Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 101, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXACT CONTROLLABILITY FOR A WAVE EQUATION WITH MIXED BOUNDARY CONDITIONS IN A NON-CYLINDRICAL DOMAIN

LIZHI CUI, HANG GAO

Abstract. In this article we study the exact controllability of a one-dimensional wave equation with mixed boundary conditions in a non-cylindrical domain. The fixed endpoint has a Dirichlet-type boundary condition, while the moving end has a Neumann-type condition. When the speed of the moving endpoint is less than the characteristic speed, the exact controllability of this equation is established by Hilbert Uniqueness Method. Moreover, we shall give the explicit dependence of the controllability time on the speed of the moving endpoint.

1. Introduction and statement of main results

Given $T > 0$. For any $0 < k < 1$, set

$$\alpha_k(t) = 1 + kt \quad \text{for } t \in [0, T]. \tag{1.1}$$

Also, define the non-cylindrical domain

$$\widehat{Q}_T^k = \{(y,t) \in \mathbb{R}^2;\ 0 < y < \alpha_k(t),\ t \in [0,T]\}$$

and write

$$V(0, \alpha_k(t)) = \{\varphi \in H^1(0, \alpha_k(t)); \varphi(0) = 0\} \quad \text{for } t \in [0,T],$$

which is a subspace of $H^1(0, \alpha_k(t))$ and we denote by $[V(0, \alpha_k(t))]'$ its conjugate space.

Consider the wave equation

$$\begin{gathered} u_{tt} - u_{yy} = 0 \quad \text{in } \widehat{Q}_T^k, \\ u(0,t) = 0, \quad u_y(\alpha_k(t), t) = v(t) \quad \text{on } (0,T), \\ u(y,0) = u^0(y), \quad u_t(y,0) = u^1(y) \quad \text{in } (0,1), \end{gathered} \tag{1.2}$$

where v is the control variable, u is the state variable and $(u^0, u^1) \in L^2(0,1) \times [V(0,1)]'$ is any given initial value. By [3] and [5], it is easy to check that the equation (1.2) has a unique solution u by transposition

$$u \in C([0,T]; L^2(0, \alpha_k(t))) \cap C^1([0,T]; [V(0, \alpha_k(t))]').$$

2000 *Mathematics Subject Classification.* 58J45, 35L05.
Key words and phrases. Exact controllability; wave equation; mixed boundary conditions; non-cylindrical domain.

Submitted October 15, 2013. Published April 11, 2014.

The main purpose of this article is to study the exact controllability of (1.2). There are numerous publications on the controllability problems of wave equations in a cylindrical domain. However, there are only a few works on the exact controllability for wave equations defined in non-cylindrical domains. We refer to [1, 2, 5, 6, 7] for some known results in this respect. In [1], the exact controllability of a multi-dimensional wave equation with constant coefficients in a non-cylindrical domain was established, while a control entered the system through the whole non-cylindrical domain. In [2, 5, 6, 7], some controllability results for the wave equations with Dirichlet boundary conditions in suitable non-cylindrical domains were investigated, respectively.

In [2] and [5], exact controllability of a wave equation in certain non-cylindrical domain was studied. But in the one-dimensional case, some conditions on the moving boundary were required, e.g.

$$\int_0^\infty |\alpha_k'(t)|dt < \infty. \tag{1.3}$$

In [6] and [7], the exact Dirichlet boundary controllability of the following systems were discussed,

$$\begin{gathered}
u_{tt} - u_{yy} = 0 \quad \text{in } \widehat{Q}_T^k, \\
u(0,t) = 0, \quad u(\alpha_k(t), t) = v(t) \quad \text{on } (0,T), \\
u(y,0) = u^0, \quad u_t(y,0) = u^1 \quad \text{in } (0,1),
\end{gathered}$$

and

$$\begin{gathered}
u_{tt} - u_{yy} = 0 \quad \text{in } \widehat{Q}_T^k, \\
u(0,t) = v(t), \quad u(\alpha_k(t), t) = 0 \quad \text{on } (0,T), \\
u(y,0) = u^0, \quad u_t(y,0) = u^1 \quad \text{in } (0,1),
\end{gathered}$$

In [6, 7] and in this research, we deal with the different case. It is easy to check that the condition

$$\int_0^\infty |\alpha_k'(t)|dt = \infty$$

is satisfied on the moving boundary

To overcome these difficulties, in this article, we transform (1.2) into an equivalent wave equation with variable coefficients in the cylindrical domain and establish the exact controllability of this equation by Hilbert Uniqueness Method. In [3], the Neumann boundary controllability problem for a multi-dimensional wave equation with variable coefficients in a cylindrical domain was studied. However in [3], in the one-dimensional case, the condition (1.3) was required. In this paper, the key point is to construct a different adjoint equation. Then we define weighted energy function for this adjoint equation and characterize the energy explicitly (see (2.4)).

Throughout this article, we set

$$T_k^* = \frac{e^{\frac{2k(1+k)}{1-k}} - 1}{k}. \tag{1.4}$$

The main result of this paper is stated as follows.

