinity Laplacian $\Delta^N_{F;\infty}$ by

$$\Delta_{F;\infty} u = F^2(Du) \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial F}{\partial p_i}(Du) \frac{\partial F}{\partial p_j}(Du), \tag{1.1}$$

$$\Delta^N_{F;\infty} u = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial F}{\partial p_i}(Du) \frac{\partial F}{\partial p_j}(Du) \tag{1.2}$$

respectively. Clearly when $F(p) = p$, they are the usual infinity Laplacian and the normalized infinity Laplacian, respectively.

The operator $\Delta_{F;\infty}$ is a kind of Aronsson operator. A general Aronsson operator $\mathscr{A}_H$ is defined by

$$\mathscr{A}_H u(x) = \langle D_x H(Du(x), u(x), x), H_p(Du(x), u(x), x) \rangle$$

2000 *Mathematics Subject Classification.* 35D40, 35J60, 35J70.
Key words and phrases. Inhomogeneous equation; normalized F-infinity Laplacian; viscosity solution.

Submitted March 18, 2014. Published April 16, 2014.

for a function $H : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$, where H_p denotes the gradient of $H(p, s, x)$ with respect to the first variable and $D_x H(Du(x), u(x), x)$ is the gradient of the map $x \mapsto H(Du(x), u(x), x)$. Clearly, $\Delta_{F;\infty}$ is the Aronsson operator $\mathscr{A}_H$ for $H(p, s, x) = \frac{1}{2}F^2(p)$.

The Aronsson equation $\mathscr{A}_H = 0$ was proposed by Aronsson in 1960's [1, 2, 3], which is the Euler-Lagrange equation associated with the variational problem for L^∞-functional

$$\mathscr{F}(u, \Omega) = \operatorname*{ess\,sup}_{x\in\Omega} H(Du(x), u(x), x), \quad u \in W^{1,\infty}(\Omega).$$

In recent years, there have been many studies of properties of the Aronsson equation, especially of the infinity Laplace equation $\Delta_\infty u = 0$ which is corresponding to the special case $H(p) = \frac{1}{2}|p|^2$, see [4, 5, 7, 9, 16, 18, 20, 21, 23, 25], etc. Uniqueness of the viscosity solution of the homogeneous infinity Laplacian equation was established by Jensen in [15]. Later, Barles and Busca gave a second proof of the uniqueness of the infinity harmonic function in [7], their proof is quite different from Jensen's work and applies to many degenerate elliptic equations without zeroth-order term.

But, largely due to the degeneracy of Aronsson operator, even the basic existence and uniqueness questions have been proven difficult. Several approaches were developed to overcome this difficulty, including the notion of viscosity solutions [11] and the method of comparison with cones [8, 12, 13, 14].

In [24], the authors studied the existence of viscosity solutions for the Dirichlet problem of the inhomogeneous equation $F^{-h}(Du)\Delta_{F;\infty} u = f$, where $0 \le h < 2$. The special case $F(p) = p$ was studied in [18] and [17]. The existence and uniqueness of the viscosity solutions of the Dirichlet problem $\Delta_\infty^N u = f$ were established by Peres, Schramm, Sheffield and Wilson in [22] using differential game theory and later reproved by Lu and Wang in [19] using the theory of partial differential equations.

In this paper, we study the existence of viscosity solutions for the Dirichlet problem of the inhomogeneous normalized F-infinity Laplacian equation.

In this paper, Ω is always assumed to be a bounded open subset of $\mathbb{R}^n$, $f \in C(\Omega)$ with $\inf_\Omega f(x) > 0$ or $\sup_\Omega f(x) < 0$ and $g \in C(\partial\Omega)$, we concentrate on the Dirichlet problem

$$\begin{gathered} \Delta_{F;\infty}^N u = f, \quad \text{in } \Omega, \\ u = g, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

We find the "radial" solution to

$$\Delta_{F;\infty}^N u = f, \tag{1.4}$$

where $f = 2a$ is a constant. Additionally, we obtain the existence and uniqueness of solutions to the Dirichlet problem in the viscosity sense. When $F(p) = \frac{1}{2}|p|^2$, these reduce to the cases discussed in [22] and [19]. We employ the classical Perron's method to get the result of existence.

The rest of this paper is organized as follows. In Section 2, we give the notations, definitions related to $\Delta_{F;\infty}^N u$. In Section 3, we give the "radial" solution of the equation $\Delta_{F;\infty}^N u = 1$, and the properties of this solution. In Section 4, we prove our main existence result by Perron's method.

2. Preliminaries

In this paper, Ω will always be a bounded open subset of $\mathbb{R}^n$. We denote the set of continuous functions on a set $V \subset \mathbb{R}^n$ by $C(V)$. If V is a subset of $\mathbb{R}^n$, ∂V denotes its boundary and $\overline{V}$ its closure. The notation $V \subset\subset \Omega$ means that V is an open subset of Ω whose closure $\overline{V}$ is a compact subset of Ω. $o(\epsilon)$ means that $\lim_{\epsilon\to 0} \frac{o(\epsilon)}{\epsilon} = 0$. $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product. $|\cdot|$ denotes the Euclidean norm.

$\mathcal{S}_{n\times n}$ denotes the set of all $n \times n$ symmetric matrices with real entries. $u \in \mathrm{USC}(\Omega)$ denotes the set of all upper semi-continuous functions and $u \in \mathrm{LSC}(\Omega)$ denotes the set of all lower semi-continuous functions.

$u \prec_{x_0} \phi$ means $u - \phi$ has a local maximum at x_0. On the other hand, $u \succ_{x_0} \phi$ means $u - \phi$ has a local minimum at x_0. Almost always in this paper, $u \prec_{x_0} \phi$ (resp. $u \succ_{x_0} \phi$) is understood as $u(x) \le \phi(x)$ (resp. $u(x) \ge \phi(x)$) for all $x \in \Omega$ in interest and $u(x_0) = \phi(x_0)$, as subtracting a constant from ϕ does not cause any problem in the standard viscosity solution argument applied in the paper.

We define $F^* : \mathbb{R}^n \to [0, \infty)$ to be

$$F^*(x) = \sup_{\xi\neq 0} \frac{\langle x, \xi\rangle}{F(\xi)}, \quad \text{for any } x \in \mathbb{R}^n, \tag{2.1}$$

then F^* has same properties (a), (b), (c) as F. Let

$$\alpha = \inf_{\xi\neq 0} \frac{|\xi|}{F(\xi)}, \quad \beta = \sup_{\xi\neq 0} \frac{|\xi|}{F(\xi)},$$

then, by (2.1) and the conditions (a), (b) on F, we have $0 < \alpha \le \beta$ and

$$\alpha|x| \le F^*(x) \le \beta|x|, \text{for any } x \in \mathbb{R}^n. \tag{2.2}$$

From (2.2), we easily get

$$F^*(-x) \le \frac{\beta}{\alpha} F^*(x), \quad \text{for any } x \in \mathbb{R}^n. \tag{2.3}$$

Definition 2.1. For $y \in \mathbb{R}^n$ and $r > 0$, we define $B_r^+(y)$ by $B_r^+(y) = \{x \in \mathbb{R}^n : F^*(x-y) < r\}$, $B_r^-(y)$ by $B_r^-(y) = \{x \in \mathbb{R}^n : F^*(y-x) < r\}$, $S_r^+(y)$ by $S_r^+(y) = \{x \in \mathbb{R}^n : F^*(x-y) = r\}$, $S_r^-(y)$ by $S_r^-(y) = \{x \in \mathbb{R}^n : F^*(y-x = r\}$.

