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OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR SECOND-ORDER NONLINEAR INTEGRO-DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this article, we study the asymptotic behavior of non-oscillatory solutions of second-order integro-dynamic equations as well as the oscillatory behavior of forced second order integro-dynamic equations on time scales. The results are new for the continuous and discrete cases. Examples are provided to illustrate the relevance of the results.

1. Introduction

We are concerned with the asymptotic behavior of non-oscillatory solutions of the second-order integro-dynamic equation on time scales of the form

$$(r(t)(x^{\Delta}(t))^{\alpha})^{\Delta} + \int_0^t a(t,s)F(s,x(s))\Delta s = 0 \tag{1.1}$$

and the oscillatory behavior of the second-order forced integro-dynamic equation

$$(r(t)(x^{\Delta}(t)))^{\Delta} + \int_0^t a(t,s)F(s,x(s))\Delta s = e(t). \tag{1.2}$$

We take $\mathbb{T} \subseteq \mathbb{R}_+ = [0,\infty)$ to be an arbitrary time-scale with $0 \in \mathbb{T}$ and $\sup \mathbb{T} =$. By $t \geq s$ we mean as usual $t \in [s,\infty) \cap \mathbb{T}$.

We shall assume throughout that:

(i) $e, r : \mathbb{T} \to \mathbb{R}$ and $a : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ are rd-continuous and $r(t) > 0$, and $a(t,s) \geq 0$ for $t > s$, α is the ratio of positive odd integers and
$$\sup_{t \geq t_0} \int_0^{t_0} a(t,s)\Delta s := k < \infty, \quad t_0 \geq 0; \tag{1.3}$$

(ii) $F : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous and assume that there exist continuous functions $f_1, f_2 : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ such that $F(t,x) = f_1(t,x) - f_2(t,x)$ for $t \geq 0$;

(iii) there exist constants β and γ being the ratios of positive odd integers and functions $p_i \in C_{rd}(\mathbb{T},(0,\infty))$, $i = 1,2$, such that
$$\begin{aligned} xf_1(t,x) &\geq p_1(t)x^{\beta+1} \quad \text{for } x \neq 0 \text{ and } t \geq 0, \\ xf_2(t,x) &\leq p_2(t)x^{\gamma+1} \quad \text{for } x \neq 0 \text{ and } t \geq 0. \end{aligned}$$

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We consider only those solutions of equation (1.1) (resp, (1.2)) which are nontrivial and differentiable for $t \geq 0$. The term solution henceforth applies to such solutions of equation (1.1). A solution x is said to be oscillatory if for every $t_0 > 0$ we have $\inf_{t\geq t_0} x(t) < 0 < \sup_{t\geq t_0} x(t)$ and it is said to be non-oscillatory otherwise.

Dynamic equations on time-scales is a fairly new topic. For general basic ideas and background, we refer the reader to the seminal book [2].

Although the oscillation and nonoscillation theory of differential equations and difference equations is well developed, the problem for integro-differential equations of Volterra type was discussed only in a few papers in the literature, see [3, 7, 10, 8, 9, 11] and their references. We refer the reader to [4, 5] for some initial papers on the oscillation and nonoscillation of integro-dynamic and integral equations on time scales.

To the best of our knowledge, there are no results on the asymptotic behavior of non-oscillatory solutions of (1.1) and the oscillatory behavior of (1.2). Therefore, the main goal of this article is to establish some new criteria for the asymptotic behavior of non-oscillatory solutions of equation (1.1) and the oscillatory behavior of equation (1.2).

2. Asymptotic behavior of the non-oscillatory solutions of (1.1)

In this section we study the asymptotic behavior of all non-oscillatory solutions of equation (1.1) with all possible types of nonlinearities. We will employ the following two lemmas, the second of which is actually a consequence of the first.

Lemma 2.1 (Young inequality [6]). *Let X and Y be nonnegative real numbers, $n > 1$ and $\frac{1}{n} + \frac{1}{m} = 1$. Then*

$$XY \leq \frac{1}{n}X^n + \frac{1}{m}Y^m.$$

Equality holds if and only if $X = Y$.

Lemma 2.2 ([1]). *If X and Y are nonnegative real numbers, then*

$$X^\lambda + (\lambda - 1)Y^\lambda - \lambda XY^{\lambda-1} \geq 0 \quad \text{for } \lambda > 1, \tag{2.1}$$

$$X^\lambda - (1 - \lambda)Y^\lambda - \lambda XY^{\lambda-1} \leq 0 \quad \text{for } \lambda < 1, \tag{2.2}$$

where the equality holds if and only if $X = Y$.

We define

$$R(t, t_0) = \int_{t_0}^{t} \Big(\frac{s}{r(s)}\Big)^{1/\alpha} \Delta s, \quad t > t_0 \geq 0.$$

Note that due to monotonicity

$$\lim_{t\to\infty} R(t, t_0) \neq 0. \tag{2.3}$$

Our first result is the following.