Theorem 1.1. *For any given $T > T_k^*$, the equation (1.2) is exactly controllable at the time T; i.e., for any initial value $(u^0, u^1) \in L^2(0,1) \times [V(0,1)]'$ and target*

$(u_d^0, u_d^1) \in L^2(0, \alpha_k(T)) \times [V(0, \alpha_k(T))]'$, there exists a control $v \in [H^1(0,T)]'$ such that the corresponding solution u of (1.2) *satisfies*

$$u(T) = u_d^0 \quad \textit{and} \quad u_t(T) = u_d^1. \tag{1.5}$$

Remark 1.2. It is easy to check that

$$T_0^* := \lim_{k\to 0} T_k^* = \lim_{k\to 0} \frac{e^{\frac{2k(1+k)}{1-k}} - 1}{k} = 2.$$

It is well known that the wave equation (1.2) in the cylindrical domain is exactly controllable at any time $T > T_0^*$. As we know, T_0^* is sharp. However, we do not know whether the controllability time T_k^* is sharp.

To establish the exact controllability of (1.2), we first transform (1.2) into an equivalent wave equation with variable coefficients in a cylindrical domain. To this aim, for any $(y,t) \in \widehat{Q}_T^k$, set $y = \alpha_k(t)x$ and $u(y,t) = u(\alpha_k(t)x, t) = w(x,t)$. Then it is easy to check that (1.2) is transformed into the wave equation

$$\begin{gathered} w_{tt} - [\frac{\beta_k(x,t)}{\alpha_k(t)} w_x]_x + [\frac{\gamma_k(x)}{\alpha_k(t)}] w_{tx} = 0 \quad \text{in } Q, \\ w(0,t) = 0, \quad w_x(1,t) = \overline{v}(t) \quad \text{on } (0,T), \\ w(x,0) = w^0(x), \quad w_t(x,0) = w^1(x) \quad \text{in } (0,1), \end{gathered} \tag{1.6}$$

where

$$\begin{gathered} Q = (0,1) \times (0,T), \quad \overline{v}(t) = \alpha_k(t) v(t), \quad \beta_k(x,t) = \frac{1 - k^2x^2}{\alpha_k(t)}, \\ \gamma_k(x) = -2kx, w^0 = u^0 \quad \text{and } w^1 = u^1 + kxu_x^0. \end{gathered} \tag{1.7}$$

By a method similar to the one used in [3], it is easy to check that the equation (1.6) has a unique solution w by transposition

$$w \in C([0,T]; L^2(0,1)) \cap C^1([0,T]; [V(0,1)]').$$

Moreover, the exact controllability of (1.2) (Theorem 1.1) is reduced to the following exact controllability result for (1.6).

Theorem 1.3. *Suppose that $T > T_k^*$. Then for any initial value $(w^0, w^1) \in L^2(0,1) \times [V(0,1)]'$ and target $(w_d^0, w_d^1) \in L^2(0,1) \times [V(0,1)]'$, there exists a control $\overline{v} \in [H^1(0,T)]'$ such that the corresponding solution w of* (1.6) *satisfies*

$$w(T) = w_d^0 \quad \textit{and} \quad w_t(T) = w_d^1.$$

To prove Theorem 1.3, we adopt Hilbert Uniqueness Method. The key is to define a weighted energy function for a wave equation with variable coefficients in cylindrical domains.

The rest of this paper is organized as follows. In Section 2, we derive an explicit energy equality for a wave equation with variable coefficients in cylindrical domains and further deduce two key inequalities for this equation. Section 3 is devoted to a proof of Theorem 1.3.

2. Two inequalities for the wave equation with variable coefficients

First we introduce some notation. Denote by $|\cdot|$ and $\|\cdot\|$ the norms of the spaces $L^2(0,1)$ and $V(0,1)$, respectively. Also, we use L^2, V and V' to represent the spaces $L^2(0,1), V(0,1)$ and $[V(0,1)]'$, respectively. Denote by $\langle\cdot,\cdot\rangle$ the duality product between the linear space F and its dual space F'.

Consider the wave equation with variable coefficients

$$\begin{gathered}\alpha_k(t)z_{tt}-[\beta_k(x,t)z_x]_x+\gamma_k(x)z_{tx}=0 \quad \text{in } Q,\\ z(0,t)=0,\quad \beta_k(1,t)z_x(1,t)-\gamma_k(1)z_t(1,t)=0 \quad \text{on } (0,T),\\ z(x,0)=z^0(x),\quad z_t(x,0)=z^1(x) \quad \text{in } (0,1),\end{gathered} \tag{2.1}$$

where $(z^0,z^1)\in V\times L^2$ is any given initial value, and α_k, β_k and γ_k are the functions given in (1.7). By a similar method in [3] and [8], it is easy to check that (2.1) has a unique solution z by transposition

$$z\in C([0,T];V)\cap C^1([0,T];L^2).$$

Define the following energy function for (2.1),

$$E(t)=\frac{1}{2}\int_0^1[\alpha_k(t)|z_t(x,t)|^2+\beta_k(x,t)|z_x(x,t)|^2]dx \quad \text{for } t\in[0,T],$$

where z is the solution of (2.1). It follows that

$$E_0\triangleq E(0)=\frac{1}{2}\int_0^1[|z^1(x)|^2+\beta_k(x,0)|z_x^0(x)|^2]dx.$$

To prove Theorem 1.3, we need the following two key inequalities.