For $u \in C(\Omega)$, $x_0 \in \Omega$, and $r > 0$ with $\overline{B_r^+(x_0) \cup B_r^-(x_0)} \subset \Omega$, we define $g(r) = \max_{F^*(x-x_0)=r} u(x)$ and $h(r) = \min_{F^*(x_0-x)=r} u(x)$. In addition, x_r^+ denotes any point with $F^*(x_r^+ - x_0) = r$ such that $u(x_r^+) = g(r)$, while x_r^- denotes any point with $F^*(x_0 - x_r^-) = r$ such that $u(x_r^-) = h(r)$.

If $x_0 \in \Omega$ and $u \in C(\Omega)$ such that u is twice differentiable at x_0, we define the set of maximum directions of u at x_0 to be the set

$$E^+(x_0) = \{e = \lim_k \frac{x_{r_k}^+ - x_0}{r_k} \text{ for some sequence } r_k \downarrow 0\}$$

and the set of minimum directions of u at x_0 to be the set

$$E^-(x_0) = \{e = \lim_k \frac{x_{r_k}^- - x_0}{r_k} \text{ for some sequence } r_k \downarrow 0\}.$$

Definition 2.2. If $u \in C(\Omega)$ is twice differentiable at x_0, we define the upper F-infinity Laplacian of u at x_0 to be $\Delta^+_{F;\infty} u(x_0) = \langle D^2u(x_0)e, e\rangle$, where e is any maximum direction of u at x_0 .

Similarly, the lower F-infinity Laplacian of u at x_0 is defined to be $\Delta^-_{F;\infty}u(x_0) = \langle D^2u(x_0)e, e\rangle$, where e is any minimum direction of u at x_0 .

Remark 2.3. From Proposition 2.5 which will be proved below, the definition of $\Delta^+_{F;\infty}u(x_0)$ (resp. $\Delta^-_{F;\infty}u(x_0)$) is independent of the choice of maximum (resp. minimum) direction of u at x_0.

Lemma 2.4 ([6, page 7])**.** *For any $y \in \mathbb{R}^n \setminus \{0\}$ and $w \in \mathbb{R}^n$, we have*

$$w \cdot DF(y) \le F(w), \tag{2.4}$$

and equality holds if and only of $w = \alpha y$ for some $\alpha \ge 0$.

Proposition 2.5. *Suppose $u \in C(\Omega)$ is twice differentiable at x_0.*

(1) If $Du(x_0) \ne 0$, then

$$\Delta^+_{F;\infty}u(x_0) = \Delta^-_{F;\infty}u(x_0) = \langle D^2u(x_0)DF(Du(x_0)), DF(Du(x_0))\rangle.$$

(2) If $Du(x_0) = 0$, then

$$\Delta^+_{F;\infty}u(x_0) = \max\{\langle D^2u(x_0)e, e\rangle : F^*(e) = 1\},$$

$$\Delta^-_{F;\infty}u(x_0) = \min\{\langle D^2u(x_0)e, e\rangle : F^*(e) = 1\}.$$

Proof. (1) There exists a positive-valued function ρ with $\rho(r) \to 0$ as $r \downarrow 0$, defined for all small positive numbers r, such that

$$|u(x) - u(x_0) - Du(x_0) \cdot (x - x_0)| \le \rho(r)r \tag{2.5}$$

for all x with $F^*(x - x_0) = r$.

Take $\tilde{x}^+_r = x_0 + rDF(Du(x_0))$. Then

$$\begin{aligned} &u(x_0) + Du(x_0) \cdot (x^+_r - x_0) - \rho(r)r \\ &\le u(x^+_r) \le u(x_0) + Du(x_0) \cdot (\tilde{x}^+_r - x_0) + \rho(r)r. \end{aligned}$$

The second inequality is due to the choice of $\tilde{x}^+_r$ and Lemma 2.4. So, $Du(x_0)\cdot(x^+_r - \tilde{x}^+_r) \le 2\rho(r)r$.

On the other hand, the chain of inequalities

$$\begin{aligned} &u(x_0) + Du(x_0) \cdot (\tilde{x}^+_r - x_0) - \rho(r)r \\ &\le u(\tilde{x}^+_r) \le u(x^+_r) \le u(x_0) + Du(x_0) \cdot (x^+_r - x_0) + \rho(r)r \end{aligned}$$

implies $Du(x_0) \cdot (x^+_r - \tilde{x}^+_r) \ge -2\rho(r)r$. So

$$|Du(x_0) \cdot \frac{x^+_r - x_0}{r} - F(Du(x_0))| = |Du(x_0) \cdot \frac{x^+_r - \tilde{x}^+_r}{r}| \le 2\rho(r). \tag{2.6}$$

Thus,

$$\lim_{r\downarrow 0} Du(x_0) \cdot \frac{x^+_r - x_0}{r} = F(Du(x_0)). \tag{2.7}$$

Then, for any $r_k \downarrow 0$ such that $\lim_k \frac{x^+_{r_k} - x_0}{r_k} = DF(y_0)$ exists, we must have $Du(x_0)\cdot DF(y_0) = F(Du(x_0))$. So, by Lemma 2.4, $DF(y_0) = DF(Du(x_0))$. Thus, for any $e \in E^+(x_0)$, $e = DF(Du(x_0))$ holds. Similarly, $E^-(x_0) = \{-DF(Du(x_0))\}$. Therefore,

$$\Delta^+_{F;\infty}u(x_0) = \Delta^-_{F;\infty}u(x_0) = \langle D^2u(x_0)DF(Du(x_0)), DF(Du(x_0))\rangle.$$

(2) If $Du(x_0) = 0$, then there exists a positive-valued function ρ with $\rho(r) \to 0$ as $r \downarrow 0$, defined for all small positive numbers r, such that

$$|u(x) - u(x_0) - \langle D^2u(x_0)(x - x_0), x - x_0\rangle| \le \rho(r)r^2 \tag{2.8}$$

for all x with $F^*(x - x_0) = r$.

Let $\lambda^+ = \max\{\langle D^2u(x_0)e, e\rangle : F^*(e) = 1\}$ and $e^+ \in S_1^+(0)$ be such that $\lambda^+ = \langle D^2u(x_0)e^+, e^+\rangle$. Take $\tilde{x}_r^+ = x_0 + re^+$. Then

$$\begin{aligned}
&u(x_0) + \langle D^2u(x_0)(x_r^+ - x_0), x_r^+ - x_0\rangle - \rho(r)r^2\\
&\le u(x_r^+)\\
&\le u(x_0) + \langle D^2u(x_0)(\tilde{x}_r^+ - x_0), \tilde{x}_r^+ - x_0\rangle + \rho(r)r^2.
\end{aligned}$$

So,

$$\langle D^2u(x_0)(x_r^+ - x_0), x_r^+ - x_0\rangle - \langle D^2u(x_0)(\tilde{x}_r^+ - x_0), \tilde{x}_r^+ - x_0\rangle \le 2\rho(r)r^2.$$

On the other hand, the chain of inequalities

$$\begin{aligned}
&u(x_0) + \langle D^2u(x_0)(\tilde{x}_r^+ - x_0), \tilde{x}_r^+ - x_0\rangle - \rho(r)r^2\\
&\le u(\tilde{x}_r^+) \le u(x_r^+)\\
&\le u(x_0) + \langle D^2u(x_0)(x_r^+ - x_0), x_r^+ - x_0\rangle + \rho(r)r^2
\end{aligned}$$

implies

$$\langle D^2u(x_0)(x_r^+ - x_0), x_r^+ - x_0\rangle - \langle D^2u(x_0)(\tilde{x}_r^+ - x_0), \tilde{x}_r^+ - x_0\rangle \ge -2\rho(r)r^2.$$

So

$$|\langle D^2u(x_0)(\frac{x_r^+ - x_0}{r}), \frac{x_r^+ - x_0}{r}\rangle - \lambda^+| \le 2\rho(r). \tag{2.9}$$

Then, take any $r_k \downarrow 0$ such that $\lim_k \frac{x_{r_k}^+ - x_0}{r_k} = e \in E^+(x_0)$, we see $\Delta_{F;\infty}^+ u(x_0) = \lambda^+$.