Theorem 2.3. *Let conditions* (i)–(iii) *hold with $\gamma = 1$ and $\beta > 1$ and suppose*

$$\lim_{t\to\infty} \frac{1}{R(t, t_0)} \int_{t_0}^{t} \Big(\frac{1}{r(v)} \int_{t_0}^{v} \int_{t_0}^{u} a(u, s)p_1^{\frac{1}{1-\beta}}(s)p_2^{\frac{\beta}{\beta-1}}(s)\Delta s\Delta u\Big)^{1/\alpha} \Delta v < \infty \tag{2.4}$$

for some $t_0 \geq 0$. If x is a non-oscillatory solution of (1.1), then

$$x(t) = O(R(t, t_0)), \quad \text{as } t \to \infty. \tag{2.5}$$

Proof. Let x be a non-oscillatory solution of equation (1.1). Hence x is either eventually positive or eventually negative. First assume x is eventually positive, say $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Using conditions (ii) and (iii) with $\beta > 1$ and $\gamma = 1$ in equation (1.1), for $t \geq t_1$, we obtain

$$\big(r(t)(x^{\Delta}(t))^{\alpha}\big)^{\Delta} \leq -\int_0^{t_1} a(t,s)F(s,x(s))\Delta s + \int_{t_1}^{t} a(t,s)[p_2(s)x(s) - p_1(s)x^{\beta}]\Delta s. \tag{2.6}$$

If we apply (2.1) with $\lambda = \beta$, $X = p_1^{1/\beta}x$, and $Y = (\frac{1}{\beta}p_2 p_1^{-1/\beta})^{\frac{1}{\beta-1}}$ we have

$$p_2(t)x(t) - p_1(t)x^{\beta}(t) \leq (\beta-1)\beta^{\frac{\beta}{1-\beta}} p_1^{\frac{1}{1-\beta}}(t) p_2^{\frac{\beta}{\beta-1}}(t), \quad t \geq t_1. \tag{2.7}$$

Substituting (2.7) into (2.6) gives

$$\begin{aligned} &\big(r(t)(x^{\Delta}(t))^{\alpha}\big)^{\Delta} \\ &\leq -\int_0^{t_1} a(t,s)F(s,x(s))\Delta s + (\beta-1)\beta^{\frac{\beta}{1-\beta}} \int_{t_1}^{t} a(t,s) p_1^{\frac{1}{1-\beta}}(s) p_2^{\frac{\beta}{\beta-1}}(s)\Delta s \end{aligned} \tag{2.8}$$

for all $t \geq t_1 \geq 0$. Let

$$m := \max\{|F(t,x(t))| : t \in [0,t_1] \cap \mathbb{T}\}.$$

By assumption (i), we have

$$\big| -\int_0^{t_1} a(t,s)F(s,x(s))\Delta s\big| \leq \int_0^{t_1} a(t,s)|F(s,x(s))|\Delta s \leq mk := b. \tag{2.9}$$

Hence from (2.8) and (2.9), we obtain

$$\Big(r(t)\left(x^{\Delta}(t)\right)^{\alpha}\Big)^{\Delta} \leq b + (\beta-1)\beta^{\frac{\beta}{1-\beta}} \int_{t_1}^{t} a(t,s) p_1^{\frac{1}{1-\beta}}(s) p_2^{\frac{\beta}{\beta-1}}(s)\Delta s\,.$$

Integrating this inequality from t_1 to t leads to

$$\begin{aligned} &\left(x^{\Delta}(t)\right)^{\alpha} \\ &\leq \frac{r(t_1)\left|\left(x^{\Delta}(t_1)\right)^{\alpha}\right|}{r(t)} + b\frac{t-t_1}{r(t)} + \frac{(\beta-1)\beta^{\frac{\beta}{1-\beta}}}{r(t)} \int_{t_1}^{t}\int_{t_1}^{u} a(u,s) p_1^{\frac{1}{1-\beta}}(s) p_2^{\frac{\beta}{\beta-1}}(s)\Delta s\Delta u \end{aligned}$$

or

$$\left(x^{\Delta}(t)\right)^{\alpha} \leq \frac{c_0 t}{r(t)} + \frac{(\beta-1)\beta^{\frac{\beta}{1-\beta}}}{r(t)} \int_{t_1}^{t}\int_{t_1}^{u} a(t,s) p_1^{\frac{1}{1-\beta}}(s) p_2^{\frac{\beta}{\beta-1}}(s)\Delta s\Delta u$$

where

$$c_0 = \frac{r(t_1)|(x^{\Delta}(t_1))^{\alpha}|}{t_1} + b.$$

By employing the well-known inequality

$$(a_1+b_1)^{\lambda} \leq \sigma_{\lambda}\left(a_1^{\lambda} + b_1^{\lambda}\right) \quad \text{for } a_1 \geq 0,\ b_1 \geq 0, \text{ and } \lambda > 0, \tag{2.10}$$

where $\sigma_{\lambda} = 1$ if $\lambda < 1$ and $\sigma_{\lambda} = 2^{\lambda-1}$ if $\lambda \geq 1$ we see that there exists positive constants c_1 and c_2 depending on α such that

$$x^{\Delta}(t) \leq c_1\big(\frac{t}{r(t)}\big)^{1/\alpha} + c_2\Big(\frac{1}{r(t)}\int_{t_1}^{t}\int_{t_1}^{u} a(t,s) p_1^{\frac{1}{1-\beta}}(s) p_2^{\frac{\beta}{\beta-1}}(s)\Delta s\Delta u\Big)^{1/\alpha}.$$

