Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 107, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF MULTIPLE SOLUTIONS TO ELLIPTIC PROBLEMS OF KIRCHHOFF TYPE WITH CRITICAL EXPONENTIAL GROWTH

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Abstract. In this article, we study elliptic problems of Kirchhoff type in dimension $N \geq 2$, whose nonlinear term has a critical exponential growth. Using variational tools, we establish the existence of at least two nontrivial and nonnegative solutions.

1. Introduction and statement of main results

In article, we establish some multiplicity results for the equation

$$\begin{aligned} &- A'\Big(\int_{\mathbb{R}^N} \frac{|\nabla u|^N + |u|^N}{N}\, dx\Big) \big(\operatorname{div}(|\nabla u|^{N-2}\nabla u) - |u|^{N-2}u\big) \\ &= B'\Big(\int_{\mathbb{R}^N} F(x,u)\, dx\Big) f(x,u) + h, \quad \text{in } \mathbb{R}^N,\ N \geq 2, \end{aligned} \tag{1.1}$$

where $A'(\cdot)$, $B'(\cdot)$ denote the derivatives of two C^1-functions $A(\cdot)$ and $B(\cdot)$; $f(\cdot,\cdot) : \mathbb{R}^N \times \mathbb{R} \to [0, +\infty[$ is a Carathéodory function such that f is radially symmetric with respect to x, i.e if $x, y \in \mathbb{R}^N$ satisfy $|x| = |y|$, then $f(x, s) = f(y, s)$, for all $s \in \mathbb{R}$. Moreover, we assume that $f(x,t) = 0$ for all $t \leq 0$ and all $x \in \mathbb{R}^N$;

$$F(x,u) = \int_0^u f(x,t)dt;$$

$h : \mathbb{R}^N \to [0, +\infty[$ is some radial function such that $h \neq 0$ and $h \in L^{N'}(\mathbb{R}^N)$ with $N' = \frac{N}{N-1}$.

Kirchhoff-type problems have become a very interesting topic of research in recent years and many papers dealing with such kind of equations were published. We can, for instance; see [4, 9, 10, 11, 12, 13, 14, 37] and references therein. The interest for these problems with various proposed nonlocal terms could be explained by their contributions to modeling many physical and biological phenomena. First, let us mention that quasilinear equations of the model

$$-M\Big(\int_\Omega |\nabla u|^p\, dx\Big) \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = f(x,u) \quad \text{in } \Omega,$$

2000 *Mathematics Subject Classification.* 35D30, 35J20, 35J61, 58E05.

Key words and phrases. Trudinger-Moser inequality; exponential growth; Kirchhoff equation; critical point; energy functional; Ekeland's lemma; Mountain Pass Theorem.

Submitted December 10, 2013. Published April 15, 2014.

where Ω is a domain of $\mathbb{R}^N$, is essentially related to the stationary analog of the Kirchhoff equation

$$u_{tt} - M\Big(\int_\Omega |\nabla u|^2\Big)\Delta u = f(x,t),$$

where $M(s) = as + b, a, b > 0$. This last equation was proposed by Kirchhoff [20] as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. The Kirchhoff model takes into account the length changes of the string produced by transverse vibrations. Later, Lions [26] gave an abstract functional analysis framework to the Kirchhoff model. Next, equations of the model

$$-M\Big(\int_\Omega |u|^p\,dx\Big)\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = f(x,u) \quad \text{in } \Omega,$$

arise in many physical phenomena such as systems of particles in thermodynamical equilibrium via gravitational potential, thermal runaway in ohmic heating, and shear bands in metal deformed under high strain rates. On the other hand, equations of Kirchhoff-type appear in some biological studies; more precisely, such kind of equations could describe the evolution of the density of a population living in some domain Ω. The reader interested in the physical and biological aspects of the Kirchhoff-type problems could be referred to [11, 37]. In the present work, we are interested in the case when the nonlinearity term $f(x,s)$ has maximal growth on s which allows us to treat the problem (1.1) variationally. Explicitly, in view of the Trudinger-Moser inequality, we will assume that f satisfies critical growth of exponential type such as $f(x,s)$ behaves like $\exp(\alpha(x)|s|^{\frac{N}{N-1}})$ as $|s| \to +\infty$. Elliptic equations involving nonlinearities of exponential growth have been studied by many authors; see, for example [1, 2, 3, 5, 7, 8, 17, 18, 21, 22, 23, 24, 25, 28, 29, 30, 31, 33, 34, 36] and references therein. Studying problems of Kirchhoff-type and involving nonlinearities having a critical exponential growth is a new research subject. Up to our best knowledge, only Figueiredo and Severo [19] studied a problem involving nonlocal terms and a nonlinear term with a critical exponential growth. In [19], the authors studied the problem

$$\begin{gathered}
-m\Big(\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = f(x,u) \quad \text{in } \Omega,\\
u = 0 \quad \text{on } \partial\Omega
\end{gathered}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2, m : [0,+\infty[\to [0,+\infty[$ is some continuous function satisfying that $\inf_{t\geq 0} m(t) > 0$, $\frac{m(t)}{t}$ is nonincreasing for $t > 0$, $m(t) \leq a_1 + a_2 t^\sigma$ for all $t \geq t_0$ for some positive constants, a_1, a_2, t_0 and σ, and

$$\int_0^{t+s} m(u)du \geq \int_0^t m(u)du + \int_0^s m(u)du, \quad \forall s,t \geq 0.$$