Similarly, we have $\Delta_{F;\infty}^- u(x_0) = \min\{\langle D^2u(x_0)e, e\rangle : F^*(e) = 1\}$. □

We are then concerned with the viscosity solutions of (1.4) given in the following definition.

Definition 2.6. $u : \Omega \to \mathbb{R}$ is called a viscosity subsolution of the partial differential equation $\Delta_{F;\infty}^N u(x) = f(x)$ in Ω, if for any $x_0 \in \Omega$ and any test function $\phi \in C^2(\Omega)$ with $u \prec_{x_0} \phi$, there holds

$$\Delta_{F;\infty}^+ \phi(x_0) \ge f(x_0).$$

In this case, we say $\Delta_{F;\infty}^N u \ge f$ in the viscosity sense.

Similarly, $u : \Omega \to \mathbb{R}$ is called a viscosity supersolution of the partial differential equation $\Delta_{F;\infty}^N u(x) = f(x)$ in Ω, if for any $x_0 \in \Omega$ and any test function $\phi \in C^2(\Omega)$ with $u \succ_{x_0} \phi$, there holds

$$\Delta_{F;\infty}^- \phi(x_0) \le f(x_0).$$

In this case, we say $\Delta_{F;\infty}^N u \le f$ in the viscosity sense.

A viscosity solution of the partial differential equation $\Delta_{F;\infty}^N u(x) = f(x)$ in Ω is both a viscosity subsolution and viscosity supersolution of the equation.

Furthermore, viscosity solutions of the Dirichlet problem (1.3) are defined as follows.

Definition 2.7. A function $u : \Omega \to \mathbb{R}$ is called a viscosity subsolution (resp., supersolution) of (1.3) if u is a viscosity subsolution (resp., supersolution) in Ω of (1.4) and $u \le g$ (resp., $u \ge g$) on $\partial\Omega$. Furthermore, $u : \Omega \to \mathbb{R}$ is a viscosity solution of (1.3) if it is both a viscosity subsolution and a viscosity supersolution of (1.3).

We will need the concepts of superjets and subjets in our approach.

Definition 2.8. Suppose $u \in C(\Omega)$. The second-order superjet of u at x_0 is defined to be the set

$$J_\Omega^{2,+}u(x_0) = \{(D\phi(x_0), D^2\phi(x_0)) : \phi \text{ is } C^2 \text{ and } u \prec_{x_0} \phi\},$$

whose closure is defined to be

$$\bar{J}_\Omega^{2,+}u(x_0) = \Big\{(p, X) \in \mathbb{R}^n \times \mathcal{S}_{n\times n} : \exists (x_n, p_n, X_n) \in \Omega \times \mathbb{R}^n \times \mathcal{S}_{n\times n} \text{ such that } (p_n, X_n) \in J_\Omega^{2,+}u(x_n) \text{ and } (x_n, u(x_n), p_n, X_n) \to (x_0, u(x_0), p, X)\Big\}.$$

The second-order subjet of u at x_0 is defined to be the set

$$J_\Omega^{2,-}u(x_0) = \{(D\phi(x_0), D^2\phi(x_0)) : \phi \text{ is } C^2 \text{ and } u \succ_{x_0} \phi\},$$

whose closure is defined to be

$$\bar{J}_\Omega^{2,-}u(x_0) = \Big\{(p, X) \in \mathbb{R}^n \times \mathcal{S}_{n\times n} : \exists (x_n, p_n, X_n) \in \Omega \times \mathbb{R}^n \times \mathcal{S}_{n\times n} \text{ such that } (p_n, X_n) \in J_\Omega^{2,-}u(x_n) \text{ and } (x_n, u(x_n), p_n, X_n) \to (x_0, u(x_0), p, X)\}.$$

Lemma 2.9 ([10])**.** *(i)*

$$F^*(DF(p)) = 1 \text{ for } p \in \mathbb{R}^n \setminus \{0\}, \tag{2.10}$$

$$F(DF^*(x)) = 1 \text{ for } x \in \mathbb{R}^n \setminus \{0\}; \tag{2.11}$$

(ii) the map $FDF : \mathbb{R}^n \to \mathbb{R}^n$ is invertible and

$$FDF = (F^*DF^*)^{-1}. \tag{2.12}$$

Here, and in what follows, FDF and F^*DF^* are continued by 0 at 0.

Remark 2.10. We note we only assume F to be positively homogenous of degree 1, not homogenous of degree 1, so $F(-x) \neq F(x)$ in general, thus $F^*(-x) \neq F^*(x)$ in general either.

Lemma 2.11. *(1) I is an index set, $f \in C(\Omega)$, for any $\lambda \in I$, $\Delta_{F;\infty}^N u_\lambda \ge f$ in Ω in the viscosity sense, $u(x) = \sup_{x\in\Omega} u_\lambda(x) < \infty$, then $\Delta_{F;\infty}^N u \ge f$ in Ω in the viscosity sense. (2) I is an index set, $f \in C(\Omega)$, for any $\lambda \in I$, $\Delta_{F;\infty}^N u_\lambda \le f$ in Ω in the viscosity sense, $u(x) = \inf_{x\in\Omega} u_\lambda(x) > -\infty$, then $\Delta_{F;\infty}^N u \le f$ in Ω in the viscosity sense.*

Proof. Because the proof of (2) is similar to that of (1), we only present the proof of (1). Suppose $\Delta_{F;\infty}^N u \ge f$ in the viscosity sense is not true in Ω. Then there exists a point $x_0 \in \Omega$ and a test function $\phi \in C^2(\Omega)$ such that $u \prec_{x_0} \phi$ and $\Delta_{F;\infty}^+ \phi(x_0) < f(x_0)$. If we replace ϕ by ϕ_δ defined by

$$\phi_\delta(x) = \phi(x) + \delta|x - x_0|^2$$

with $\delta > 0$, then $u - \phi_\delta$ has a strict maximum at point x_0; i.e., $u(x_0) = \phi_\delta(x_0)$, $u(x) < \phi_\delta(x), x \neq x_0$, and we have

$$\Delta^+_{F;\infty}\phi_\delta(x_0) = \Delta^+_{F;\infty}\phi(x_0) + O(\delta) < f(x_0),$$

if $\delta > 0$ is taken small enough. So we can assume that the original test function ϕ satisfies

$$\phi(x) \geq u(x) + \delta|x - x_0|^2$$

for some $\delta > 0$.

We claim that $\Delta^+_{F;\infty}\phi(x) < f(x)$ in an open neighborhood $B_r(x_0)$ of x_0. In fact, we prove the claim via a dichotomy.

If $D\phi(x_0) \neq 0$, then $D\phi(x) \neq 0$ in a neighborhood $B_R(x_0)$ of x_0. The continuity of f and $D^2\phi$ implies that in a neighborhood $B_r(x_0) \subset B_R(x_0)$ of x_0,

$$\Delta^+_{F;\infty}\phi(x) = \langle D^2\phi(x)DF(D\phi(x)), DF(D\phi(x))\rangle < f(x).$$

If $D\phi(x_0) = 0$, then $\Delta^+_{F;\infty}\phi(x_0) = \max\{\langle D^2\phi(x_0)e, e\rangle : F^*(e) = 1\} < f(x_0)$. So in a neighborhood $B_r(x_0)$ of x_0,

$$\Delta^+_{F;\infty}\phi(x) \leq \max\{\langle D^2\phi(x)e, e\rangle : F^*(e) = 1\} < f(x).$$

The claim is proved.

