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SOLUTIONS TO KIRCHHOFF EQUATIONS WITH COMBINED NONLINEARITIES

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Abstract. We prove the existence of multiple positive solutions for the Kirchhoff equation

$$-\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = h(x)u^q + f(x,u), \quad x\in\Omega,$$
$$u=0, \quad x\in\partial\Omega,$$

Here Ω is an open bounded domain in R^N $(N=1,2,3)$, $h(x)\in L^\infty(\Omega)$, $f(x,s)$ is a continuous function which is asymptotically linear at zero and is asymptotically 3-linear at infinity. Our main tools are the Ekeland's variational principle and the mountain pass lemma.

1. Introduction and main results

In this article, we study the existence of positive solutions for the Kirchhoff equation

$$\begin{gathered}-\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = h(x)u^q + f(x,u), \quad x\in\Omega,\\ u=0, \quad x\in\partial\Omega,\end{gathered} \tag{1.1}$$

where Ω is a bounded smooth domain in R^N $(N=1,2,3)$, $a>0$, $b>0$, $0<q<1$.

To state the assumptions, we recall some results about the following two eigenvalue problems:

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u=0 \text{ on } \Omega, \tag{1.2}$$

and

$$-\Big(\int_\Omega |\nabla u|^2dx\Big)\Delta u = \mu u^3 \text{ in } \Omega, \quad u=0 \text{ on } \Omega. \tag{1.3}$$

Let λ_1 be the principal eigenvalue of (1.2) and let $\phi_1>0$ be its associated eigenfunction. It is known that λ_1 can be characterized by

$$\lambda_1 = \inf\Big\{\int_\Omega |\nabla u|^2dx:\ u\in H_0^1(\Omega),\ \int_\Omega |u|^2dx = 1\Big\},$$

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where $H_0^1(\Omega)$ is the usual Sobolev space defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| = \big(\int_\Omega |\nabla u|^2 dx\big)^{1/2}$. Moreover, define

$$\mu_1 = \inf\Big\{\|u\|^4 : \ u \in H_0^1(\Omega), \ \int_\Omega |u|^4 dx = 1\Big\}.$$

As shown in [13], there exists $\mu_1 > 0$ which is the principle eigenvalue of (1.3) and there is a corresponding eigenfunction of $\varphi_1 > 0$ in Ω.

In this article, we assume that h, f satisfy the following conditions:

(H1) $h \in L^\infty(\Omega)$ and $h(x) \not\equiv 0$;

(F1) $f \in C(\Omega \times \mathbb{R})$, $f(x,0) = 0$ for all $x \in \Omega$, $f(x,s) \geq 0$ for all $x \in \Omega$ and $s \geq 0$;

(F2)
$$\lim_{s\to 0^+} \frac{f(x,s)}{a\lambda_1 s + b\mu_1 s^3} = \alpha \in [0,1), \quad \lim_{s\to +\infty} \frac{f(x,s)}{a\lambda_1 s + b\mu_1 s^3} = \beta \in (1,+\infty)$$
uniformly for a.e. $x \in \Omega$.

It is obvious that the values of $f(x,s)$ for $s < 0$ are irrelevant for us to seek for positive solutions of (1.1), and we may define

$$f(x,s) = 0 \quad \text{for } x \in \Omega, \ s \leq 0.$$

The problem

$$\begin{gathered} -(a + b\int_\Omega |\nabla u|^2 dx)\Delta u = g(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.4}$$

is related to the stationary analogue of the Kirchhoff equation which was proposed by Kirchhoff in 1883 [9] as an generalization of the well-known d'Alembert's equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = g(x,u)$$

for free vibrations of elastic strings. Kirchhoffs model takes into account the changes in length of the string produced by transverse vibrations. Here, L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. In [1], it was pointed out that the problem (1.4) models several physical systems, where u describes a process which depends on the average of itself. Nonlocal effect also finds its applications in biological systems. After [2] and [14], there are abundant results about Kirchhoff's equations.