Concerning the nonlinear term, the authors assume that f has a critical exponential growth and satisfies that $\int_0^s f(x,t)dt \leq K_0 f(x,s)$ for all $(x,s) \in \Omega \times [s_0,+\infty[$ for some positive constants s_0 and K_0. Furthermore, it was assumed that for each $x \in \Omega$, $\frac{f(x,s)}{s^3}$ is increasing for $s > 0$. This article is a contribution in this new direction. In our paper, we treat a more general problem which is defined in all the space $\mathbb{R}^N$, $N \geq 2$ and presenting a nonlocal term in the right-hand side of the equation. We will try to adapt some arguments developed in [16].

Now, we state our main hypotheses in this work.

(H1) $A : [0,+\infty[\to \mathbb{R}$ is a C^1-function satisfying that $A(0) = 0$ and

- if $s > 0$, then $A'(s) > 0$,
- there exist $C_0 > 0$, $\alpha_0 > 0$ and $s_0 > 0$ such that
$$A(s) \geq C_0\ s^{\alpha_0}, \quad \forall 0 \leq s \leq s_0.$$

(H2) $B : \mathbb{R} \to \mathbb{R}$ is a C^1-function satisfying that there exist $C_1 > 0$, $\alpha_1 > 0$ and $s_1 > 0$ such that
$$B(s) \leq C_1\ s^{\alpha_1}, \quad \forall 0 \leq s \leq s_1.$$

(H3) There exist $C_2 > 0$, $\alpha > N - 1$, $\beta > 0$ and a bounded radial function $\gamma : \mathbb{R}^N \to [0, +\infty[$ such that for all $s \geq 0$ and all $x \in \mathbb{R}^N$,
$$|f(x,s)| \leq C_2\Big(|s|^\alpha + |s|^\beta\big(\exp\big(\gamma(x)|s|^{\frac{N}{N-1}}\big) - S_{N-2}(\gamma(x), s)\big)\Big),$$
where
$$S_{N-2}(\gamma(x), s) = \sum_{k=0}^{N-2} \frac{(\gamma(x))^k}{k!}|s|^{\frac{kN}{N-1}}.$$

(H4) There exist $\lambda_0 > 0$, $a_0 > 0$, $k_0 > 0$ and $M_0 > 0$ such that
- $\lambda_0 A(s) \geq A'(s)s$ for all $s \geq M_0$,
- $A(s) \geq k_0\ s^{a_0}$ for all $s \geq M_0$.

(H5) The function B satisfies that
- there exist $\lambda_1 > 0$ and $M_1 > 0$ such that
$$\lambda_1 B(s) \leq B'(s)s, \quad \forall\ s \geq M_1,$$
- if $s \geq 0$, then $B'(s) \geq 0$.

(H6) There exist a bounded nonempty open set Ω of $\mathbb{R}^N$, $M_2 > 0$, $\theta > 0$, and $K \in L^1(\mathbb{R}^N)$ such that
$$0 < \theta F(x,s) \leq f(x,s)s, \quad \forall s \geq M_2,\ \forall x \in \Omega,$$
$$\theta F(x,s) \leq f(x,s)s + K(x), \quad \forall s \geq 0,\ \forall x \in \mathbb{R}^N.$$

Definition 1.1. A function $u \in W^{1,N}(\mathbb{R}^N)$ is said to be a weak solution of the problem (1.1) if it satisfies
$$\begin{aligned} &A'\big(\frac{\|u\|^N}{N}\big)\Big(\int_{\mathbb{R}^N} |\nabla u|^{N-2}\nabla u \cdot \nabla v\, dx + \int_{\mathbb{R}^N} |u|^{N-2}uv\, dx\Big) \\ &= B'\Big(\int_{\mathbb{R}^N} F(x,u)\, dx\Big)\int_{\mathbb{R}^N} f(x,u)v\, dx + \int_{\mathbb{R}^N} hv dx, \quad \forall v \in W^{1,N}(\mathbb{R}^N). \end{aligned}$$

The main result of the present work is given by the following two theorems.

Theorem 1.2. *Assume that* (H1)–(H3) *hold true. If* $\alpha_0 N < \alpha_1 \inf(\alpha + 1, \beta + 1)$, *then there exists* $\eta > 0$ *such that the problem* (1.1) *admits at least one nontrivial and nonnegative weak solution provided that* $|h|_{L^{N'}(\mathbb{R}^N)} < \eta$.

Theorem 1.3. *Assume that* (H1)–(H6) *hold true. In addition, we assume*

(H7) *there exists* $R_0 > 0$ *such that* $\gamma(x) = 0$ *for* $|x| \leq R_0$.