Theorem 2.1. *For any $T>0$, there exists a positive constant C_1 depending only only T, such that solutions z of* (2.1) *satisfy*

$$\int_0^T|z_t(1,t)|^2dt\le C_1(\|z^0\|^2+|z^1|^2)\quad \textit{for any } (z^0,z^1)\in V\times L^2. \tag{2.2}$$

Theorem 2.2. *Suppose that $T>T_k^*$. Then there exists a positive constant C_2 depending only on T, such that solutions z of* (2.1) *satisfy*

$$\int_0^T|z_t(1,t)|^2dt\ge C_2(\|z^0\|^2+|z^1|^2)\quad \textit{for any } (z^0,z^1)\in V\times L^2. \tag{2.3}$$

First, we prove two lemmas, which will be used in the proofs of these inequalities. The first lemma is related to an equivalent expression of the energy $E(t)$.

Lemma 2.3. *Suppose that z is any solution of* (2.1)*. Then we have*

$$E(t)=\frac{1}{\alpha_k(t)}E_0-\frac{k}{\alpha_k(t)}\int_0^t\alpha_k(s)|z_t(1,s)|^2ds,\quad 0\le t\le T. \tag{2.4}$$

Proof. Multiplying both sides of the first equation of (2.1) by z_t and integrating on $(0,1)\times(0,t)$, we obtain

$$\begin{aligned}0&=\int_0^t\int_0^1\big\{\alpha_k(s)z_{tt}(x,s)z_t(x,s)-[\beta_k(x,s)z_x(x,s)]_xz_t(x,s)\\&\quad+\gamma_k(x)z_{tx}(x,s)z_t(x,s)\big\}dx\,ds\\&\triangleq J_1+J_2+J_3.\end{aligned}$$

Next, we calculate the above three integrals. It is easy to check that

$$\begin{aligned} J_1 &= \int_0^t \int_0^1 \frac{1}{2}\alpha_k(s)[|z_t(x,s)|^2]_t \,dx\,ds \\ &= \frac{1}{2}\int_0^1 \alpha_k(s)|z_t(x,s)|^2 dx\Big|_0^t - \frac{k}{2}\int_0^t \int_0^1 |z_t(x,s)|^2 \,dx\,ds. \end{aligned} \tag{2.5}$$

Further, by the second equation of (2.1), it holds that

$$\begin{aligned} J_2 &= -\int_0^t \beta_k(x,s)z_x(x,s)z_t(x,s)ds\Big|_0^1 + \int_0^t \int_0^1 \beta_k(x,s)z_x(x,s)z_{xt}(x,s)\,dx\,ds \\ &= -\int_0^t \beta_k(x,s)z_x(x,s)z_t(x,s)ds\Big|_0^1 + \frac{1}{2}\int_0^1 \beta_k(x,s)|z_x(x,s)|^2 dx\Big|_0^t \\ &\quad - \frac{1}{2}\int_0^t \int_0^1 \beta_{k,t}(x,s)|z_x(x,s)|^2 \,dx\,ds \\ &= -\int_0^t \gamma_k(1)|z_t(1,s)|^2 ds + \frac{1}{2}\int_0^1 \beta_k(x,s)|z_x(x,s)|^2 dx\Big|_0^t \\ &\quad - \frac{1}{2}\int_0^t \int_0^1 \beta_{k,t}(x,s)|z_x(x,s)|^2 \,dx\,ds. \end{aligned}$$

By (1.7), it is obvious that

$$\beta_{k,t}(x,t) = -\frac{k(1-k^2x^2)}{(1+kt)^2} = -\frac{k}{(1+kt)}\beta_k(x,t).$$

This implies that

$$\begin{aligned} J_2 &= -\int_0^t \int_0^1 [\beta_k(x,s)z_x(x,s)]_x z_t(x,s)\,dx\,ds \\ &= -\int_0^t \gamma_k(1)|z_t(1,s)|^2 ds + \frac{1}{2}\int_0^1 \beta_k(x,s)|z_x(x,s)|^2 dx\Big|_0^t \\ &\quad + \frac{1}{2}\int_0^t \frac{k}{(1+ks)}\int_0^1 \beta_k(x,s)|z_x(x,s)|^2 \,dx\,ds. \end{aligned} \tag{2.6}$$

Further, by the definition of γ_k, we find that

$$\begin{aligned} J_3 &= \frac{1}{2}\int_0^t \gamma_k(x)|z_t(x,s)|^2 ds\Big|_0^1 - \frac{1}{2}\int_0^t \int_0^1 \gamma_{k,x}(x)|z_t(x,s)|^2 \,dx\,ds \\ &= \frac{1}{2}\int_0^t \gamma_k(1)|z_t(1,s)|^2 ds - \frac{1}{2}\int_0^t \int_0^1 \gamma_{k,x}(x)|z_t(x,s)|^2 \,dx\,ds. \end{aligned}$$

Since $\gamma_{k,x}(x) = -2k$, it follows that

$$J_3 = \frac{1}{2}\int_0^t \gamma_k(1)|z_t(1,s)|^2 ds + k\int_0^t \int_0^1 |z_t(x,s)|^2 \,dx\,ds. \tag{2.7}$$