Integrating this inequality from t_1 to $t \geq t_1$, we obtain

$$\begin{aligned} |x(t)| &\leq |x(t_1)| + c_1 R(t,t_1) \\ &\quad + c_2 \int_{t_1}^{t} \Big(\frac{1}{r(v)} \int_{t_1}^{v} \int_{t_1}^{u} a(u,s) p_1^{\frac{1}{1-\beta}}(s) p_2^{\frac{\beta}{\beta-1}}(s) \Delta s \Delta u\Big)^{1/\alpha} \Delta v \\ &\leq |x(t_1)| + c_1 R(t,t_0) \\ &\quad + c_2 \int_{t_0}^{t} \Big(\frac{1}{r(v)} \int_{t_0}^{v} \int_{t_0}^{u} a(u,s) p_1^{\frac{1}{1-\beta}}(s) p_2^{\frac{\beta}{\beta-1}}(s) \Delta s \Delta u\Big)^{1/\alpha} \Delta v. \end{aligned} \tag{2.11}$$

Dividing both sides of (2.11) by $R(t,t_0)$ and using (2.3) and (2.4), we see that (2.5) holds. The proof is similar if x is eventually negative. $\square$

Next, we present the following simple result.

Theorem 2.4. *Let conditions* (i) *and* (ii) *hold with $f_2 = 0$ and $x f_1(t,x) > 0$ for $x \neq 0$ and $t \geq 0$. If x is a non-oscillatory solution of equation* (1.1)*, then* (2.5) *holds.*

Proof. Let $x(t)$ be a non-oscillatory solution of equation (1.1) with $f_2 = 0$. First assume x is eventually positive, say $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. From (1.1) we find that

$$(r(t)(x^{\Delta}(t))^{\alpha})^{\Delta} = -\int_0^t a(t,s) f_1(s,x(s)) \Delta s \leq \int_0^{t_1} a(t,s) f_1(s,x(s)) \Delta s.$$

Using (1.3) (see (2.9)) in the above inequality, we obtain $(r(t)(x^{\Delta}(t))^{\alpha})^{\Delta} \leq b$. The rest of the proof is similar to that of Theorem 2.3 and hence is omitted. $\square$

Theorem 2.5. *Let conditions* (i)–(iii) *hold with $\beta = 1$ and $\gamma < 1$ and suppose*

$$\lim_{t\to\infty} \frac{1}{R(t,t_0)} \int_{t_0}^{t} \Big(\frac{1}{r(v)} \int_{t_0}^{v} \int_{t_0}^{u} a(u,s) p_1^{\frac{\gamma}{\gamma-1}}(s) p_2^{\frac{1}{1-\gamma}}(s) \Delta s \Delta u\Big)^{1/\alpha} \Delta v < \infty \tag{2.12}$$

for some $t_0 \geq 0$. If x is a non-oscillatory solution of equation (1.1)*, then* (2.5) *holds.*

Proof. Let x be a non-oscillatory solution of (1.1). First assume x is eventually positive, say $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. From conditions (ii) and (iii) with $\beta = 1$ and $\gamma < 1$ in equation (1.1) we have

$$\big(r(t)(x^{\Delta}(t))^{\alpha}\big)^{\Delta} \leq -\int_0^{t_1} a(t,s) F(s,x(s)) \Delta s + \int_{t_1}^{t} a(t,s)[p_2(s) x^{\gamma}(s) - p_1(s) x] \Delta s \tag{2.13}$$

for all t$\geq t_1$. Hence,

$$\big(r(t)(x^{\Delta}(t))^{\alpha}\big)^{\Delta} \leq b + \int_{t_1}^{t} a(t,s)[p_2(s) x^{\gamma}(s) - p_1(s) x] \Delta s,$$

where b is as in (2.9). Applying (2.2) with $\lambda = \gamma$, $X = p_2^{1/\gamma} x$ and $Y = (\frac{1}{\gamma} p_1 p_2^{\frac{-1}{\gamma}})^{\frac{1}{\gamma-1}}$, we obtain

$$p_2(t) x^{\gamma}(t) - p_1(t) x(t) \leq (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} p_1^{\frac{\gamma}{\gamma-1}}(t) p_2^{\frac{1}{1-\gamma}}(t), \quad t \geq t_1. \tag{2.14}$$