Some interesting studies by variational methods can be found in [4, 12, 13, 19, 18, 17, 3, 16, 15] references therein and for Kirchhoff-type problem (1.4), they consider it in a bounded domain Ω. For example, Perera and Zhang [13] obtain nontrivial solutions of (1.4) with asymptotically 4-linear terms by using Yang index. In [19], they revisit problem (1.4) and establish the existence of a positive, a negative and a sign-changing solution by means of invariant sets of descent flow. Similar results can also be found in Mao and Zhang [12] and in Yang and Zhang [18]. Yang and Zhang in [17] obtain the existence of nontrivial solutions for (1.4) by using the local linking theory. Sun and Tang [16] prove the existence of a mountain pass type positive solution for problem (1.4) with the nonlinearity which is asymptotically linear near zero and superlinear at infinity. Sun and Liu [15] obtain a nontrivial solution via Morse theory by computing the relevant critical groups for problem (1.4) with the nonlinearity which is superlinear near zero but asymptotically 4-linear at infinity and asymptotically near zero but 4-linear at infinity. In [11], the authors obtain the existence of positive solutions for (1.1) with $h \equiv 0$ and $f(x,t) = \nu h(x,t)$ by using the topological degree argument and variational method, where h is a continuous function which is asymptotically linear at zero and is asymptotically 3-linear at infinity. Inspired by [11], we shall study the existence of positive solutions for problem (1.1) with $h \not\equiv 0$ and f which is asymptotically linear at zero and asymptotically 3-linear infinity by using the Ekeland's variational principle and Mountain Pass Lemma different from [11]. In [11], when $N = 1, 2, 3$, the authors studied equation (1.1) with $h \equiv 0$ and obtain the existence results of positive solution for equation (1.1) under the conditions: $a, b > 0$, and f satisfies (F1) and (F2) with $\alpha > 1$ and $\beta < 1$; $a \geq 0$, $b > 0$, and f satisfies (F1) and (F2) with $\alpha < 1$ and $\beta > 1$, respectively. But equation (1.1) with $h \not\equiv 0$ has not been studied. We shall obtain the existence of two positive solution for equation (1.1) because of the nonlinearity term $h(x)t^q (0 < q < 1)$. By the way, recently, Cheng, Wu and Liu [5] apply variant mountain pass theorem and Ekeland variational principle to study the existence of multiple nontrivial solutions for a class of Kirchhoff type problems with concave nonlinearity similar to our problem. But in their article, the nonlinear term is superlinear at infinity.

In this article, we denote by $\|\cdot\|_p$ the $L^p(\Omega)$-normal ($1 \leq p \leq \infty$). We say that $u \in H_0^1(\Omega)$ is a positive (nonnegative) weak solution to problem (1.1) if $u > 0$ ($u \geq 0$) a.e. Ω and satisfies

$$\Big(a + b\int_\Omega |\nabla u|^2 dx\Big)\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega h(x) u^q v dx + \int_\Omega f(x,u) v dx$$

for all $v \in H_0^1(\Omega)$. By assumption (F1), we know that to seek a nonnegative weak solution of (1.1) is equivalent to finding a nonzero critical point of the following functional on $H_0^1(\Omega)$:

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{4}\Big(\int_\Omega |\nabla u|^2 dx\Big)^2 - \frac{1}{q+1}\int_\Omega h(x)(u^+)^{q+1} dx - \int_\Omega F(x,u^+) dx,$$

where $u^+ = \max\{0, u\}$, $F(x,s) = \int_0^s f(x,\sigma) d\sigma$. By (F1) and (F2), I is a C^1 functional. By the strong maximum principle, the nonzero critical points of I are positive solutions to problem (1.1) if $h(x) \geq 0$.

Our results are as follows.

Theorem 1.1. *Suppose that $N = 1, 2, 3$, $a > 0$, $b > 0$, $0 < q < 1$, h and f satisfy* (H1), (F1), (F2). *Assume further that exists $v \in H_0^1(\Omega)$ such that*

(H2) $\int_\Omega h(x)(v^+)^{q+1} dx > 0$.