For any ϵ with $0 < \epsilon < \delta r^2$, there exists $\lambda \in I$ such that $u_\lambda(x_0) > u(x_0) - \epsilon$. Let $\hat{\phi}(x) = \phi(x) - \epsilon$. Then $\hat{\phi}(x_0) < u_\lambda(x_0)$ and

$$\hat{\phi}(x) \geq u(x) - \epsilon + \delta|x - x_0|^2 > u(x) \geq u_\lambda(x)$$

on $\partial B_r(x_0)$. So there exists $x_* \in B_r(x_0)$ such that $u_\lambda - \hat{\phi}$ has maximum at x_*. As $\Delta^+_{F;\infty}u_\lambda \geq f$ in Ω in the viscosity sense and $u_\lambda \prec_{x_*} \hat{\phi}$, we have

$$\Delta^+_{F;\infty}\hat{\phi}(x_*) \geq f(x_*),$$

which is a contradiction with the claim we just have derived,

$$\Delta^+_{F;\infty}\hat{\phi}(x) = \Delta^+_{F;\infty}\phi(x) < f(x)$$

in $B_r(x_0)$. □

3. Solutions of the equation $\Delta^N_{F;\infty}u = 2a$

Let $u(x) = a[F^*(x)]^2 + BF^*(x) + C$, where $a \neq 0$, B, C are all constants. Suppose $\{x \in \mathbb{R}^n \setminus \{0\} : 2aF^*(x) + B > 0\}$ is a nonempty domain, in this domain, we calculate:

$$\frac{\partial u}{\partial x_i} = [2aF^*(x) + B]\frac{\partial F^*}{\partial x_i}, \tag{3.1}$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = 2a\frac{\partial F^*}{\partial x_i} \cdot \frac{\partial F^*}{\partial x_j} + [2aF^*(x) + B]\frac{\partial^2 F^*}{\partial x_i \partial x_j}. \tag{3.2}$$

As F is positively homogeneous of degree 1, $\frac{\partial F}{\partial p_i}$ is positively homogeneous of degree 0. So by (2.11) and (2.12), we have

$$\frac{\partial F}{\partial p_i}(DF^*(x)) = \frac{x_i}{F^*(x)}. \tag{3.3}$$

Thus, by (2.11), (3.1) and (3.3), we obtain

$$F(Du(x)) = 2aF^*(x) + B, \tag{3.4}$$

$$\frac{\partial F}{\partial p_i}(Du(x)) = \frac{x_i}{F^*(x)}. \tag{3.5}$$

Since F^* is of class $C^2(\mathbb{R}^n \setminus \{0\})$ and positively homogeneous of degree 1, we have

$$\sum_{i=1}^{n} \frac{\partial F^*}{\partial x_i} x_i = F^*(x), \quad \sum_{i=1}^{n} \frac{\partial^2 F^*}{\partial x_i \partial x_j} x_i = 0, \quad \text{for all } x \neq 0. \tag{3.6}$$

Using (3.2), (3.4), (3.5) and (3.6), through direct calculation, we obtain

$$\Delta^N_{F;\infty} u = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial F}{\partial p_i}(Du(x)) \cdot \frac{\partial F}{\partial p_j}(Du(x)) = 2a.$$

Thus, we proved that $u(x) = a[F^*(x)]^2 + BF^*(x) + C$ is a solution of the equation

$$\Delta^N_{F;\infty} u = 2a \tag{3.7}$$

in the domain $\{x \in \mathbb{R}^n \setminus \{0\} : 2aF^*(x) + B > 0\}$.

Since (3.7) is invariant by translation,

$$\Psi_{x_0,BC}(x) = a[F^*(x - x_0)]^2 + BF^*(x - x_0) + C$$

is its C^2 solution in

$$D^+(x_0, B) := \{x \in \mathbb{R}^n \setminus \{x_0\} : 2aF^*(x - x_0) + B > 0\}.$$

In particular, we have the following lemma.

Lemma 3.1. *$\Psi_{x_0,BC}(x)$ is a viscosity solution of (3.7) in $D^+(x_0, B)$.*

Proof. The fact that a classical solution is a viscosity solution follows easily from the definition of a viscosity solution. □

Remark 3.2. Similarly, let

$$\Phi_{x_0,BC}(x) = -a[F^*(x_0 - x)]^2 + BF^*(x_0 - x) + C,$$
$$D^-(x_0, B) = \{x \in \mathbb{R}^n \setminus \{x_0\} : 2aF^*(x_0 - x) + B > 0\},$$

then $\Phi_{x_0,BC}(x)$ is a viscosity solution of equation

$$\Delta^N_{F;\infty} u = -2a \tag{3.8}$$

in $D^-(x_0, B)$.

For simplicity, taking $a = 1/2$. Letting $B = 0$, $\Psi_{x_0}(x) = \frac{1}{2}[F^*(x - x_0)]^2 + C$ and $D(x_0) = D^+(x_0, B) = \mathbb{R}^n \setminus \{x_0\}$.

4. A strict comparison principle

Theorem 4.1. *For $j = 1, 2$, suppose $u_j \in C(\overline{\Omega})$ and*

$$\Delta^N_{F;\infty} u_1 \leq f_1, \quad \Delta^N_{F;\infty} u_2 \geq f_2$$

in Ω, where $f_1 < f_2$, and $f_j \in C(\Omega)$. Then $\sup_\Omega (u_2 - u_1) \leq \max_{\partial\Omega}(u_2 - u_1)$.

Proof. Without the loss of generality, we may assume $u_2 \leq u_1$ on $\partial\Omega$ and intend to prove $u_2 \leq u_1$ in Ω. Furthermore, for any small $\delta > 0$, let $u_\delta = u_2 - \delta$. Then $u_\delta < u_1$ on $\partial\Omega$ and $\Delta^N_{F;\infty} u_\delta \geq f_2$ in Ω. If we can show that $u_\delta < u_1$ in Ω for every small $\delta > 0$, then it follows that $u_2 \leq u_1$ in Ω. So we may additionally assume $u_2 < u_1$ on $\partial\Omega$ in the following proof.

We apply the sup- and inf-convolution technique here. Take any

$$A \geq \max\{\|u_1\|_{L^\infty(\Omega)}, \|u_2\|_{L^\infty(\Omega)}\}.$$

For any sufficiently small real number $\epsilon > 0$, we take $\delta = 3\sqrt{A\epsilon}$ and $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$. We define, on $\mathbb{R}^n$,

$$u_{1,\epsilon}(x) = \inf_{y\in\Omega}(u_1(y) + \frac{1}{2\epsilon}|x-y|^2), \tag{4.1}$$

$$u_2^\epsilon(x) = \sup_{y\in\Omega}(u_2(y) - \frac{1}{2\epsilon}|x-y|^2) \tag{4.2}$$

For any $y \in \Omega$ such that $|y-x| \geq 2\sqrt{A\epsilon}$, $u_1(y) + \frac{1}{2\epsilon}|x-y|^2 \geq u_1(x)$ holds. So, in Ω_δ,

$$u_{1,\epsilon}(x) = \inf_{y\in\Omega, |x-y|\leq 2\sqrt{A\epsilon}}(u_1(y) + \frac{1}{2\epsilon}|x-y|^2) = \inf_{|z|\leq 2\sqrt{A\epsilon}}(u_1(x+z) + \frac{1}{2\epsilon}|z|^2), \tag{4.3}$$

as $x + z \in \Omega$ for any $x \in \Omega_\delta$ and $|z| \leq 2\sqrt{A\epsilon}$. Similarly, for $x \in \Omega_\delta$,

$$u_2^\epsilon(x) = \sup_{y\in\Omega, |x-y|\leq 2\sqrt{A\epsilon}}(u_2(y) - \frac{1}{2\epsilon}|x-y|^2) = \sup_{|z|\leq 2\sqrt{A\epsilon}}(u_2(x+z) - \frac{1}{2\epsilon}|z|^2), \tag{4.4}$$