Using (2.14) in (2.13) we have

$$\Big(r(t)\left(x^{\Delta}(t)\right)^{\alpha}\Big)^{\Delta} \leq b + (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} \int_{t_1}^{t} a(t,s) p_1^{\frac{\gamma}{\gamma-1}}(s) p_2^{\frac{1}{1-\gamma}}(s) \Delta s \quad t \geq t_1.$$

The rest of the proof is similar to that of Theorem 2.3 and hence is omitted. $\square$

Theorem 2.6. *Let conditions* (i)–(iii) *hold with $\beta > 1$ and $\gamma < 1$ and assume that there exists a positive rd-continuous function $\xi : \mathbb{T} \to \mathbb{T}$ such that*

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{R(t,t_0)} \int_{t_0}^{t} \Big(\frac{1}{r(v)} \int_{t_0}^{v} \int_{t_0}^{u} a(u,s) \\ &\times \big[c_1 \xi^{\frac{\beta}{\beta-1}}(s) p_1^{\frac{1}{1-\beta}}(s) + c_2 \xi^{\frac{\gamma}{\gamma-1}}(s) p_2^{\frac{1}{1-\gamma}}(s) \big] \Delta s\, \Delta u \Big)^{1/\alpha} \Delta v < \infty \end{aligned} \tag{2.15}$$

for some $t_0 \geq 0$, where $c_1 = (\beta-1)\beta^{\frac{\beta}{1-\beta}}$ and $c_2 = (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}$. If x is a non-oscillatory solution of equation (1.1)*, then* (2.5) *holds.*

Proof. Let x be a non-oscillatory solution of equation (1.1). First assume x is eventually positive, say $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Using (ii) and (iii) in equation (1.1) we obtain

$$\begin{aligned} \big(r(t)(x^\Delta(t))^\alpha\big)^\Delta &\leq -\int_0^{t_1} a(t,s)F(s,x(s))\Delta s + \int_{t_1}^{t} a(t,s)[\xi(s)x(s) - p_1(s)x^\beta(s)]\,\Delta s \\ &\quad + \int_{t_1}^{t} a(t,s)[p_2(s)x^\gamma(s) - \xi(s)x(s)]\,\Delta s. \end{aligned}$$

As in the proof of Theorems 2.3 and 2.5, one can easily show that

$$\begin{aligned} &\big(r(t)(x^\Delta(t))^\alpha\big)^\Delta \\ &\leq -\int_0^{t_1} a(t,s)F(s,x(s))\Delta s \\ &\quad + \int_{t_1}^{t} a(t,s)\Big[(\beta-1)\beta^{\frac{\beta}{1-\beta}} \xi^{\frac{\beta}{\beta-1}}(s) p_1^{\frac{1}{1-\beta}}(s) + (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} \xi^{\frac{\gamma}{1-\gamma}}(s) p_2^{\frac{1}{1-\gamma}}(s)\Big]\,\Delta s. \end{aligned}$$

The rest of the proof is similar to that of Theorem 2.3 and hence is omitted. $\square$

Theorem 2.7. *Let conditions* (i)–(iii) *hold with $\beta > 1$ and $\gamma < 1$ and suppose that there exists a positive rd-continuous function $\xi : \mathbb{T} \to \mathbb{T}$ such that*

$$\lim_{t\to\infty} \frac{1}{R(t,t_0)} \int_{t_0}^{t} \Big(\frac{1}{r(v)} \int_{t_0}^{v} \int_{t_0}^{u} a(u,s) \xi^{\frac{\beta}{\beta-1}}(s) p_1^{\frac{1}{1-\beta}}(s)\,\Delta s\,\Delta u \Big)^{1/\alpha} \Delta v < \infty$$

and

$$\lim_{t\to\infty} \frac{1}{R(t,t_0)} \int_{t_0}^{t} \Big(\frac{1}{r(v)} \int_{t_0}^{v} \int_{t_0}^{u} a(u,s) \xi^{\frac{\gamma}{\gamma-1}}(s) p_2^{\frac{1}{1-\gamma}}(s)\,\Delta s\,\Delta u \Big)^{1/\alpha} \Delta v < \infty$$

for some $t_0 \geq 0$. If x is a non-oscillatory solution of equation (1.1)*, then* (2.5) *holds.*

For the cases when both f_1 and f_2 are superlinear ($\beta > \gamma > 1$) or else sublinear ($1 > \beta > \gamma > 0$), we have the following result.