Then there exists a constant $m > 0$ such that if $\|h\|_\infty < m$, problem (1.1) *has a solution $u_1 \in H_0^1(\Omega)$, $u_1 \geq 0$ and $I(u_1) < 0$. Moreover, if $h(x) \geq 0$, then $u_1 > 0$ a. e. in Ω.*

Theorem 1.2. *Suppose that $N = 1, 2, 3$, $a > 0$, $b > 0$, $0 < q < 1$, h and f satisfy* (H1), (F1), (F2). *Assume further $\beta\mu_1$ is not an eigenvalue of* (1.3). *Then there exists a constant $m > 0$ such that if $\|h\|_\infty < m$, problem* (1.1) *has a nonnegative solution $u_2 \in H_0^1(\Omega)$ with $u_2 > 0$ and $I(u_2) > 0$ if $h(x) \geq 0$.*

Remark 1.3. Theorem 1.1 for problem (1.1) with $a, b > 0$ generalizes [10, Theorem 1.1] where (1.1) with $a = 1$ and $b = 0$.

Corollary 1.4. *Suppose that $N = 1, 2, 3$, $a > 0$, $b > 0$, $0 < q < 1$, h and f satisfy* (H1), (F1), (F2). *Assume further that $\beta\mu_1$ is not an eigenvalue of* (1.3) *and $h(x) \geq (\not\equiv) 0$. Then there exists a constant $m > 0$ such that for all $h \in L^\infty(\Omega)$ with $\|h\|_\infty < m$, problem* (1.1) *has at least two positive solutions $u_1, u_2 \in H_0^1(\Omega)$ such that $I(u_1) < 0 < I(u_2)$.*

Remark 1.5. If $h(x) \geq (\not\equiv) 0$, it is easy to see that (H2) is always satisfied. Therefore, Corollary 1.1 is a straightforward conclusion of Theorems 1.1 and 1.2 by applying the strong maximum principle [8].

This paper is organized as follows. In Section 1, we obtain the existence of a local minimum solution by the Ekeland's variational principle. In Section 2, by using the Mountain Pass Lemma, we obtain the existence of a mountain pass solution. In the following discussion, we denote various positive constants as C or C_i, $i = 1, 2, 3, \dots$.

2. Existence of a local minimum

In this section, we prove Theorem 1.1 by Ekeland's variational principle. We need the following Lemmas.

Lemma 2.1. *Suppose that $N = 1, 2, 3$, $a > 0$, $b > 0$, $0 < q < 1$, h and f satisfy* (H1), (F1), (F2). *Then there exists a constant $m > 0$ such that if $\|h\|_\infty < m$, we have*

- (a) *There exist $\rho, \gamma > 0$ such that $I(u)|_{\|u\|=\rho} \geq \gamma > 0$.*
- (b) *There exists an $e \in \mathbb{R} \setminus B_\rho(0)$ such that $I(e) < 0$.*

Proof. (a) By (F2), $\beta \in (1, +\infty)$ and noticing that $f(x, s)/s^{p-1} \to 0$ as $s \to +\infty$ uniformly in $x \in \Omega$ for any fixed $p \in (4, 6)$ if $N = 3$; $p \in (4, +\infty)$ if $N = 1, 2$. Given $\varepsilon \in (0, 1)$, there exist $\delta, M_\varepsilon > 0$ satisfying $0 < \delta < +\infty$ such that

$$f(x, s) < \Big(\alpha + \varepsilon\Big)(a\lambda_1 s + b\mu_1 s^3), \quad 0 < s < \delta,$$

and

$$f(x, s) < M_\varepsilon s^{p-1}, \quad \delta < s,$$

where $p \in (4, 6)$ if $N = 3$; $p \in (4, +\infty)$ if $N = 1, 2$. Together with (F1) and $f(x, s) = 0$ for $x \in \Omega$, $s \leq 0$, we obtain

$$f(x, s) < a\lambda_1(\alpha + \varepsilon)|s| + b\mu_1(\alpha + \varepsilon)|s|^3 + M_\varepsilon s^{p-1}, \quad s \in R.$$

This yields

$$F(x, s) \leq \frac{a\lambda_1}{2}(\alpha + \varepsilon)|s|^2 + \frac{b\mu_1}{4}(\alpha + \varepsilon)|s|^4 + A|s|^p, \quad s \in R, \tag{2.1}$$

where $A = M_\varepsilon/p$. Furthermore, by (F2), for the above ε, we have

$$f(x, s) > (\beta - \varepsilon)(a\lambda_1 s + b\mu_1 s^3), \quad s > \delta_\infty.$$

Thus, we obtain

$$F(x, s) > (\beta - \varepsilon)\Big(\frac{a\lambda_1}{2}s^2 + \frac{b\mu_1}{4}s^4\Big), \quad s > \delta_\infty.$$