If $\alpha_0 N < \alpha_1 \inf(\alpha + 1, \beta + 1)$, $a_0 N > 1$ *and* $\lambda_1 \theta > \lambda_0 N$, *then there exists* $\eta > 0$ *such that the problem* (1.1) *admits at least two nontrivial and nonnegative weak solutions provided that* $|h|_{L^{N'}(\mathbb{R}^N)} < \eta$.

2. Preliminaries

Here, we state some interesting properties of the space $W^{1,N}(\mathbb{R}^N)$ that will be useful throughout this paper. Let Ω be a bounded domain of $\mathbb{R}^N, N \geq 2$. First, we recall that the important Trudinger-Moser inequality (see [27, 35]) asserts that

$$\exp\big(\alpha|u|^{\frac{N}{N-1}}\big) \in L^1(\Omega) \quad \text{for } u \in W_0^{1,N}(\Omega) \text{ and } \alpha > 0.$$

Then, there exists a positive constant $C > 0$ depending only on N such that

$$\sup_{|\nabla u|_{L^N(\Omega)} \leq 1} \int_\Omega \exp\Big(\alpha|u|^{\frac{N}{N-1}}\big)\,dx \leq C|\Omega| \quad \text{if } \alpha \leq \alpha_N, \quad \forall u \in W_0^{1,N}(\Omega),$$

where $\alpha_N = NW_{N-1}^{\frac{1}{N-1}}$ and W_{N-1} is the measure of the unit sphere in $\mathbb{R}^N$. In the case of $\mathbb{R}^N, N \geq 2$, we have the following result (for $N = 2$, see [7, 33], and for $N \geq 2$, see [1, 30])

$$\int_{\mathbb{R}^N} [\exp(\alpha|u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u)]\,dx < +\infty \quad \text{for } u \in W^{1,N}(\mathbb{R}^N) \text{ and } \alpha > 0,$$

where

$$S_{N-2}(\alpha, u) = \sum_{k=0}^{N-2} \frac{\alpha^k}{k!}|u|^{\frac{kN}{N-1}}.$$

Moreover, if $|\nabla u|_{L^N(\mathbb{R}^N)} \leq 1$, $|u|_{L^N(\mathbb{R}^N)} \leq M < +\infty$ and $\alpha < \alpha_N$, then there exists a constant $C = C(N, M, \alpha) > 0$, which depends only on N, M and α such that

$$\int_{\mathbb{R}^N} \big[\exp(\alpha|u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u)\big]\,dx \leq C.$$

Furthermore, using above results together with Hölder's inequality, if $\alpha > 0$ and $q > 0$, then

$$\int_{\mathbb{R}^N} |u|^q \big[\exp(\alpha|u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u)\big]\,dx < +\infty, \quad \forall u \in W^{1,N}(\mathbb{R}^N). \tag{2.1}$$

More precisely, if $\|u\|_{W^{1,N}(\mathbb{R}^N)} \leq M$ with $\alpha M^{\frac{N}{N-1}} < \alpha_N$, then there exists $C = C(\alpha, M, q, N) > 0$ such that

$$\int_{\mathbb{R}^N} |u|^q \big[\exp(\alpha|u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u)\big]\,dx \leq C\|u\|^q_{W^{1,N}(\mathbb{R}^N)}. \tag{2.2}$$

3. Proof of Theorem 1.2

First, observe that the appropriate space in which the problem (1.1) will be studied is $W_r^{1,N}(\mathbb{R}^N)$ which consists of all the functions in $W^{1,N}(\mathbb{R}^N)$ which are radial. The space $W_r^{1,N}(\mathbb{R}^N)$ will be equipped with the classical norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} (|\nabla u|^N + |u|^N)\,dx\Big)^{1/N}.$$

It should be useful to remind that the continuous embedding $W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ holds for all $q \in [N, +\infty[$.

We start by introducing the energy functional corresponding to the problem (1.1): $J : W_r^{1,N}(\mathbb{R}^N) \to \mathbb{R}$,

$$J(u) = A\big(\frac{\|u\|^N}{N}\big) - B\Big(\int_{\mathbb{R}^N} F(x, u)\,dx\Big) - \int_{\mathbb{R}^N} hu\,dx.$$

By (H1) and (H2), it is clear that there exist $c_1 > 0$ and $c_2 > 0$ such that for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$, we have

$$|F(x,s)| \le c_1|s|^{\alpha+1} + c_2|s|^{\beta+1}\big(\exp(\gamma_\infty|s|^{\frac{N}{N-1}}) - S_{N-2}(\gamma_\infty, s)\big), \tag{3.1}$$

where $\gamma_\infty = \sup_{x\in\mathbb{R}^N}\gamma(x)$. Taking (2.1) into account, it yields

$$F(x,u) \in L^1(\mathbb{R}^N), \quad \text{for all } (x,u) \in \mathbb{R}^N \times W_r^{1,N}(\mathbb{R}^N).$$