Let

$$f_1^\epsilon(x) = \sup_{x+z\in\Omega, |z|\leq 2\sqrt{A\epsilon}} f_1(x+z) = \sup_{|z|\leq 2\sqrt{A\epsilon}} f_1(x+z), \tag{4.5}$$

$$f_{2,\epsilon}(x) = \inf_{x+z\in\Omega, |z|\leq 2\sqrt{A\epsilon}} f_2(x+z) = \inf_{|z|\leq 2\sqrt{A\epsilon}} f_2(x+z), \tag{4.6}$$

for $x \in \Omega_\delta$. Clearly, f_1^ϵ is upper-semicontinuous. It is continuous due to the equicontinuity of the one parameter family of the functions $x \mapsto f_1(x+z)$ in any compact subset of Ω. $f_{2,\epsilon}$ is continuous for a similar reason.

We notice that, for every z with $|z| \leq 2\sqrt{A\epsilon}$ and $x \in \Omega_\delta$,

$$\Delta^N_{F;\infty}(u_1(x+z) + \frac{1}{2\epsilon}|z|^2) \leq f_1(x+z) \leq f_1^\epsilon(x), \tag{4.7}$$

$$\Delta^N_{F;\infty}(u_2(x+z) - \frac{1}{2\epsilon}|z|^2) \geq f_2(x+z) \geq f_{2,\epsilon}(x). \tag{4.8}$$

Lemma 2.11 implies that $\Delta^N_{F;\infty} u_{1,\epsilon} \leq f_1^\epsilon$ and $\Delta^N_{F;\infty} u_2^\epsilon \geq f_{2,\epsilon}$ in Ω_δ in the viscosity sense.

By [5, Proposition 6.4], we have the following result.

Proposition 4.2. *$-u_{1,\epsilon}$ and u_2^ϵ are semi-convex in $\mathbb{R}^n$. $u_{1,\epsilon} \leq u_1$ and $u_2^\epsilon \geq u_2$ in Ω. $u_{1,\epsilon}$ and u_2^ϵ converge locally uniformly to u_1 and u_2 in Ω, as $\epsilon \to 0$. $u_{1,\epsilon}$ and u_2^ϵ are both differentiable at the maximum points of $u_2^\epsilon - u_{1,\epsilon}$.*

As a result, if we take the value of ϵ smaller if necessary, then $u_{1,\epsilon} > u_2^\epsilon$ on $\partial\Omega_\delta$, $\Delta^N_{F;\infty} u_{1,\epsilon} \leq f_1^\epsilon$ and $\Delta^N_{F;\infty} u_2^\epsilon \geq f_{2,\epsilon}$ in Ω_δ, and $f_1^\epsilon < f_{2,\epsilon}$ in Ω_δ.

If we can prove $u_2^\epsilon \leq u_{1,\epsilon}$ in Ω_δ for any small $\epsilon > 0$ and $\delta = 3\sqrt{A\epsilon}$, then $u_2 \leq u_1$ in Ω holds. So we may without loss of generality assume that $-u_1$ and u_2 are semi-convex in $\mathbb{R}^n$.

Suppose $u_1(x_0) < u_2(x_0)$ for some $x_0 \in \Omega$. Without the loss of generality, we assume that $u_2(x_0) - u_1(x_0) = \max_\Omega(u_2 - u_1)$. Then $\exists \delta > 0$ such that for any $h \in \mathbb{R}^n$ with $|h| < \delta$, we have $u_1(x_0) < u_2(x_0+h)$, while $u_2(\cdot + h) < u_1(\cdot)$ in $\Omega \setminus \Omega_\delta$, and $f_2(x+h) > f_1(x)$, for all $x \in \Omega_\delta$. For any small positive number ϵ and $h \in \mathbb{R}^n$ with $|h| < \delta$, we define

$$w_{\epsilon,h}(x,y) = u_2(x+h) - u_1(y) - \frac{1}{2\epsilon}|x-y|^2, \tag{4.9}$$

for all $(x,y) \in \overline{\Omega}_\delta \times \overline{\Omega}_\delta$. Let

$$M_0 = \max_{\Omega}(u_2 - u_1), \tag{4.10}$$

$$M_h = \max_{\overline{\Omega}_\delta}(u_2(\cdot + h) - u_1(\cdot)), \tag{4.11}$$

$$M_{\epsilon,h} = \max_{\overline{\Omega}_\delta \times \overline{\Omega}_\delta} w_{\epsilon,h} = u_2(x_{\epsilon,h}) - u_1(y_{\epsilon,h}) - \frac{1}{2\epsilon}|x_{\epsilon,h} - y_{\epsilon,h}|^2 \tag{4.12}$$

for some $(x_{\epsilon,h}, y_{\epsilon,h}) \in \overline{\Omega}_\delta \times \overline{\Omega}_\delta$. Our assumption implies $M_h > 0$ for all h with $0 \le |h| < \delta$, and clearly $\lim_{h\to 0} M_h = M_0$.

As the semi-convex functions $u_2(\cdot + h)$ and $-u_1$ are locally Lipschitz continuous, the function M_h is Lipschitz continuous in $h \in \mathbb{R}^n$ with $|h| < \delta$, if δ is taken smaller.

By [11, Lemma 3.1], we know that

$$\lim_{\epsilon\downarrow 0} M_{\epsilon,h} = M_h, \tag{4.13}$$

$$\lim_{\epsilon\downarrow 0} \frac{1}{2\epsilon}|x_{\epsilon,h} - y_{\epsilon,h}|^2 = 0, \tag{4.14}$$

$$\lim_{\epsilon\downarrow 0}(u_2(x_{\epsilon,h} + h) - u_1(y_{\epsilon,h})) = M_h. \tag{4.15}$$

As a result of the second equality, $\lim_{\epsilon\downarrow 0} |x_{\epsilon,h} - y_{\epsilon,h}| = 0$.

As $M_h > 0 \ge \max_{\partial\Omega_\delta}(u_2(\cdot + h) - u_1(\cdot))$, we know $x_{\epsilon,h}, y_{\epsilon,h} \in \Omega_1$ for some $\Omega_1 \subset\subset \Omega_\delta$ and all small $\epsilon > 0$.

Then [11, Theorem 3.2] implies that there exist $X = X_{\epsilon,h}, Y = Y_{\epsilon,h} \in \mathcal{S}_{n\times n}$ such that $(\frac{x_{\epsilon,h} - y_{\epsilon,h}}{\epsilon}, X) \in \bar{J}^{2,+}_\Omega u_2(x_\epsilon + h)$, $(\frac{x_{\epsilon,h} - y_{\epsilon,h}}{\epsilon}, Y) \in \bar{J}^{2,-}_\Omega u_1(y_\epsilon)$ and

$$-\frac{3}{\epsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{3}{\epsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{4.16}$$

In particular, $X \le Y$.

Again, we solve the problem via a dichotomy.