Theorem 2.8. *Let conditions* (i)–(iii) *hold with $\beta > \gamma$ and assume*

$$\lim_{t\to\infty} \frac{1}{R(t,t_0)} \int_{t_0}^{t} \Big(\frac{1}{r(v)} \int_{t_0}^{v} \int_{t_0}^{u} a(u,s) p_1^{\frac{\gamma}{\gamma-\beta}}(s) p_2^{\frac{\beta}{\beta-\gamma}}(s)\,\Delta s\,\Delta u \Big)^{1/\alpha} \Delta v < \infty \tag{2.16}$$

for some $t_0 \geq 0$. If x is a non-oscillatory solution of equation (1.1)*, then* (2.5) *holds.*

Proof. Let x be a non-oscillatory solution of (1.1). First assume x is eventually positive, say $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Using conditions (ii) and (iii) in equation (1.1) we have

$$\begin{aligned}&\big(r(t)(x^\Delta(t))^\alpha\big)^\Delta \\ &\leq -\int_0^{t_1} a(t,s)F(s,x(s))\Delta s + \int_{t_1}^t a(t,s)[p_2(s)x^\gamma(s) - p_1(s)x^\beta(s)]\,\Delta s.\end{aligned} \tag{2.17}$$

By applying Lemma 2.1 with

$$n = \frac{\beta}{\gamma}, \quad X = x^\gamma(s), \quad Y = \frac{\gamma p_2(s)}{\beta p_1(s)}, \quad m = \frac{m}{\beta-\gamma}$$

we obtain

$$\begin{aligned}p_2(s)x^\gamma(s) \,-\, p_1(s)x^\beta(s) &= \frac{\beta}{\gamma}p_1(s)[x^\gamma(s)\frac{\gamma}{\beta}\frac{p_2(s)}{p_1(s)} - \frac{\gamma}{\beta}(x^\gamma(s))^{\beta/\gamma}] \\ &= \frac{\beta}{\gamma}p_1(s)[XY - \frac{1}{n}X^n] \\ &\leq \frac{\beta}{\gamma}p_1(s)(\frac{1}{m}Y^m) \\ &= (\frac{\beta-\gamma}{\gamma})[\frac{\gamma}{\beta}p_2(s)]^{\frac{\beta}{\beta-\gamma}}(p_1(s))^{\frac{\gamma}{\gamma-\beta}}.\end{aligned}$$

The rest of the proof is similar to that of Theorem 2.3 and hence is omitted. □

Remark 2.9. If in addition to the hypotheses of Theorems 2.3–2.8,

$$\lim_{t\to\infty} R(t,t_0) < \infty,$$

then every non-oscillatory solution of (1.1) is bounded.

Remark 2.10. The results given above hold for equations of the form

$$(r(t)(x^\Delta(t))^\alpha)^\Delta + \int_0^t a(t,s)F(s,x(s))\Delta s = e(t) \tag{2.18}$$

if the additional condition

$$\lim_{t\to\infty} \frac{1}{R(t,t_0)} \int_{t_0}^t \Big(\frac{1}{r(v)} \int_{t_0}^v |e(s)|\,\Delta s\Big)^{1/\alpha} \Delta v < \infty$$

is satisfied.

3. Oscillation results for (1.2)

This section we study of the oscillatory properties of (1.2). For this end hypotheses (i) and (ii) are replaced by the assumptions:

(I) $e, r : \mathbb{T} \to \mathbb{R}$ and $a : \mathbb{T}\times\mathbb{T} \to \mathbb{R}$ are rd-continuous, $r(t) > 0$ and $a(t,s) \geq 0$ for $t > s$ and there exist rd-continuous functions $k, m : \mathbb{T} \to \mathbb{R}^+$ such that

$$a(t,s) \leq k(t)m(s), \quad t \geq s \tag{3.1}$$

with

$$k_1 := \sup_{t\geq 0} k(t) < \infty, \quad k_2 := \sup_{t\geq 0} \int_0^t m(s)\Delta s < \infty.$$

In this case condition (1.3) is satisfied with $k = k_1 k_2$.

(II) $F : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous and assume that there exists rd-continuous function, $q : \mathbb{T} \to (0, \infty)$ and a real number β with $0 < \beta \le 1$ such that

$$xF(t, x) \le q(t)x^{\beta+1}, \quad \text{for } x \neq 0 \text{ and } t \ge 0. \tag{3.2}$$

In what follows

$$g_\pm(t, p) = e(t) \mp k_1(1-\beta)\beta^{\beta/(1-\beta)} \int_0^t p^{\beta/(\beta-1)}(s)q(s)^{1/(1-\beta)}m^{1/(1-\beta)}(s)\Delta s, \tag{3.3}$$

where $0 < \beta < 1$, $p \in C_{rd}(\mathbb{T}, (0, \infty))$.

We first give sufficient conditions under which non-oscillatory solutions x of equation (1.2) satisfy

$$x(t) = O(t), \quad \text{as } t \to \infty. \tag{3.4}$$

Theorem 3.1. *Let $0 < \beta < 1$, conditions* (I) *and* (II) *hold, assume the function $t/r(t)$ is bounded, and for some $t_0 \ge 0$,*

$$\int_{t_0}^\infty \frac{s}{r(s)}\Delta s < \infty. \tag{3.5}$$

Let $p \in C_{rd}(\mathbb{T}, (0, \infty))$ such that

$$\int_{t_0}^\infty sp(s)\,\Delta s < \infty. \tag{3.6}$$

If

$$\begin{gathered} \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^t \frac{1}{r(u)}\int_{t_0}^u g_-(s, p)\Delta s\Delta u < \infty, \\ \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^t \frac{1}{r(u)}\int_{t_0}^u g_+(s, p)\,\Delta s\,\Delta u > -\infty, \end{gathered} \tag{3.7}$$

then every non-oscillatory solution $x(t)$ of (1.2) *satisfies*

$$\limsup_{t\to\infty} \frac{|x(t)|}{t} < \infty.$$

Proof. Let x be a non-oscillatory solution of (1.1). First assume x is eventually positive, say $x(t) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$.