Hence, the functional J is well defined on $W_r^{1,N}(\mathbb{R}^N)$. Moreover, by standard arguments (see [6]), we could easily establish that J is of class C^1 in $W_r^{1,N}(\mathbb{R}^N)$ and that we have

$$\begin{aligned}
&\langle J'(u), v\rangle \\
&= A'\big(\frac{\|u\|^N}{N}\big)\Big(\int_{\mathbb{R}^N}|\nabla u|^{N-2}\nabla u\cdot\nabla v\,dx + \int_{\mathbb{R}^N}|u|^{N-2}uv\,dx\Big) \\
&\quad - B'\Big(\int_{\mathbb{R}^N}F(x,u)\,dx\Big)\int_{\mathbb{R}^N}f(x,u)v\,dx - \int_{\mathbb{R}^N}hv\,dx, \quad \forall\ u,v \in W_r^{1,N}(\mathbb{R}^N).
\end{aligned}$$

Here, according to Definition 1.1 and by the virtue of the known so-called principle of symmetric criticality (see [32]), every critical point of the functional J is in fact a weak solution of the problem (1.1).

Lemma 3.1. *Assume that* (H1)–(H3) *hold. Then, there exist* $\mu > 0, \rho > 0$ *and* $\eta > 0$ *such that*

$$J(u) \ge \mu, \quad \textit{for all } u \in W_r^{1,N}(\mathbb{R}^N) \textit{ such that } \|u\| = \rho,$$

provided that $|h|_{L^{N'}(\mathbb{R}^N)} < \eta$.

Proof. For $0 < M < 1$ small enough, by (3.1) and (2.2), we have

$$\int_{\mathbb{R}^N}|F(x,u)|\,dx \le c_3\|u\|^{\inf(\alpha+1,\beta+1)}, \quad \text{for } \|u\| \le M.$$

Now, consider $0 < \rho < M$ be such that $c_3\rho^{\inf(\alpha+1,\beta+1)} < s_1$. Then, we obtain

$$\begin{aligned}
B(\int_{\mathbb{R}^N}F(x,u)\,dx) \le C_1(\int_{\mathbb{R}^N}F(x,u)\,dx)^{\alpha_1} &\le C_1(c_3\rho^{\inf(\alpha+1,\beta+1)})^{\alpha_1} \\
&\le c_4\ \rho^{\alpha_1\inf(\alpha+1,\beta+1)}.
\end{aligned} \tag{3.2}$$

On the other hand, if we assume that $\frac{\rho^N}{N} < s_0$, for $\|u\| = \rho$, we have

$$A(\frac{\|u\|^N}{N}) = A(\frac{\rho^N}{N}) \ge C_0\frac{\rho^{\alpha_0 N}}{N^{\alpha_0}}. \tag{3.3}$$

Using (3.2) and (3.3), we obtain

$$J(u) \ge c_5\rho^{\alpha_0 N} - c_4\rho^{\alpha_1\inf(\alpha+1,\beta+1)} - |h|_{L^{N'}(\mathbb{R}^N)}\rho, \quad \text{for } \|u\| = \rho.$$

Now, since $\alpha_0 N < \alpha_1\inf(\alpha+1,\beta+1)$, then we can choose ρ small enough such that $\eta = c_5\ \rho^{\alpha_0 N-1} - c_4\ \rho^{\alpha_1\inf(\alpha+1,\beta+1)-1}$ is positive. Hence, Lemma 3.1 holds with $\mu = \rho(\eta - |h|_{L^{N'}(\mathbb{R}^N)})$. □

Proof of Theorem 1.2 completed. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be a radial function such that $\varphi \ge 0, \varphi \ne 0$ and $\int_{\mathbb{R}^N}h\varphi\,dx > 0$. For $0 < t < 1$, we have

$$J(t\varphi) = A\Big(\frac{t^N\|\varphi\|^N}{N}\Big) - B\Big(\int_{\mathbb{R}^N}F(x,t\varphi)\,dx\Big) - t\int_{\mathbb{R}^N}h\varphi\,dx.$$

Then

$$
\begin{aligned}
\frac{d}{dt}J(t\varphi) &= t^{N-1}\|\varphi\|^N A'\Big(\frac{t^N\|\varphi\|^N}{N}\Big) \\
&\quad - B'\Big(\int_{\mathbb{R}^N} F(x,t\varphi)\,dx\Big)\int_{\mathbb{R}^N} f(x,t\varphi)\varphi\,dx - \int_{\mathbb{R}^N} h\varphi\,dx.
\end{aligned}
$$