Case 1. Suppose that $\exists h$ with $|h| < \delta$, and $\epsilon_k \to 0$ such that $x_{\epsilon_k,h} \ne y_{\epsilon_k,h}$. Then it is easy to see that

$$\begin{aligned} f_2(x_{\epsilon_k,h}) &\le \langle X(DF(\frac{x_{\epsilon_k,h} - y_{\epsilon_k,h}}{\epsilon_k})), DF(\frac{x_{\epsilon_k,h} - y_{\epsilon_k,h}}{\epsilon_k})\rangle \\ &\le \langle Y(DF(\frac{x_{\epsilon_k,h} - y_{\epsilon_k,h}}{\epsilon_k})), DF(\frac{x_{\epsilon_k,h} - y_{\epsilon_k,h}}{\epsilon_k})\rangle \\ &\le f_1(y_{\epsilon_k,h}). \end{aligned}$$

For a subsequence of $\{\epsilon_k\}$, $x_{\epsilon_k,h} \to x_h$ and $y_{\epsilon_k,h} \to y_h$. As $\lim_{\epsilon\downarrow 0} |x_{\epsilon_k,h} - y_{\epsilon_k,h}| = 0$, we know that $x_h = y_h$, which leads to a contradiction with the assumption $f_1(x_h) < f_2(x_h)$.

Case 2. For every $h \in \mathbb{R}^n$ with $|h| < \delta$, $x_{\epsilon,h} = y_{\epsilon,h}$ holds for every small $\epsilon > 0$. Then $M_{\epsilon,h} = u_2(x_{\epsilon,h} + h) - u_1(y_{\epsilon,h}) = M_h$. We simply write $x_{\epsilon,h} = y_{\epsilon,h} = x_h$. The semi-convexity of $u_2(\cdot + h)$ and $-u_1(\cdot)$ implies that the two functions are differentiable at the maximum point x_h of their sum. The definition of x_h shows that

$$u_2(x_h + h) - u_1(x_h) \geq u_2(y + h) - u_1(x_h) - \frac{1}{2\epsilon}|x_h - y|^2, \tag{4.17}$$

which in turn implies

$$u_2(x_h + h) \geq u_2(y + h) - \frac{1}{2\epsilon}|x_h - y|^2, \tag{4.18}$$

for small $\epsilon > 0$. So $Du_2(x_h + h) = Du_1(x_h) = 0$.

For small $h, k \in \mathbb{R}^n$,

$$\begin{aligned} M_h &= u_2(x_h + h) - u_1(x_h) \geq u_2(x_k + h) - u_1(x_k) \\ &= M_k + u_2(x_k + h) - u_2(x_k + k) \geq M_k - o(|h - k|), \end{aligned}$$

as $Du_2(x_k + k) = 0$. So $DM_h = 0$ *a.e.* as M_h is Lipschitz continuous, which implies $M_h = M_0$ for all small $h \in \mathbb{R}^n$.

At x_0, either $f_1(x_0) < 0$ or $f_2(x_0) > 0$ holds due to the fact $f_1 < f_2$. Without loss of generality, we assume that $f_2(x_0) > 0$. The proof for the case $f_1(x_0) < 0$ is parallel. So u_2 is ∞-subharmonic in a neighborhood of x_0.

For any h with $|h| < \delta$,

$$u_2(x_0 + h) - u_1(x_0) \leq u_2(x_h + h) - u_1(x_h) = u_2(x_0) - u_1(x_0). \tag{4.19}$$

So $u_2(x_0)$ is a local maximum of u_2. As $\Delta_{F;\infty} u_2 \geq 0$, the maximum principle for infinity harmonic functions implies that u_2 is constant near x_0. So we have

$$\Delta^N_{F;\infty} u_2(x_0) = \max\{\langle D^2 u_2(x_0)e, e\rangle : F^*(e) = 1\} = 0 < f_2(x_0), \tag{4.20}$$

which is a contradiction. □

Theorem 4.3 (Comparison Principle). *Suppose $u, v \in C(\overline{\Omega})$ satisfy*

$$\Delta^N_{F;\infty} u \geq f(x), \tag{4.21}$$

$$\Delta^N_{F;\infty} v \leq f(x) \tag{4.22}$$

in the viscosity sense in the domain Ω, where f is a continuous positive function defined on Ω. Then

$$\sup_{\Omega}(u - v) \leq \max_{\partial\Omega}(u - v). \tag{4.23}$$

Proof. Without loss of generality, we may assume that $u \leq v$ on $\partial\Omega$ and intend to prove $u \leq v$ in Ω. For a small $\delta > 0$, we take

$$u_\delta(x) = (1 + \delta)u(x) - \delta\|u\|_{L^\infty(\partial\Omega)}. \tag{4.24}$$

Then $u_\delta \leq u \leq v$ on $\partial\Omega$, and it is easily checked by the standard viscosity solution theory that

$$\Delta^N_{F;\infty} u_\delta(x) = (1 + \delta)\Delta^N_{F;\infty} u(x) \geq (1 + \delta)f(x) > f(x) \geq \Delta^N_{F;\infty} v(x) \tag{4.25}$$

in Ω in the viscosity sense.

Applying the preceding strict comparison theorem to v and u_δ, we have $u_\delta \leq v$ in Ω for any small $\delta > 0$. Sending δ to 0, we have $u \leq v$ in Ω as desired. □

5. Existence theorem

In this section, we prove existence of (1.3) by Perron's method. Firstly we prove some lemmas.

Lemma 5.1. *Let U be bounded, $u \in \mathrm{USC}(\overline{U})$ and $\Delta_{F;\infty} u \geq 0$ in U. If $x_0 \in \mathbb{R}^n$, $a \in \mathbb{R}$, $b \geq 0$ and*

$$u(x) \leq C(x) = a + bF^*(x - x_0) \quad \text{for } x \in \partial(U \setminus \{x_0\}), \tag{5.1}$$

then

$$u(x) \leq C(x) \quad \text{for } x \in U. \tag{5.2}$$

Proof. Firstly we assume $b > 0$. Assume that $u(\hat{x}) - C(\hat{x}) > 0$ at some point $\hat{x} \in U \setminus \{x_0\}$. Choose R so large that $F^*(x - x_0) \leq R$ on ∂U and put $w = a + bF^*(x - x_0) + \epsilon(R^2 - [F^*(x - x_0)]^2)$. Then $u \leq w$ on $\partial(U \setminus \{x_0\})$, whereas $u(\hat{x}) - w(\hat{x}) > 0$ if ϵ is sufficiently small. We may assume that $\hat{x}$ is the maximum of $u - w$ on $U \setminus \{x_0\}$. Through direct calculation, we have

$$\frac{\partial w}{\partial x_i} = [b - 2\epsilon F^*(x - x_0)] \frac{\partial F^*}{\partial x_i}(x - x_0), \tag{5.3}$$

$$\frac{\partial^2 w}{\partial x_i \partial x_j} = [b - 2\epsilon F^*(x - x_0)] \frac{\partial^2 F^*}{\partial x_i \partial x_j}(x - x_0) - 2\epsilon \frac{\partial F^*}{\partial x_i}(x - x_0) \cdot \frac{\partial F^*}{\partial x_j}(x - x_0). \tag{5.4}$$

Since $b > 0$, we have $b - 2\epsilon F^*(\hat{x} - x_0) > 0$, if we choose ϵ sufficiently small. So the 1-positively homogeneous of F, (2.11), (2.12) and (5.3) imply

$$F(Dw)(\hat{x}) = b - 2\epsilon F^*(\hat{x} - x_0), \quad DF(Dw)(\hat{x}) = \frac{\hat{x} - x_0}{F^*(\hat{x} - x_0)}. \tag{5.5}$$

Using (3.6), (5.4) and (5.5), we obtain $\Delta_{F;\infty} w(\hat{x}) = -2\epsilon(b - 2\epsilon F^*(\hat{x} - x_0))^2$, and this is strictly negative. This contradicts the assumption $\Delta_{F;\infty} u \geq 0$.