By (2.5)-(2.7) and the definition of $E(t)$, we see that

$$\begin{aligned} E(t) &= E_0 + \frac{1}{2}\int_0^t \gamma_k(1)|z_t(1,s)|^2 ds - \frac{1}{2}\int_0^t \frac{k}{(1+ks)}\int_0^1 \beta_k(x,s)|z_x(x,s)|^2 \,dx\,ds \\ &\quad - \frac{k}{2}\int_0^t \int_0^1 |z_t(x,s)|^2 \,dx\,ds \end{aligned}$$

$$\begin{aligned}&= E_0 - \int_0^t k|z_t(1,s)|^2 ds - \frac{1}{2}\int_0^t \frac{k}{(1+ks)}\int_0^1 \beta_k(x,s)|z_x(x,s)|^2\,dx\,ds\\&\quad - \frac{1}{2}\int_0^t \frac{k}{(1+ks)}\int_0^1 \alpha_k(x,s)|z_t(x,s)|^2\,dx\,ds\\&= E_0 - \int_0^t k|z_t(1,s)|^2 ds - \int_0^t \frac{k}{(1+ks)}E(s)ds,\end{aligned}$$

which implies that

$$E_t(t) = -\frac{k}{1+kt}E(t) - k|z_t(1,t)|^2, \quad 0 \le t \le T.$$

It follows that

$$[(1+kt)E(t)]_t = -k(1+kt)|z_t(1,t)|^2, \quad 0 \le t \le T,$$

which completes the proof of Lemma 2.3. □

Remark 2.4. By (2.4), it is easy to check that $E(t) < \frac{1}{\alpha_k(t)}E_0 < E_0$.

By the multiplier method, we have the following estimate for any solution of (2.1).

Lemma 2.5. *Let $q \in C^1([0,1])$. Then any solution z of (2.1) satisfies*

$$\begin{aligned}&[\frac{1}{2}\int_0^T \beta_k(x,t)q(x)|z_x(x,t)|^2 dt]\big|_0^1 + \frac{1}{2}\int_0^T \alpha_k(t)q(1)|z_t(1,t)|^2 dt\\&= \frac{1}{2}\int_0^T\int_0^1 q_x(x)[\alpha_k(t)|z_t(x,t)|^2 + \beta_k(x,t)|z_x(x,t)|^2]\,dx\,dt\\&\quad - \int_0^T\int_0^1 \alpha_{k,t}(t)q(x)z_t(x,t)z_x(x,t)\,dx\,dt\\&\quad - \frac{1}{2}\int_0^T\int_0^1 \beta_{k,x}(x,t)q(x)|z_x(x,t)|^2\,dx\,dt\\&\quad + \Big\{\int_0^1 [\alpha_k(t)q(x)z_t(x,t)z_x(x,t) + \frac{1}{2}\gamma_k(x)q(x)|z_x(x,t)|^2]dx\Big\}\Big|_0^T.\end{aligned} \tag{2.8}$$

Proof. Multiplying the first equation of (2.1) by qz_x and integrating on Q, we obtain

$$\begin{aligned}0 &= \int_0^T\int_0^1 \alpha_k(t)z_{tt}(x,t)q(x)z_x(x,t)\,dx\,dt\\&\quad - \int_0^T\int_0^1 [\beta_k(x,t)z_x(x,t)]_x q(x)z_x(x,t)\,dx\,dt\\&\quad + \int_0^T\int_0^1 \gamma_k(x)z_{tx}(x,t)q(x)z_x(x,t)\,dx\,dt\\&\triangleq L_1 + L_2 + L_3.\end{aligned}$$

Now, we calculate L_1, L_2 and L_3. First, it is easy to check that

$$\begin{aligned} L_1 &= \int_0^1 \alpha_k(t)q(x)z_t(x,t)z_x(x,t)dx\Big|_0^T - \int_0^T\int_0^1 \alpha_{k,t}(t)q(x)z_t(x,t)z_x(x,t)\,dx\,dt \\ &\quad - \frac{1}{2}\int_0^T \alpha_k(t)q(x)|z_t(x,t)|^2dt\Big|_0^1 + \frac{1}{2}\int_0^T\int_0^1 \alpha_k(t)q_x(x)|z_t(x,t)|^2\,dx\,dt \\ &= \int_0^1 \alpha_k(t)q(x)z_t(x,t)z_x(x,t)dx\Big|_0^T - \int_0^T\int_0^1 \alpha_{k,t}(t)q(x)z_t(x,t)z_x(x,t)\,dx\,dt \\ &\quad - \frac{1}{2}\int_0^T \alpha_k(t)q(1)|z_t(1,t)|^2dt + \frac{1}{2}\int_0^T\int_0^1 \alpha_k(t)q_x(x)|z_t(x,t)|^2\,dx\,dt. \end{aligned} \tag{2.9}$$