Obviously, we have

$$\lim_{t\to 0^+}\int_{\mathbb{R}^N} f(x,t\varphi)\varphi\,dx = 0.$$

Since $N-1>0$, then one can easily find $\delta>0$ small enough such that

$$\frac{d}{dt}J(t\varphi) < 0 \quad \forall\, 0<t<\delta.$$

Having in mind that $J(0)=0$, then there exists $0<t_0<\inf(\delta,\frac{\rho}{\|\varphi\|})$ such that $J(t_0\varphi)<0$. Set

$$d_\rho = \inf\Big\{J(u), u\in\overline{B(0,\rho)}\Big\},$$

where $\overline{B(0,\rho)} = \{u\in W_r^{1,N}(\mathbb{R}^N), \|u\|\le\rho\}$. By Ekeland's variational principle (see [15]), there exists a sequence $(u_n)\subset\overline{B(0,\rho)}$ such that $J'(u_n)\to 0$ and $J(u_n)\to d_\rho$ as $n\to+\infty$. Since (u_n) is bounded in $W_r^{1,N}(\mathbb{R}^N)$, then there exists $u\in W_r^{1,N}(\mathbb{R}^N)$ such that $u_n\rightharpoonup u$ weakly in $W_r^{1,N}(\mathbb{R}^N)$. We claim that, up to a subsequence, (u_n) is strongly convergent to u. We start by proving that

$$\sup_{n\in\mathbb{N}}\Big(\int_{\mathbb{R}^N}|f(x,u_n)|^{\frac{N}{N-1}}\,dx\Big) < +\infty.$$

By (H3), there exists $c_6>0$ such that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}|f(x,u_n)|^{\frac{N}{N-1}}\,dx \\
&\le c_6\int_{\mathbb{R}^N}|u_n|^{\frac{\alpha N}{N-1}}\,dx + c_6\int_{\mathbb{R}^N}|u_n|^{\frac{\beta N}{N-1}}\Big(\exp\Big(\frac{\gamma_\infty N}{N-1}|u_n|^{\frac{N}{N-1}}\Big) \\
&\quad - S_{N-2}\Big(\frac{\gamma_\infty N}{N-1},u_n\Big)\Big)dx.
\end{aligned}
\tag{3.4}
$$

Since $\frac{\alpha N}{N-1}>N$, we have

$$\sup_{n\in\mathbb{N}}\Big(\int_{\mathbb{R}^N}|u_n|^{\frac{\alpha N}{N-1}}\,dx\Big) < +\infty.$$

On the other hand, notice that we can assume ρ defined in Lemma 3.1 is such that $\frac{\gamma_\infty N}{N-1}\rho^{\frac{N}{N-1}} < \alpha_N$. Consequently, by (2.2), it follows that there exists a positive constant c_7 such that

$$\int_{\mathbb{R}^N}|u_n|^{\frac{\beta N}{N-1}}\Big(\exp\Big(\frac{\gamma_\infty N}{N-1}|u_n|^{\frac{N}{N-1}}\Big) - S_{N-2}\Big(\frac{\gamma_\infty N}{N-1},u_n\Big)\Big)dx \le c_7, \quad \forall\, n\in\mathbb{N}.$$

By (3.4), we immediately deduce that

$$\sup_{n\in\mathbb{N}}\Big(\int_{\mathbb{R}^N}|f(x,u_n)|^{\frac{N}{N-1}}\,dx\Big) < +\infty.$$

Let $R>0$ and consider $B_R=\{x\in\mathbb{R}^N, |x|<R\}$. By Hölder's inequality, we have

$$\int_{B_R}|f(x,u_n)(u_n-u)|\,dx \le \Big(\int_{\mathbb{R}^N}|f(x,u_n)|^{\frac{N}{N-1}}\,dx\Big)^{\frac{N-1}{N}}\Big(\int_{B_R}|u_n-u|^N\,dx\Big)^{1/N}.$$

Taking into account that the embedding $W^{1,N}(\mathbb{R}^N)$ into $L^N(B_R)$ is compact, it follows

$$\lim_{n\to+\infty}\int_{B_R} f(x,u_n)(u_n-u)\,dx = 0. \tag{3.5}$$

By the radial lemma (see [6]), there exists a positive constant $C_N > 0$ depending only on N such that

$$|u_n(x)| \le \frac{C_N}{|x|}|u_n|_{L^N(\mathbb{R}^N)}, \quad \forall x \neq 0.$$

We have

$$\begin{aligned}
&\int_{|x|\ge R} |u_n|^{\frac{\beta N}{N-1}}\Big(\exp\Big(\frac{\gamma_\infty N}{N-1}|u_n|^{\frac{N}{N-1}}\Big) - S_{N-2}\Big(\frac{\gamma_\infty N}{N-1}, u_n\Big)\Big)\,dx\\
&= \sum_{j=N-1}^{+\infty}\frac{(\frac{\gamma_\infty N}{N-1})^j}{j!}\int_{|x|\ge R}|u_n|^{\frac{(\beta+j)N}{N-1}}\,dx\\
&\le c_8\sum_{j=N-1}^{+\infty}\frac{(\frac{\gamma_\infty N}{N-1})^j}{j!}\int_R^{+\infty}\frac{r^{N-1}}{r^{\frac{(\beta+j)N}{N-1}}}dr\\
&\le c_9\sum_{j=N-1}^{+\infty}\frac{(\frac{\gamma_\infty N}{N-1})^j}{j!}\int_R^{+\infty}\frac{dr}{r^{\frac{\beta N}{N-1}+1}}\\
&\le \frac{c_{10}}{R^{\frac{\beta N}{N-1}}}.
\end{aligned} \tag{3.6}$$