Using condition (3.2) in (1.2) we have

$$\big(r(t)(x^\Delta(t))\big)^\Delta \le e(t) - \int_0^{t_1} a(t, s)F(s, x(s))\Delta s + \int_{t_1}^t a(t, s)q(s)x^\beta(s)\Delta s, \tag{3.8}$$

for $t \ge t_1$. Let

$$c := \max_{0\le t\le t_1} |F(t, x(t)| < \infty.$$

By assumption (3.1), we obtain

$$\big| -\int_0^{t_1} a(t, s)F(s, x(s))\Delta s\big| \le c\int_0^{t_1} a(t, s)\Delta s \le ck_1k_2 =: b, \quad t \ge t_1.$$

Hence from (3.8) we have

$$\begin{aligned} \big(r(t)(x^\Delta(t))\big)^\Delta &\le e(t) + b + k_1\int_{t_1}^t [m(s)q(s)x^\beta(s) - p(s)x(s)]\Delta s \\ &\quad + k_1\int_{t_1}^t p(s)x(s)\Delta s, \quad t \ge t_1. \end{aligned} \tag{3.9}$$

Applying (2.2) of Lemma 2.2 with

$$\lambda = \beta, \quad X = (qm)^{1/\beta} x, \quad Y = \big(\frac{1}{\beta} p (mq)^{-1/\beta}\big)^{\frac{1}{\beta-1}}$$

we have

$$m(s)q(s)x^\beta(s) - p(s)x(s) \le (1-\beta)\beta^{\beta/(1-\beta)} p^{\beta/(\beta-1)}(s) m^{1/(1-\beta)}(s) q^{1/(1-\beta)}(s).$$

Thus, we obtain

$$\big(r(t)(x^\Delta(t))\big)^\Delta \le g_+(t,p) + b + k_1 \int_{t_1}^t p(s)x(s)\Delta s \quad \text{for } t \ge t_1. \tag{3.10}$$

Integrating (3.10) from t_1 to t we have

$$r(t)x^\Delta(t) \le r(t_1)x^\Delta(t_1) + \int_{t_1}^t g_+(s,p)\Delta s + b(t-t_1) + k_1 \int_{t_1}^t \int_{t_1}^u p(s)x(s)\Delta s\,\Delta u, \tag{3.11}$$

for $t \ge t_1$. Employing [10, Lemma 3] to interchange the order of integration, we obtain

$$r(t)x^\Delta(t) \le r(t_1)x^\Delta(t_1) + \int_{t_1}^t g_+(s,p)\Delta s + b(t-t_1) + k_1\, t \int_{t_1}^t p(s)x(s)\Delta s, \quad t \ge t_1$$

and so,

$$x^\Delta(t) \le \frac{r(t_1)x^\Delta(t_1)}{r(t)} + \frac{1}{r(t)} \int_{t_1}^t g_+(s)\Delta s + \frac{b(t-t_1)}{r(t)} + \frac{k_1\, t}{r(t)} \int_{t_1}^t p(s)x(s)\Delta s, \quad t \ge t_1.$$

Integrating this inequality from t_1 to t and using (3.5) and the fact that the function $t/r(t)$ is bounded for $t \ge t_1$, say by k_3 we see that

$$\begin{aligned} x(t) \le x(t_1) + r(t_1)x^\Delta(t_1) \int_{t_1}^t \frac{1}{r(s)}\Delta s + \int_{t_1}^t \frac{1}{r(u)} \int_{t_1}^u g_+(s)\Delta s \Delta u \\ + b\int_{t_1}^t \frac{s}{r(s)}\Delta s + k_1 k_3 \int_{t_1}^t \int_{t_1}^u p(s)x(s)\Delta s \Delta u, \quad t \ge t_1. \end{aligned}$$