Further,

$$\begin{aligned} L_2 &= -\int_0^T\int_0^1 [\beta_k(x,t)z_x(x,t)]_x q(x)z_x(x,t)\,dx\,dt \\ &= -\int_0^T \beta_k(x,t)q(x)|z_x(x,t)|^2dt\Big|_0^1 \\ &\quad + \int_0^T\int_0^1 [\beta_k(x,t)q_x(x)|z_x(x,t)|^2 + \beta_k(x,t)z_x(x,t)q(x)z_{xx}(x,t)]\,dx\,dt \\ &= -\int_0^T \beta_k(x,t)q(x)|z_x(x,t)|^2dt\Big|_0^1 + \int_0^T\int_0^1 \beta_k(x,t)q_x(x)|z_x(x,t)|^2\,dx\,dt \\ &\quad + \frac{1}{2}\int_0^T \beta_k(x,t)q(x)|z_x(x,t)|^2dt\Big|_0^1 - \frac{1}{2}\int_0^T\int_0^1 [\beta_k(x,t)q(x)]_x|z_x(x,t)|^2\,dx\,dt. \end{aligned}$$

It follows that

$$\begin{aligned} L_2 &= -\frac{1}{2}\int_0^T \beta_k(x,t)q(x)|z_x(x,t)|^2dt\Big|_0^1 \\ &\quad + \frac{1}{2}\int_0^T\int_0^1 [\beta_k(x,t)q_x(x)|z_x(x,t)|^2 - \beta_{k,x}(x,t)q(x)|z_x(x,t)|^2]\,dx\,dt. \end{aligned} \tag{2.10}$$

Further,

$$L_3 = \frac{1}{2}\int_0^1 \gamma_k(x)q(x)|z_x(x,t)|^2dx\Big|_0^T. \tag{2.11}$$

By (2.9)-(2.11), we get the desired result in Lemma 2.5. □

Next, we prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Choose $q(x) = x$. Notice that $\alpha_{k,t}(t) = k$, $\beta_{k,x}(x,t) = \frac{-2k^2x}{1+kt}$ and $\gamma_k(x) = -2kx$. By (2.8), it follows that

$$\begin{aligned} &\Big(\frac{1}{2} + \frac{2k^2}{1-k^2}\Big)\int_0^T \alpha_k(t)|z_t(1,t)|^2dt \\ &= \int_0^T E(t)dt - \int_0^T\int_0^1 kxz_t(x,t)z_x(x,t)\,dx\,dt + \int_0^T\int_0^1 \frac{k^2x^2}{1+kt}|z_x(x,t)|^2\,dx\,dt \\ &\quad + \Big\{\int_0^1 \big[\alpha_k(t)xz_t(x,t)z_x(x,t) - kx^2|z_x(x,t)|^2\big]dx\Big\}\Big|_0^T. \end{aligned} \tag{2.12}$$

Now, we estimate the terms in the right-hand side of (2.12). Using the Young inequality, we obtain

$$
\begin{aligned}
&\int_0^T E(t)dt + \int_0^T \int_0^1 \frac{k^2x^2}{1+kt} |z_x(x,t)|^2 \,dx\,dt - \int_0^T \int_0^1 kxz_t(x,t)z_x(x,t)\,dx\,dt \\
&\leq \int_0^T E(t)dt + \int_0^T \int_0^1 \frac{k^2x^2}{1+kt} |z_x(x,t)|^2 \,dx\,dt \\
&\quad + \frac{1}{2}\int_0^T \int_0^1 \frac{k^2x^2}{1+kt} |z_x(x,t)|^2 \,dx\,dt + \frac{1}{2}\int_0^T \int_0^1 \alpha_k(t)|z_t(x,t)|^2\,dx\,dt \\
&= \int_0^T E(t)dt + \frac{3}{2}\int_0^T \int_0^1 \frac{k^2x^2}{1-k^2x^2} \beta_k(x,t)|z_x(x,t)|^2 \,dx\,dt \\
&\quad + \frac{1}{2}\int_0^T \int_0^1 \alpha_k(t)|z_t(x,t)|^2\,dx\,dt \\
&\leq \int_0^T E(t)dt + \frac{3}{2}\frac{k^2}{1-k^2}\int_0^T \int_0^1 \beta_k(x,t)|z_x(x,t)|^2 \,dx\,dt \\
&\quad + \frac{1}{2}\int_0^T \int_0^1 \alpha_k(t)|z_t(x,t)|^2\,dx\,dt \\
&\leq \int_0^T E(t)dt + \Big(\frac{3k^2}{1-k^2}+1\Big)\int_0^T E(t)dt \\
&= \frac{2+k^2}{1-k^2}\int_0^T E(t)dt.
\end{aligned}
\tag{2.13}
$$

Further, for any $t \in [0,T]$ and $0 < \varepsilon < 1$, by the Young inequality, it holds that

$$
\begin{aligned}
&\Big|\int_0^1 [\alpha_k(t)xz_t(x,t)z_x(x,t) - kx^2|z_x(x,t)|^2]dx\Big| \\
&\leq \sqrt{1+kt}\Big[\frac{1}{2\varepsilon}\int_0^1 \alpha_k(t)|z_t(x,t)|^2dx + \frac{\varepsilon}{2}\int_0^1 x^2|z_x(x,t)|^2dx\Big] \\
&\quad + k\int_0^1 x^2|z_x(x,t)|^2dx \\
&\leq \frac{\sqrt{1+kt}}{2\varepsilon}\int_0^1 \alpha_k(t)|z_t(x,t)|^2dx + \Big(\frac{\sqrt{1+kt}}{2}\varepsilon + k\Big)\int_0^1 x^2|z_x(x,t)|^2dx \\
&\leq \frac{\sqrt{1+kt}}{\varepsilon}\frac{1}{2}\int_0^1 \alpha_k(t)|z_t(x,t)|^2dx \\
&\quad + \frac{2\big(\frac{\sqrt{1+kt}}{2}\varepsilon + k\big)(1+kt)}{1-k^2}\frac{1}{2}\int_0^1 \beta_k(x,t)|z_x(x,t)|^2dx.
\end{aligned}
$$