Let $\epsilon > 0$. By (3.6), there exists $R_1(\epsilon) > 1$ large enough such that, for all $n \in \mathbb{N}$,

$$\int_{|x|\ge R_1(\epsilon)} |u_n|^{\frac{\beta N}{N-1}}\Big(\exp\Big(\frac{\gamma_\infty N}{N-1}|u_n|^{\frac{N}{N-1}}\Big) - S_{N-2}\Big(\frac{\gamma_\infty N}{N-1}, u_n\Big)\Big)\,dx \le \epsilon. \tag{3.7}$$

Next, we have

$$\int_{|x|\ge R}|u_n|^{\frac{\alpha N}{N-1}}\,dx \le c_{11}\int_R^{+\infty}\frac{dr}{r^{\frac{\alpha N}{N-1}-N+1}} \le \frac{c_{12}}{R^{\frac{\alpha N}{N-1}-N}}.$$

Since $\frac{\alpha N}{N-1} > N$, then one can find $R_2(\epsilon) > 1$ large enough such that

$$\int_{|x|\ge R_2(\epsilon)}|u_n|^{\frac{\alpha N}{N-1}}\,dx \le \epsilon, \quad \forall n\in\mathbb{N}. \tag{3.8}$$

Put $R(\epsilon) = \sup(R_1(\epsilon), R_2(\epsilon))$. By (3.7) and (3.8), it yields

$$\int_{|x|\ge R(\epsilon)}|f(x,u_n)|^{\frac{N}{N-1}}\,dx \le 2\epsilon, \quad \forall n\in\mathbb{N}. \tag{3.9}$$

By Hölder's inequality and (3.9), we obtain

$$\begin{aligned}
&\int_{|x|\ge R(\epsilon)}|f(x,u_n)(u_n-u)|\,dx\\
&\le \Big(\int_{|x|\ge R(\epsilon)}|f(x,u_n)|^{\frac{N}{N-1}}\,dx\Big)^{\frac{N-1}{N}}|u_n-u|_{L^N(\mathbb{R}^N)}\\
&\le c_{13}\epsilon^{\frac{N-1}{N}}, \quad \forall n\in\mathbb{N}.
\end{aligned} \tag{3.10}$$

Now, (3.10) together with (3.5) imply

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} f(x,u_n)(u_n-u)\,dx=0. \tag{3.11}$$

Next, arguing as in the establishment of the boundedness of the sequence

$$\Big(\int_{\mathbb{R}^N}|f(x,u_n)|^{\frac{N}{N-1}}\,dx\Big),$$

we can show that

$$\sup_{n\in\mathbb{N}}\Big(\int_{\mathbb{R}^N}|F(x,u_n)|\,dx\Big)<+\infty.$$

That fact together with (3.11) leads to

$$B'(\int_{\mathbb{R}^N}F(x,u_n)\,dx)\int_{\mathbb{R}^N}f(x,u_n)(u_n-u)\,dx\to 0\quad\text{as } n\to+\infty.$$

Using the weak convergence of (u_n) to u in $W^{1,N}_r(\mathbb{R}^N)$, it immediately follows

$$\int_{\mathbb{R}^N}h(u_n-u)\,dx\to 0\quad\text{as } n\to+\infty.$$

Taking these results into account and having in mind that $J'(u_n)(u_n-u)\to 0$, we deduce that

$$A'(\frac{\|u_n\|^N}{N})(\int_{\mathbb{R}^N}|\nabla u_n|^{N-2}\nabla u_n\nabla(u_n-u)\,dx+\int_{\mathbb{R}^N}|u_n|^{N-2}u_n(u_n-u)\,dx)\to 0,$$

as $n\to+\infty$. If $\|u_n\|\to 0$, then $u_n\to 0$ strongly in $W^{1,N}_r(\mathbb{R}^N)$ and there is nothing to prove. Otherwise, $\|u_n\|\to t>0$ and $A'(\frac{\|u_n\|^N}{N})\to A'(\frac{t^N}{N})>0$. In that case, we obtain

$$\lim_{n\to+\infty}\Big(\int_{\mathbb{R}^N}|\nabla u_n|^{N-2}\nabla u_n\nabla(u_n-u)\,dx+\int_{\mathbb{R}^N}|u_n|^{N-2}u_n(u_n-u)\,dx\Big)=0,$$

which implies that (u_n) is strongly convergent to u in $W^{1,N}_r(\mathbb{R}^N)$. Finally, we conclude that $J(u)=d_\rho\le J(t_0\varphi)<0$ and that $J'(u)=0$. Hence, u is a nontrivial weak solution of (1.1) with negative energy. Set $u^-=\min(u,0)$. We have $\langle J'(u),u^-\rangle=0$. Thus,

$$A'(\frac{\|u\|^N}{N})\|u^-\|^N-B'(\int_{\mathbb{R}^N}F(x,u)\,dx)\int_{\mathbb{R}^N}f(x,u)u^-\,dx=\int_{\mathbb{R}^N}hu^-\,dx\le 0.$$

Having in mind that $f(x,u)u^-=0$, we deduce that $\|u^-\|=0$ and therefore $u\ge 0$. $\square$

4. Proof of Theorem 1.3

A first solution with negative energy is given by Theorem 1.2. The existence of a second weak solution will be proved using the well known Mountain Pass Theorem. We start by the following lemma.