If $b = 0$, we substitute b by $\delta > 0$ in (5.1) and let $\delta \to 0$. $\square$

Lemma 5.2. *Let U be bounded, $u \in \mathrm{USC}(\overline{U})$ and $\Delta_{F;\infty} u \geq 0$ in U. Then the function defined for $y \in U$ and $r < \alpha d(y, \partial U)$ by*

$$L_r^+(y) := \inf\{k \geq 0 : u(z) \leq u(y) + kr, \forall z \in S_r^+(y)\} \tag{5.6}$$

is nondecreasing in r.

Proof. $L_r^+(y)$ is the smallest nonnegative constant for which

$$u(x) \leq u(y) + L_r^+(y) F^*(x - y)$$

holds for $F^*(x-y) = r$. Lemma 5.1 then implies the inequality holds for $F^*(x-y) \leq r$. Thus $(u(x) - u(y))/F^*(x - y) \leq L_r^+(y)$ for $F^*(x - y) \leq r$. This implies that $L_r^+(y)$ is nondecreasing as a function of r for fixed y. $\square$

Lemma 5.3. *Let U be bounded, $u \in \mathrm{USC}(\overline{U})$ and $\Delta_{F;\infty} u \geq 0$ in U. Then u is locally Lipschitz continuous.*

Proof. Firstly we show u is bounded below on compact subsets of U. Let $x \in U$, $0 < r < \frac{\alpha}{2} d(x, \partial U)$, y be any point in the set $B(x, \frac{r}{\beta}) := \{z \in \mathbb{R}^n : |x - z| < \frac{r}{\beta}\}$. Obviously, $B(x, \frac{r}{\beta}) \subset U$, $B_r^+(y) \subset U$ and $x \in B_r^+(y)$.

If $L_r^+(y) = 0$, then $u(x) \leq u(y)$ by (2.2) and Lemma 5.2.

If $L_r^+(y) > 0$, then $L_r^+(y) = \max_{z\in S_r^+(y)} \frac{u(z)-u(y)}{r}$. From (2.2) and Lemma 5.2, we have

$$\begin{aligned} u(x) &\le u(y) + \max_{z\in S_r^+(y)} \frac{u(z)-u(y)}{r} F^*(x-y) \\ &\le u(y) + \max_{z\in S_r^+(y)} \frac{u(z)-u(y)}{r} \beta|x-y|. \end{aligned} \tag{5.7}$$

Since $|x-y| < r/\beta$ in (5.7), we find

$$\frac{r}{r-\beta|x-y|} u(x) - \max_{z\in S_r^+(y)} u(z) \frac{\beta|x-y|}{r-\beta|x-y|} \le u(y). \tag{5.8}$$

Using the upper semi-continuity of u, we know $u(y)$ is locally bounded below. Let L_r^+ be given by (5.6). Using the upper semi-continuity of u and the local boundedness below just proved, $L_r^+(y)$ is locally bounded above for fixed r.

We now know that $L_r^+(y) \ge 0$ is bounded above for fixed r and y in a compact subset of $d(y, \partial U) > 2r/\alpha$. Interchanging x and y in (5.7) and putting the resulting relations together yields

$$|u(x) - u(y)| \le \beta \max(L_r^+(y), L_r^+(x))|x-y|, \tag{5.9}$$

for $|x-y| \le r/\beta$ and $2r/\alpha < \max(\operatorname{dist}(x,\partial U), \operatorname{dist}(y, \partial U))$. We conclude that u is locally Lipschitz continuous. $\square$

Now we are ready to prove the existence of a viscosity solution of the Dirichlet boundary problem (1.3) by constructing a solution as the infimum of a family of admissible supersolutions.

Theorem 5.4. *Suppose Ω is a bounded open subset of $\mathbb{R}^n$, $f \in C(\Omega)$, $\inf_\Omega f(x) > 0$ or $\sup_\Omega f(x) < 0$, and $g \in C(\partial\Omega)$. Then there exists a unique $u \in C(\overline{\Omega})$ such that $u = g$ on $\partial\Omega$ and $\Delta^N_{F;\infty} u(x) = f(x)$ in Ω in the viscosity sense.*

Proof. Let $\tilde{\Omega} = \{x \in \mathbb{R}^n : -x \in \Omega\}$, then $u \in C(\overline{\Omega})$ satisfies $\Delta^N_{F;\infty} u(x) = f(x), x \in \Omega$ and $u(x) = g(x), x \in \partial\Omega$ in the viscosity sense if and only if $w(x) = -u(-x) \in C(\tilde{\Omega})$ satisfies the Dirichlet boundary problem

$$\begin{gathered} \Delta^N_{F;\infty} w(x) = -f(-x), \quad x \text{in } \tilde{\Omega}, \\ w(x) = -g(-x), \quad x \text{ on } \partial\tilde{\Omega}, \end{gathered} \tag{5.10}$$

in the viscosity sense. Thus, it is sufficient to consider the case $\inf_\Omega f(x) > 0$ only, since $-\sup_{x\in\Omega} f(x) = \inf_{x\in\tilde{\Omega}}\{-f(-x)\}$.

In the following, we assume $\inf_\Omega f(x) > 0$. We define the admissible sets S and T to be

$$S = \{v \in \mathrm{C}(\overline{\Omega}) : \Delta^N_{F;\infty} v \le f \text{ and } v \ge g \text{ on } \partial\Omega\},$$
$$T = \{w \in \mathrm{C}(\overline{\Omega}) : \Delta^N_{F;\infty} w \ge f \text{ and } w \le g \text{ on } \partial\Omega\},$$

where $\Delta^N_{F;\infty} v \le f$ and $\Delta^N_{F;\infty} w \ge f$ are satisfied in the viscosity sense. Firstly, we show S and T are nonempty. The constant function

$$\Phi(x) = \|g\|_{L^\infty(\partial\Omega)} + 1, \quad x \in \overline{\Omega}$$

is clearly an element of the set S. So the admissible set S is nonempty.

For any fixed point $z \in \partial\Omega$, take $\Psi(x) = \frac{a}{2}[F^*(x-z)]^2 - C$, where $a > \|f\|_{L^\infty(\Omega)}$ and $C > 0$ sufficiently large such that $\Psi \leq g$ on $\partial\Omega$. Because $\Delta^N_{F;\infty}\psi = a > \|f\|_{L^\infty(\Omega)} \geq f$ in Ω, $\Psi \in T$. That is T is nonempty.

Take

$$u(x) = \inf_{v \in S} v(x), \quad x \in \overline{\Omega},$$

$$\bar{u}(x) = \sup_{w \in T} w(x), \quad x \in \overline{\Omega}.$$

By Theorem 4.3, we have $w \leq v$, $\forall v \in S$, for all $w \in T$. Since $\Phi = \|g\|_{L^\infty(\partial\Omega)} + 1 \in S$ and $\Psi \in T$, we obtain $u(x) \geq \Psi(x) > -\infty$ and $\bar{u}(x) \leq \Phi(x) < \infty$. Thus, by Lemma 2.11, u is a viscosity supersolution of (1.3) in Ω, $\bar{u}$ is a viscosity subsolution of (1.3) in Ω, and the inequality $\bar{u} \leq g \leq u$ holds on $\partial\Omega$. As the infimum of a family of upper semi-continuous functions, u is upper semi-continuous on $\overline{\Omega}$. We have $\Delta^N_{F;\infty}u \geq f$ in Ω in the viscosity sense. Suppose not, there exists a C^2 function ϕ and a point x_0 such that $u \prec_{x_0} \phi$, but $\Delta^+_{F;\infty}\phi(x_0) < f(x_0)$. For any small $\epsilon > 0$, we define

$$\phi_\epsilon(x) = \phi(x_0) + \langle D\phi(x_0), x - x_0\rangle + \frac{1}{2}\langle D^2\phi(x_0)(x - x_0), x - x_0\rangle + \epsilon|x - x_0|^2. \tag{5.11}$$

Clearly, $u \prec_{x_0} \phi \prec_{x_0} \phi_\epsilon$, and $\Delta^+_{F;\infty}\phi_\epsilon(x) < f(x)$ for all x close to x_0, if ϵ is taken small enough, thanks to the continuity of f. Moreover, x_0 is a strict local maximum point of $u - \phi_\epsilon$. In other words, $\phi_\epsilon > u$ for all x near but other than x_0 and $\phi_\epsilon(x_0) = u(x_0)$.