Choose

$$
\varepsilon = \frac{1-k}{\sqrt{1+kt}}.
$$

Then it is easy to check that $\varepsilon \in (0,1)$ and

$$
\frac{\sqrt{1+kt}}{\varepsilon} = \frac{2\left(\frac{\sqrt{1+kt}}{2}\varepsilon + k\right)(1+kt)}{1-k^2} = \frac{1+kt}{1-k}.
$$

By (2.4), it follows that

$$\begin{aligned}
&\Big|\int_0^1 [\alpha_k(t)xz_t(x,t)z_x(x,t) - kx^2|z_x(x,t)|^2]dx\Big| \\
&= \frac{1+kt}{1-k}E(t) \\
&\le \frac{1+kt}{1-k}\frac{1}{1+kt}E_0 = \frac{1}{1-k}E_0.
\end{aligned}$$

This implies that

$$\Big|\Big\{\int_0^1 [\alpha_k(t)xz_t(x,t)z_x(x,t) - kx^2|z_x(x,t)|^2]dx\Big\}\Big|_0^T\Big| \le \frac{2}{1-k}E_0. \tag{2.14}$$

By (2.12), (2.13), (2.14) and Remark 2.4, we find that

$$\begin{aligned}
&\big(\frac{1}{2} + \frac{2k^2}{1-k^2}\big)\int_0^T \alpha_k(t)|z_t(1,t)|^2 dt \\
&\le \frac{2+k^2}{1-k^2}\int_0^T E_0 dt + \frac{2}{1-k}E_0 = \big(\frac{2+k^2}{1-k^2}T + \frac{2}{1-k}\big)E_0.
\end{aligned} \tag{2.15}$$

By (1.7), it follows that

$$\beta_k(x,0) = 1 - k^2x^2.$$

It is obvious that

$$E_0 = \frac{1}{2}\int_0^1 [|z^1|^2 + \beta_k(x,0)|z_x^0|^2]dx \le \frac{1}{2}(\|z^0\|^2 + |z^1|^2). \tag{2.16}$$

By (2.15) and (2.16), noting that $1 \le \alpha_k(t) \le (1+kT)$ for $0 \le t \le T$, one can find a positive constant $C_1 = \frac{1}{2}(\frac{1}{2} + \frac{2k^2}{1-k^2})^{-1}(\frac{2+k^2}{1-k^2}T + \frac{2}{1-k})$ such that

$$\int_0^T |z_t(1,t)|^2 dt \le C_1(\|z^0\|^2 + |z^1|^2),$$

which completes the proof. $\square$

Proof of Theorem 2.2. By the Young inequality, for any $\varepsilon \in (0, \frac{1}{2})$, it is easy to check that

$$\begin{aligned}
&\int_0^T E(t)dt + \int_0^T\int_0^1 \frac{k^2x^2}{1+kt}|z_x(x,t)|^2\,dx\,dt - \int_0^T\int_0^1 kxz_t(x,t)z_x(x,t)\,dx\,dt \\
&\ge \int_0^T\int_0^1 \Big\{\frac{1-\varepsilon}{2}\alpha_k(t)|z_t(x,t)|^2 + \big[\frac{1}{2}\beta_k(x,t) + \big(1 - \frac{1}{2\varepsilon}\big)\frac{k^2x^2}{1+kt}\big]|z_x(x,t)|^2\Big\}\,dx\,dt \\
&= \int_0^T\int_0^1 \Big\{(1-\varepsilon)\frac{\alpha_k(t)}{2}|z_t(x,t)|^2 + [1 + \big(2 - \frac{1}{\varepsilon}\big)\frac{k^2x^2}{1-k^2x^2}]\frac{\beta_k(x,t)}{2}|z_x(x,t)|^2\Big\}\,dx\,dt \\
&\ge \int_0^T\int_0^1 \Big\{(1-\varepsilon)\frac{\alpha_k(t)}{2}|z_t(x,t)|^2 + [1 + \big(2 - \frac{1}{\varepsilon}\big)\frac{k^2}{1-k^2}]\frac{\beta_k(x,t)}{2}|z_x(x,t)|^2\Big\}\,dx\,dt.
\end{aligned}$$

Take $\varepsilon = \frac{k}{1+k} \in (0, \frac{1}{2})$, then we find that

$$1 - \varepsilon = 1 + \big(2 - \frac{1}{\varepsilon}\big)\frac{k^2}{1-k^2}.$$

It follows that

$$\begin{aligned} &\int_0^T E(t)dt + \int_0^T \int_0^1 \frac{k^2x^2}{1+kt}|z_x(x,t)|^2\,dx\,dt - \int_0^T \int_0^1 kxz_t(x,t)z_x(x,t)\,dx\,dt \\ &\ge \big(1-\frac{k}{1+k}\big)\int_0^T E(t)dt = \frac{1}{1+k}\int_0^T E(t)dt. \end{aligned} \tag{2.17}$$