Lemma 4.1. *Assume that the hypotheses of Theorem 1.3 hold. Then, the functional J satisfies the Palais-Smale condition.*

Proof. Let $(u_n) \subset W^{1,N}(\mathbb{R}^N)$ be such that $(J(u_n))$ is bounded and $J'(u_n) \to 0$. We claim that, up to a subsequence, (u_n) is strongly convergent. By (H4), we have

$$\lambda_0 N A\Big(\frac{\|u\|^N}{N}\Big) \geq A'\Big(\frac{\|u\|^N}{N}\Big)\|u_n\|^N - c_{14}, \quad \forall n \in \mathbb{N}. \tag{4.1}$$

On the other hand, by (H6), we have

$$0 \leq \theta \int_{\mathbb{R}^N} F(x,u_n)\,dx \leq \int_{\mathbb{R}^N} f(x,u_n)u_n\,dx + c_{15}.$$

Let $0 < \epsilon < \inf(\theta, \frac{c_{15}}{M_1})$. If

$$\int_{\mathbb{R}^N} F(x,u_n)\,dx \geq \frac{c_{15}}{\epsilon},$$

then we obtain

$$0 \leq (\theta - \epsilon)\int_{\mathbb{R}^N} F(x,u_n)\,dx \leq \int_{\mathbb{R}^N} f(x,u_n)u_n\,dx.$$

This inequality together with (H5) imply

$$\begin{aligned}
&B'\Big(\int_{\mathbb{R}^N} F(x,u_n)\,dx\Big)\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx - \lambda_1(\theta-\epsilon)B\Big(\int_{\mathbb{R}^N} F(x,u_n)\,dx\Big)\\
&\geq B'\Big(\int_{\mathbb{R}^N} F(x,u_n)\,dx\Big)\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx\\
&\quad - (\theta-\epsilon)B'\Big(\int_{\mathbb{R}^N} F(x,u_n)\,dx\Big)\int_{\mathbb{R}^N} F(x,u_n)\,dx \geq 0.
\end{aligned}$$

If $\int_{\mathbb{R}^N} F(x,u_n)\,dx \leq c_{15}/\epsilon$, it is clear that there exists a positive constant $c_\epsilon > 0$ such that

$$B'\Big(\int_{\mathbb{R}^N} F(x,u_n)\,dx\Big)\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx - \lambda_1(\theta-\epsilon)B\Big(\int_{\mathbb{R}^N} F(x,u_n)\,dx\Big) \geq -c_\epsilon.$$

Hence, we deduce that

$$\begin{aligned}
&B'(\int_{\mathbb{R}^N} F(x,u_n)\,dx)\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx - \lambda_1(\theta-\epsilon)B(\int_{\mathbb{R}^N} F(x,u_n)\,dx)\\
&\geq -c_\epsilon, \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{4.2}$$

Now, choose $0 < \epsilon < \inf(\theta, c_{15}/M_1)$ small enough such that $\lambda_1(\theta-\epsilon) > \lambda_0 N$. Since (u_n) is a (PS) sequence of J, then there exists a positive constant $c_{16} > 0$ such that

$$\lambda_1(\theta-\epsilon)J(u_n) - \langle J'(u_n), u_n\rangle \leq c_{16}(1+\|u_n\|), \quad \forall n \in \mathbb{N}.$$

Thus,

$$\begin{aligned}
&\Big(\lambda_1(\theta-\epsilon)A\Big(\frac{\|u_n\|^N}{N}\Big) - A'\Big(\frac{\|u_n\|^N}{N}\Big)\|u_n\|^N\Big)\\
&+ B'\Big(\int_{\mathbb{R}^N} f(x,u_n)\,dx\Big)\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx - \lambda_1(\theta-\epsilon)B\Big(\int_{\mathbb{R}^N} F(x,u_n)\,dx\Big)\\
&\leq c_{17}(1+\|u_n\|), \quad \forall n \in \mathbb{N}.
\end{aligned}$$

By (4.2), we have

$$\lambda_1(\theta-\epsilon)A\Big(\frac{\|u_n\|^N}{N}\Big) - A'\Big(\frac{\|u_n\|^N}{N}\Big)\|u_n\|^N \leq c_\epsilon + c_{17}(1+\|u_n\|), \quad \forall n \in \mathbb{N}.$$