Once again, using [10, Lemma 3] we have

$$\begin{aligned} x(t) \le x(t_1) + r(t_1)x^\Delta(t_1) \int_{t_1}^t \frac{1}{r(s)}\,\Delta s + \int_{t_1}^t \frac{1}{r(u)} \int_{t_1}^u g_+(s)\,\Delta s\,\Delta u \\ + b\int_{t_1}^t \frac{s}{r(s)}\Delta s + k_1 k_3 t \int_{t_1}^t p(s)x(s)\,\Delta s, \quad t \ge t_1 \end{aligned} \tag{3.12}$$

and so,

$$\frac{x(t)}{t} \le c_1 + c_2 \int_{t_1}^t s p(s)\Big(\frac{x(s)}{s}\Big)\Delta s, \quad t \ge t_1; \tag{3.13}$$

note (3.5) and (3.7), $c_2 = k_1 k_3$ and c_1 is an upper bound for

$$\frac{1}{t}\Big[x(t_1) + r(t_1)x^\Delta(t_1) \int_{t_1}^t \frac{1}{r(s)}\,\Delta s + \int_{t_1}^t \frac{1}{r(u)} \int_{t_1}^u g_+(s)\,\Delta s\,\Delta u + b\int_{t_1}^t \frac{s}{r(s)}\,\Delta s]$$

for $t \ge t_1$. Applying Gronwall's inequality [2, Corollary 6.7] to inequality (3.13) and then using condition (3.6) we have

$$\limsup_{t\to\infty} \frac{x(t)}{t} < \infty. \tag{3.14}$$

If $x(t)$ is eventually negative, we can set $y = -x$ to see that y satisfies equation (1.2) with $e(t)$ replaced by $-e(t)$ and $F(t,x)$ replaced by $-F(t,-y)$. It follows in a similar manner that

$$\limsup_{t\to\infty} \frac{-x(t)}{t} < \infty. \tag{3.15}$$

The proof is complete. $\square$

Next, by employing Theorem 3.1 we present the following oscillation result for equation (1.2).

Theorem 3.2. *Let $0 < \beta < 1$, conditions (I), (II), (3.5), (3.6), and (3.7) hold, assume the function $t/r(t)$ is bounded , and there is a function $p \in C_{rd}(\mathbb{T},(0,\infty))$ such that (3.6) holds. If for every $0 < M < 1$,*

$$\begin{gathered}\limsup_{t\to\infty} \Big[Mt + \int_{t_0}^t \frac{1}{r(u)} \int_{t_0}^u g_-(s,p)\Delta s\Delta u\Big] = \infty,\\ \liminf_{t\to\infty} \Big[Mt + \int_{t_0}^t \frac{1}{r(u)} \int_{t_0}^u g_+(s,p)\Delta s\Delta u\Big] = -\infty,\end{gathered} \tag{3.16}$$

then (1.2) is oscillatory.

Proof. Let x be a non-oscillatory solution of equation (1.2), say $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. The proof when $x(t)$ is eventually negative is similar. Proceeding as in the proof of Theorem 3.1 we arrive at (3.12). Therefore,

$$\begin{aligned}x(t) \leq x(t_1) + r(t_1)x^{\Delta}(t_1) \int_{t_1}^{\infty} \frac{1}{r(s)}\Delta s + \int_{t_1}^t \frac{1}{r(u)} \int_{t_1}^u g_+(s,p)\Delta s\Delta u\\ + b\int_{t_1}^{\infty} \frac{s}{r(s)}\Delta s + k_1k_3t \int_{t_1}^{\infty} sp(s)\big(\frac{x(s)}{s}\big)\Delta s, \quad t \geq t_1.\end{aligned}$$

Clearly, the conclusion of Theorem 3.1 holds. This together with (3.5) imply that

$$x(t) \leq M_1 + M\,t + \int_{t_1}^t \frac{1}{r(u)} \int_{t_1}^u g_+(s,p)\Delta s\Delta u, \tag{3.17}$$

where M_1 and M are positive real numbers. Note that we make $M < 1$ possible by increasing the size of t_1. Finally, taking liminf in (3.17) as t$\to \infty$ and using (3.16) result in a contradiction with the fact that $x(t)$ is eventually positive. $\square$

Corollary 3.3. *Let $0 < \beta < 1$ and condition (I), (II), (3.5), and (3.6) hold, assume the function $t/r(t)$ is bounded, and for some $t_0 \geq 0$ suppose*

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^t \frac{1}{r(u)} \int_{t_0}^u e(s)\,\Delta s\Delta u < \infty, \quad \liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^t \frac{1}{r(u)} \int_{t_0}^u e(s)\Delta s\Delta u > -\infty \tag{3.18}$$

and

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^t \frac{1}{r(u)} \int_{t_0}^u p^{\beta/(\beta-1)}(s)q(s)^{1/(1-\beta)}m^{1/(1-\beta)}(s)\Delta s\Delta u < \infty. \tag{3.19}$$

If for every $0 < M < 1$,

$$\begin{gathered}\limsup_{t\to\infty} \Big[Mt + \int_{t_0}^t \frac{1}{r(u)} \int_{t_0}^u e(s)\Delta s\Delta u\Big] = \infty,\\ \liminf_{t\to\infty} \Big[Mt + \int_{t_0}^t \frac{1}{r(u)} \int_{t_0}^u e(s)\Delta s\Delta u\Big] = -\infty,\end{gathered} \tag{3.20}$$

then (1.2) *is oscillatory.*

Similar reasoning to that in the sublinear case guarantees the following theorems for the integro-dynamic equation (1.2) when $\beta = 1$.