Therefore, substituting (2.4), (2.14) and (2.17) into (2.12) indicates that

$$\begin{aligned} &(\frac{1}{2}+\frac{2k^2}{1-k^2})\int_0^T \alpha_k(t)|z_t(1,t)|^2dt \\ &\ge \frac{1}{1+k}\int_0^T E(t)dt - \frac{2}{1-k}E_0 \\ &= \frac{1}{1+k}\int_0^T [\frac{1}{1+kt}E_0 - \frac{k}{\alpha_k(t)}\int_0^t \alpha_k(s)|z_t(1,s)|^2ds]dt - \frac{2}{1-k}E_0, \end{aligned}$$

which implies that

$$\begin{aligned} &(\frac{1}{2}+\frac{2k^2}{1-k^2})\int_0^T \alpha_k(t)|z_t(1,t)|^2dt + \frac{1}{k+1}\int_0^T [\frac{k}{\alpha_k(t)}\int_0^t \alpha_k(s)|z_t(1,s)|^2ds]dt \\ &\ge \frac{1}{1+k}\int_0^T \frac{1}{1+kt}E_0dt - \frac{2}{1-k}E_0. \end{aligned}$$

It follows that

$$\begin{aligned} &(\frac{1}{2}+\frac{2k^2}{1-k^2}+\frac{kT}{1+k})\int_0^T \alpha_k(t)|z_t(1,t)|^2dt \\ &\ge \frac{1}{1+k}\int_0^T \frac{1}{1+kt}E_0dt - \frac{2}{1-k}E_0 = [\frac{1}{k(1+k)}\ln(1+kT) - \frac{2}{1-k}]E_0. \end{aligned} \tag{2.18}$$

From (2.18) and (2.16), it holds that

$$\begin{aligned} &(\frac{1}{2}+\frac{2k^2}{1-k^2}+\frac{kT}{1+k})(1+kT)\int_0^T |z_t(1,t)|^2dt \\ &\ge \frac{1-k^2}{2}[\frac{1}{k(1+k)}\ln(1+kT) - \frac{2}{1-k}](\|z^0\|^2+|z^1|^2). \end{aligned} \tag{2.19}$$

Notice that if $T > T_k^*$, then $\frac{1}{k(1+k)}\ln(1+kT) - \frac{2}{1-k} > 0$. This, together with (2.19) indicates the desired estimate in Theorem 2.2. □

3. Proof of Theorem 1.3

In this section we use the Hilbert Uniqueness Method. For Theorem 1.3, it suffices to show that for any given initial value $(w^0, w^1) \in L^2 \times V'$ and target $(w_d^0, w_d^1) \in L^2 \times V'$, one can find a control $\overline{v} = \overline{v}(t) \in [H^1(0,T)]'$ such that the corresponding solution w of (1.6) satisfies

$$w(T) = w_d^0 \quad \text{and} \quad w_t(T) = w_d^1. \tag{3.1}$$

We divide the whole proof into three parts.

Step 1. First, we define a linear operator Λ from $V \times L^2$ to $V' \times L^2$. Consider the wave equation

$$\begin{gathered} \xi_{tt} - [\frac{\beta_k(x,t)}{\alpha_k(t)}\xi_x]_x + [\frac{\gamma_k(x)}{\alpha_k(t)}]\xi_{tx} = 0 \quad \text{in } Q, \\ \xi(0,t) = 0, \quad \xi_x(1,t) = 0 \quad \text{on } (0,T), \\ \xi(x,T) = w_d^0(x), \quad \xi_t(x,T) = w_d^1(x) \quad \text{in } (0,1). \end{gathered} \tag{3.2}$$

It is easy to check that (3.2) has a unique solution

$$\xi \in C([0,T]; L^2) \cap C^1([0,T]; V')$$

and set

$$(\xi^0, \xi^1) \triangleq (\xi(x,0), \xi_t(x,0)) \in L^2 \times V'.$$

Thus

$$(w^0 - \xi^0, w^1 - \xi^1) \in L^2 \times V'. \tag{3.3}$$

On the other hand, for any $(z^0, z^1) \in V \times L^2$, we denote by z the corresponding solution of (2.1). Consider the wave equation

$$\begin{gathered} \eta_{tt} - [\frac{\beta_k(x,t)}{\alpha_k(t)}\eta_x]_x + [\frac{\gamma_k(x)}{\alpha_k(t)}]\eta_{tx} = 0 \quad \text{in } Q, \\ \eta(0,t) = 0, \quad \eta_x(1,t) = \frac{1}{\beta_k(1,t)} G_{z_t(1,t)} \quad \text{on } (0,T), \\ \eta(x,T) = \eta_t(x,T) = 0 \quad \text{in } (0,1). \end{gathered} \tag{3.4}$$

Notice that $G_{z_t(1,t)} \in (H^1(0,T))'$ is defined as

$$\langle G_{z_t(1,t)}, \phi \rangle_{(H^1(0,T))', H^1(0,T)} = -\int_0^T z_t(1,t)\phi_t(t)dt \quad \text{for any } \phi \in H^1(0,T). \tag{3.5}$$