By (4.1), we have

$$(\lambda_1(\theta-\epsilon)-\lambda_0 N)A\Big(\frac{\|u_n\|^N}{N}\Big)\le c_\epsilon+c_{14}+c_{17}(1+\|u_n\|),\quad \forall n\in\mathbb{N}.$$

Finally, using again (H4), we obtain

$$c_{18}\|u_n\|^{a_0N}\le c_{19}(1+\|u_n\|),\quad \forall n\in\mathbb{N}.$$

Taking into account that $a_0N>1$, we conclude that (u_n) is bounded in $W_r^{1,N}(\mathbb{R}^N)$. Denote by u the weak limit of (u_n) in $W_r^{1,N}(\mathbb{R}^N)$. Here, in order to prove that (u_n) is strongly convergent to u in $W_r^{1,N}(\mathbb{R}^N)$, we can follow the arguments used to establish the same result in the proof of Theorem 1.2 with some suitable modification. In fact, the arguments used in the proof of Theorem 1.2 to establish that

$$\sup_{n\in\mathbb{N}}\Big(\int_{\mathbb{R}^N}|f(x,u_n)|^{\frac{N}{N-1}}\,dx\Big)<+\infty$$

are no longer valid. It is clear, that always we have

$$\sup_{n\in\mathbb{N}}\Big(\int_{\mathbb{R}^N}|u_n|^{\frac{\alpha N}{N-1}}\,dx\Big)<+\infty.$$

On the other hand, by (H7) we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}|u_n|^{\frac{\beta N}{N-1}}\Big(\exp\Big(\frac{\gamma(x)N}{N-1}|u_n|^{\frac{N}{N-1}}\Big)-S_{N-2}\Big(\frac{\gamma(x)N}{N-1},u_n\Big)\Big)dx\\
&=\int_{|x|\ge R_0}|u_n|^{\frac{\beta N}{N-1}}\Big(\exp\Big(\frac{\gamma(x)N}{N-1}|u_n|^{\frac{N}{N-1}}\Big)-S_{N-2}\Big(\frac{\gamma(x)N}{N-1},u_n\Big)\Big)dx\\
&\le\int_{|x|\ge R_0}|u_n|^{\frac{\beta N}{N-1}}\Big(\exp\Big(\frac{\gamma_\infty N}{N-1}|u_n|^{\frac{N}{N-1}}\Big)-S_{N-2}\Big(\frac{\gamma_\infty N}{N-1},u_n\Big)\Big)dx.
\end{aligned}$$

Using (3.6), we obtain

$$\sup_{n\in\mathbb{N}}(\int_{\mathbb{R}^N}|u_n|^{\frac{\beta N}{N-1}}\Big(\exp\Big(\frac{\gamma(x)N}{N-1}|u_n|^{\frac{N}{N-1}}\Big)-S_{N-2}\Big(\frac{\gamma(x)N}{N-1},u_n\Big)\Big)dx)<+\infty.$$

Consequently,

$$\sup_{n\in\mathbb{N}}\Big(\int_{\mathbb{R}^N}|f(x,u_n)|^{\frac{N}{N-1}}\,dx\Big)<+\infty.$$

This completes the proof of Lemma 4.1. □

Proof of Theorem 1.3 completed. Let $w\in C_0^\infty(\mathbb{R}^N)$ be a radial function such that $w\ge 0$ and $\inf_{x\in\Omega}w(x)>0$. By (H_6), there exists $t_1>0$ large enough and a positive constant c_{20} such that

$$F(x,tw(x))\ge c_{20}\ t^\theta,\forall t\ge t_1,\ x\in\Omega.$$

Thus,

$$\int_{\mathbb{R}^N}F(x,tw)\,dx\ge\int_\Omega F(x,tw)\,dx\ge c_{21}t^\theta,\quad \forall t\ge t_1. \tag{4.3}$$

On the other hand, by (H4) and (H5), one can easily find $t_2>0$ large enough such that

$$\begin{aligned}
B(t)&\ge c_{22}t^{\lambda_1},\quad \forall t\ge t_2,\\
A(t)&\le c_{23}t^{\lambda_0},\quad \forall t\ge t_2.
\end{aligned}\tag{4.4}$$

Combining (4.3) and (4.4), for t large enough it follows

$$J(tw)=A\Big(\frac{t^N\|w\|^N}{N}\Big)-B\Big(\int_{\mathbb{R}^N}F(x,tw)\,dx\Big)-t\int_{\mathbb{R}^N}hw\,dx\le c_{24}t^{\lambda_0 N}-c_{25}t^{\lambda_1\theta}.$$

Since $\lambda_1\theta>\lambda_0 N$, we deduce that $J(tw)\to-\infty$ as $t\to+\infty$. Finally, taking Lemma 3.1 and Lemma 4.1 into account and according to the Mountain Pass Theorem, the functional J has a critical point with positive energy. Therefore, we conclude that the problem (1.1) admits at least two nontrivial weak solutions. This completes the proof of Theorem 1.3. □

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