We define $\hat{\phi}(x) = \phi_\epsilon(x) - \delta$ for a small positive number δ. Then $\hat{\phi}(x) < u(x)$ in a small neighborhood of x_0 which is contained in the set $\{x : \Delta^+_{F;\infty}\phi_\epsilon(x) < f(x)\}$, but $\hat{\phi}(x) \geq u(x)$ outside this neighborhood, if we take δ small enough.

Take $\hat{v} = \min\{u, \hat{\phi}\}$. Then $\hat{v}$ is upper semi-continuous on $\overline{\Omega}$. Because u is a viscosity supersolution in Ω and $\hat{\phi}$ also is in the small neighborhood of x_0, $\hat{v}$ is a viscosity supersolution of (1.4) in Ω, and along $\partial\Omega$, $\hat{v} = u \geq g$. This implies $\hat{v} \in S$, but $\hat{v} < u$ near x_0, which is a contradiction to the definition of u as the infimum of all elements in S. Therefore

$$\Delta^+_{F;\infty}u(x) \geq f(x) \tag{5.12}$$

in Ω. Hence u is a viscosity solution of (1.4).

We now show $u = g$ on $\partial\Omega$. For any point $z \in \partial\Omega$, and any $\epsilon > 0$, there is a neighborhood $B^+_r(z)$ of z such that $|g(x) - g(z)| < \epsilon$ for all $x \in B^+_r(z) \cap \partial\Omega$. Take a large number $C > 0$ such that $Cr > 2\|g\|_{L^\infty(\partial\Omega)}$. We define

$$v(x) = g(z) + \epsilon + CF^*(x - z) \tag{5.13}$$

for $x \in \Omega$. For $x \in \partial\Omega$ and $F^*(x - z) < r$, $v(x) \geq g(z) + \epsilon \geq g(x)$; while for $x \in \partial\Omega$ and $F^*(x - z) \geq r$, $v(x) \geq g(z) + \epsilon + Cr > g(z) + \epsilon + 2\|g\|_{L^\infty(\partial\Omega)} \geq g(x)$, that is $v \geq g$ on $\partial\Omega$. In addition, through direct calculation we have $\Delta^N_{F;\infty}v = 0$ in Ω and since $\inf_\Omega f(x) > 0$, $\Delta^N_{F;\infty}v = 0 \leq f(x)$ in Ω. So $v \in S$ and $v(z) = g(z) + \epsilon$. Thus $g(z) \leq u(z) \leq v(z) = g(z) + \epsilon$, for arbitrary $\epsilon > 0$. Letting $\epsilon \to 0^+$, we have $u(z) = g(z)$ for any $z \in \partial\Omega$. Indeed, as $\Delta^+_{F;\infty}u(x) = f(x) \geq 0$, $\Delta_{F;\infty}u \geq 0$, so by Lemma 5.3 u is locally Lipschitz continuous in Ω. Therefore u is continuous in Ω. The following is to prove $u \in C(\overline{\Omega})$.

By Lemma 2.11, $\bar{u}$ verifies $\Delta^N_{F;\infty}\bar{u}(x) \geq f(x)$ in the viscosity sense. Clearly, $\bar{u}$ is lower semi-continuous in $\overline{\Omega}$ as the supremum of a family of lower semi-continuous functions and $\bar{u} \le g$ on $\partial\Omega$. We now show $\bar{u} \ge g$ on $\partial\Omega$. Fix a point $z \in \partial\Omega$ and a positive number ϵ. Since g is continuous on $\partial\Omega$, there exists a positive number r such that $|g(x) - g(z)| < \epsilon$, for all $x \in \Omega \cap B_r^-(z)$. As Ω is a bounded domain, the values of $F^*(z-x)$ are bounded above and bounded below from zero for all $x \in \Omega \setminus B_r^-(z)$. We take a large number A such that $A > \sup_{x\in\Omega} F^*(z-x)$ and a large number $C \ge \|f\|_{L^\infty(\Omega)}$ such that

$$C[A^2 - (A-r)^2] \ge 2\|g\|_{L^\infty(\partial\Omega)}.$$

We define

$$w(x) = g(z) - \epsilon - C[A^2 - (A - F^*(z-x))^2], \quad x \in \overline{\Omega}$$

with A, C as chosen. For $x \in \Omega$,

$$Dw(x) = 2C(A - F^*(z-x))DF^*(z-x) \neq 0,$$

and

$$\begin{aligned}\Delta^N_{F;\infty} w(x) &= \langle D^2 w(x) DF(Dw(x)), DF(Dw(x))\rangle \\ &= 2C \ge \|f\|_{L^\infty(\Omega)} \ge f(x).\end{aligned}$$

That is, w is a viscosity subsolution of $\Delta^N_{F;\infty} u(x) = f(x)$ for all $x \in \Omega$.

On $\partial\Omega \cap B_r^-(z)$, $w(x) \le g(z) - \epsilon \le g(x)$; while on $\partial\Omega \setminus B_r^-(z)$,

$$\begin{aligned}w(x) &\le g(z) - \epsilon - C[A^2 - (A - F^*(z-x))^2] \\ &\le g(z) - \epsilon - 2\|g\|_{L^\infty(\partial\Omega)} \\ &\le -\|g\|_{L^\infty(\partial\Omega)} \le g(x).\end{aligned}$$

That is to say $w \le g$ on $\partial\Omega$. So the function w defined above is in the family T. Thus, from the definition of $\bar{u}$, we obtain $\bar{u} \ge w$. Since $w(z) = g(z) - \epsilon$, we have $\bar{u}(z) \ge g(z) - \epsilon$ for any $\epsilon > 0$, which implies that $\bar{u}(z) \ge g(z)$ for any $z \in \partial\Omega$.

As the supremum of a family of lower semi-continuous functions on $\overline{\Omega}$, $\bar{u}$ is lower semi-continuous on $\overline{\Omega}$. Therefore

$$g(z) \le \bar{u}(z) \le \liminf_{x\in\Omega\to z} \bar{u}(x), \quad \forall z \in \partial\Omega.$$

The comparison principle (Theorem 4.3) implies $v \le w$ on Ω for any $w \in S$ and $v \in T$. In particular, $\bar{u} \le u$ in Ω. So

$$g(z) \le \liminf_{x\in\Omega\to z} \bar{u}(x) \le \liminf_{x\in\Omega\to z} u(x), \quad \forall z \in \partial\Omega.$$

On the other hand, the upper semi-continuity of u on $\overline{\Omega}$ implies that

$$\limsup_{x\in\Omega\to z} u(x) \le u(z) = g(z), \forall z \in \partial\Omega.$$

So $\lim_{x\in\Omega\to z} u(x) = g(z), \forall z \in \partial\Omega$.

This shows that $u \in C(\overline{\Omega})$. The uniqueness follows from [20, Theorem 1.4]. This completes the proof. □

Remark 5.5. The condition that f does not change sign in Ω is indispensable, as a counter-example for the normalized infinity Laplacian provided in [22] shows the uniqueness of a viscosity solution subject to given boundary data fails without such a condition.

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