Now, we define the operator

$$\begin{gathered} \Lambda : V \times L^2 \to V' \times L^2, \\ (z^0, z^1) \to \big(\eta_t(x,0) + \gamma_k(x)\eta_x(x,0) - k\eta(x,0), -\eta(x,0)\big). \end{gathered}$$

Therefore,

$$\begin{aligned} &\langle \Lambda(z^0, z^1), (z^0, z^1) \rangle \\ &= \int_0^1 [\eta_t(x,0)z^0 - k\eta(x,0)z^0 + \gamma_k(x)\eta_x(x,0)z^0 - \eta(x,0)z^1]dx. \end{aligned} \tag{3.6}$$

For simplicity, we set $F = V \times L^2, F' = V' \times L^2$.

Step 2. We prove that Λ is an isomorphism. To this aim, multiplying both sides of the first equation of (3.4) by $\alpha_k(t)z$ $(0 \le t \le T)$ and integrating on Q, we obtain that

$$\begin{aligned} &-\int_0^T \beta_k(1,t)\eta_x(1,t)z(1,t)dt \\ &= \int_0^1 [\eta'(x,0)z^0 - k\eta(x,0)z^0 - \eta(x,0)z^1 + \gamma_k(x)\eta_x(x,0)z^0]dx. \end{aligned} \tag{3.7}$$

From (3.4)-(3.6) and (3.7), we conclude that

$$\int_0^T |z_t(1,t)|^2 dt = \langle \Lambda(z^0, z^1), (z^0, z^1) \rangle. \tag{3.8}$$

By Theorem 2.1, it holds that Λ is a linear bounded operator.

It remains to show that Λ is onto. To this end, define the bilinear functional on $F \times F$:

$$A((\hat{z}^0, \hat{z}^1), (z^0, z^1)) = \langle \Lambda(\hat{z}^0, \hat{z}^1), (z^0, z^1)\rangle,$$

where $(\hat{z}^0, \hat{z}^1)$, $(z^0, z^1) \in F \times F$. It is clear that A is bounded. From (3.8) and Theorem 2.2, it follows that A is coercive. Hence, applying Lax-Milgram Theorem, we derive that Λ is onto. This completes the proof of Step 2.

Step 3. We prove that the exact controllability of (1.6) is equivalent that Λ is an isomorphism. Indeed, for any given $(w^0, w^1), (w^0_d, w^1_d) \in L^2 \times V'$, we choose

$$\overline{v}(\cdot) = \frac{1}{\beta_k(1, \cdot)} G_{z_t(1,\cdot)} \in (H^1(0,T))',$$

where z is the solution of (2.1) associated to $(z^0, z^1) = \Lambda^{-1}((w^1 - \xi^1) + \gamma_k(x)(w^0_x - \xi^0_x) - k(w^0 - \xi^0), -(w^0 - \xi^0))$ and w is the solution of (1.6). From the definition of Λ, we conclude that $\Lambda(z^0, z^1) = (\eta'(x,0) + \gamma_k(x)\eta_x(x,0) - k\eta(x,0), -\eta(x,0))$, where η is the solution of (3.4). Then, η satisfies $(\eta(x,0), \eta'(x,0)) = (w^0 - \xi^0, w^1 - \xi^1)$. This implies that $w = \xi + \eta$ satisfies both (1.6) and (3.1). This completes the proof of Theorem 1.3.

Acknowledgments. This work is partially supported by the NSF of China under grants 11171060 and 11371084.

References

[1] C. Bardos, G. Chen; *Control and stabilization for the wave equation, part III: domain with moving boundary*, SIAM J. Control Optim., 19 (1981), 123–138.

[2] F. D. Araruna, G. O. Antunes, L. A. Medeiros, *Exact controllability for the semilinear string equation in the non cylindrical domains*, Control Cybernet., 33 (2004), 237–257.

[3] M. M. Cavalcanti; *Exact controllability of the wave equation with mixed boundary condition and time coefficients*, Arch. Math. (BRNO), 35 (1999), 29–57.

[4] J. L. Lions; *Exact controllability, stabilizability and perturbation for distributed systems*, SIAM Rev., 30 (1988), 1–68.

[5] M. Milla Miranda; *Exact controllability for the wave equation in domains with variable boundary*, Rev. Mat. Univ. Complut. Madrid, 9 (1996), 435–457.

[6] Lizhi Cui, Xu Liu, Hang Gao; *Exact controllability for a one-dimensional wave equation in non-cylindrical domains*, Journal of Mathematical Analysis and Applications, 402 (2013), 612–625.

[7] Lizhi Cui, Libo Song; *Controllability for a Wave Equation with Moving Boundary*, Journal of Applied Mathematics, doi:10.1155/2014/827698 (2014).

[8] J. P. Quinn, D. L. Russell; *Asymptotic stability and energy decay rates for solutions of hyperbolic equations with boundary damping*, Proc. Soc. Edinburgh Sect. A 77 (1977), 97–127.

Lizhi Cui

College of Applied Mathematics, Jilin University of Finance and Economics, Changchun 130117, China.

School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China

E-mail address: cuilz924@126.com

Hang Gao

School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China

E-mail address: hangg@nenu.edu.cn