Theorem 3.4. *Let $\beta = 1$, conditions* (I), (II), (3.5) *and* (3.18) *hold, assume the function $t/r(t)$ is bounded, and for some $t_0 \geq 0$ suppose*

$$\limsup_{t\to\infty} \int_{t_0}^{t} sm(s)q(s)\Delta s < \infty. \tag{3.21}$$

Then every non-oscillatory solution of equation (1.2) *satisfies*

$$\limsup_{t\to\infty} \frac{|x(t)|}{t} < \infty.$$

Theorem 3.5. *Let $\beta = 1$, conditions* (I), (II), (3.5), (3.18), (3.20), *and* (3.21) *hold, assume the function $t/r(t)$ is bounded. Then* (1.2) *is oscillatory.*

Remark 3.6. We note that the results of Section 3 can be obtained by using the hypothesis (i) with the additional assumption that the function $a(t,s)$ is non-increasing with respect to the first variable. In this case, $k_1 m(t)$ which appeared in the proofs and $m(t)$ which appeared in the statements of the theorems should be replaced by $a(t,t)$. The details are left to the reader.

4. Examples

As we already mentioned the results of the present paper are new for the cases when $\mathbb{T} = \mathbb{R}$ (the continuous case) or when $\mathbb{T} = \mathbb{Z}$ (the discrete case).

Example 4.1. Consider the integro-differential equations

$$\Big(\frac{1}{t}(x'(t))^3\Big)' + \int_0^t \frac{t}{t^2+s^2}[s^a\, x^5(s) - x^3(s)]ds = 0, \quad t > 0 \tag{4.1}$$

and

$$\Big(\frac{1}{t^2}(x'(t))^{1/3}\Big)' + \int_0^t \frac{t}{t^2+s^2}[s^b x^{5/7}(s) - s^c x^{3/7}(s)]ds = 0, \quad t > 0, \tag{4.2}$$

where a, b, and c are nonnegative real numbers satisfying $3a < 2$ and $3b - 2 < 5c \leq 3b$.

For (4.1), take $\alpha = 3$, $r(t) = 1/t$, $a(t,s) = t/(t^2+s^2)$, $p_1(t) = t^a$, $p_2(t) = 1$, $\beta = 5$, $\gamma = 3$, $R(t,0) = (3/5)t^{5/3}$. Since

$$\begin{aligned}
&t^{-5/3}\int_0^t \Big(v\int_0^v \frac{1}{u}\int_0^u \frac{u^2}{u^2+s^2} s^{-3a/2} ds du\Big)^{1/3} dv \\
&\leq c_1 t^{-5/3}\int_0^t \Big(v\int_0^v u^{-3a/2} du\Big)^{1/3} dv \\
&= c_2 t^{-a/2},
\end{aligned}$$

where c_1 and c_2 are certain constants, condition (2.16) holds.

For (4.2), take $\alpha = 1/3$, $r(t) = 1/t^2$, $a(t,s) = t/(t^2+s^2)$, $p_1(t) = t^b$, $p_2(t) = t^c$, $\beta = 5/7$, $\gamma = 3/7$, $R(t,0) = (1/10)t^{10}$. Condition (2.16) holds, because

$$t^{-10}\int_0^t \Big(v^2\int_0^v \frac{1}{u}\int_0^u \frac{u^2}{u^2+s^2} s^{-3a/2+5c/2} ds du\Big)^3 dv$$

$$\begin{aligned}&\le d_1 t^{-10}\int_0^t \Big(v^2\int_0^v u^{-3b/2+5c/2}du\Big)^3 dv\\ &= d_2 t^{-9b/2+15c/2},\end{aligned}$$

where d_1 and d_2 are certain constants.

As a result, we may conclude from Theorem 2.8 that every non-oscillatory solution of (4.1) and of (4.2) satisfies $x = O(t^{5/3})$ and $x = O(t^{10})$, respectively, as $t \to \infty$.

Example 4.2. Consider the integro-differential equation

$$((1+t)^3x')' + \int_0^t \frac{x^\beta(s)}{(t^2+1)(s^4+)}ds = t^4\sin t, \tag{4.3}$$

where $\beta = 1/3$ or $\beta = 1$.

We observe that $r(t) = (1+t)^3$, $k(t) = 1/(t^2+1)$, $m(s) = 1/(s^4+1)$, $q(t) = 1$, $e(t) = t^4 \sin t$. Letting $p(t) = m(t)$, we see that the integral appearing in the definition of $g_\pm(t,p)$ given by (3.3) becomes bounded. It is then not difficult to show that all conditions of Theorem 3.2 for $\beta = 1/3$ are satisfied. On the other hand, all conditions of Theorem 3.5 for $\beta = 1$ are also satisfied. Therefore, every solution of equation (4.3) is oscillatory for $\beta = 1/3$ and $\beta = 1$.

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