Together with (F1) and $f(x, s) = 0$ for $x \in \Omega$, $s \leq 0$, there exists a constant $B > 0$ such that

$$F(x, s) \geq \frac{a}{2}(\beta - \varepsilon)\lambda_1|s|^2 + \frac{b}{4}(\beta - \varepsilon)\mu_1|s|^4 - B, \quad s \in R. \tag{2.2}$$

Since $\alpha < 1$, we can choose $\varepsilon > 0$ such that $\varepsilon < 1-\alpha$. By (H1), (2.1), $\lambda_1\|u\|_2^2 \leq \|u\|^2$, $\mu_1\|u\|_4^4 \leq \|u\|^2$, the Sobolev's embedding theorem: $\|u\|_{q+1}^{q+1} \leq K\|u\|^{q+1}$, $\|u\|_{p+1}^{p+1} \leq M\|u\|^{p+1}$ and the Young inequality, we have

$$\begin{aligned}
&I(u)\\
&= \frac{a}{2}\int_\Omega |\nabla u|^2dx + \frac{b}{4}\Big(\int_\Omega |\nabla u|^2dx\Big)^2 - \frac{1}{q+1}\int_\Omega h(x)(u^+)^{q+1}dx - \int_\Omega F(x,u^+)dx\\
&\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\|h\|_\infty}{q+1}\|u^+\|_{q+1}^{q+1} - \frac{a}{2}(\alpha+\varepsilon)\lambda_1\|u^+\|_2^2\\
&\quad - \frac{b}{4}(\alpha+\varepsilon)\mu_1\|u^+\|_4^4 - A\|u^+\|_p^p\\
&\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\|h\|_\infty}{q+1}\|u\|_{q+1}^{q+1} - \frac{a}{2}(\alpha+\varepsilon)\|u\|^2 - \frac{b}{4}(\alpha+\varepsilon)\|u\|^4 - A\|u\|_p^p\\
&\geq \frac{a[1-(\alpha+\varepsilon)]}{2}\|u\|^2 + \frac{b[1-(\alpha+\varepsilon)]}{4}\|u\|^4 - \frac{\|h\|_\infty K}{q+1}\|u\|^{q+1} - AM\|u\|^p\\
&\geq \|u\|^2\big(C_1 - C_2\|h\|_\infty\|u\|^{q-1} - C_3\|u\|^{p-2}\big),
\end{aligned} \tag{2.3}$$

where $C_1 = \frac{a[1-(\alpha+\varepsilon)]}{2}$, $C_2 = \frac{K}{q+1}$ and $C_3 = AM$. Let

$$g(t) = C_2\|h\|_\infty t^{q-1} + C_3t^{p-2} \quad \text{for } t \geq 0.$$

Clearly,

$$g'(t) = C_2(q-1)\|h\|_\infty t^{q-2} + (p-2)C_3t^{p-3}.$$

From $g'(t_0) = 0$, we have

$$t_0 = (C_4\|h\|_\infty)^{\frac{1}{p-q-1}}, \quad 0 < q < 1 < 4 < p,$$

where $C_4 = \frac{C_2(1-q)}{(p-2)C_3}$. Then

$$g(t_0) = C_2\|h\|_\infty(C_4\|h\|_\infty)^{\frac{q-1}{p-q-1}} + C_3(C_4\|h\|_\infty)^{\frac{p-2}{p-q-1}} = C_5\|h\|_\infty^{\frac{p-2}{p-q-1}},$$

where $C_5 = C_2C_4^{\frac{q-1}{p-q-1}} + C_3C_4^{\frac{p-2}{p-q-1}}$ and $\frac{p-2}{p-q-1} > 0$ because $0 < q < 1 < 4 < p$. Thus, for any $p > 4$, there exists $m > 0$ such that $g(t_0) < C_1$ if $\|h\|_\infty < m$. Then, if $\|h\|_\infty < m$ and taking $\rho = t_0$, from (2.3), (a) is proved.

(b) For $t > 0$ large enough, by (2.2) and $0 < q < 1$, taking $\varepsilon > 0$ such that $\varepsilon < \min\{\beta-1, 1-\alpha\}$, we have

$$\begin{aligned}
I(t\varphi_1) &= \frac{at^2}{2}\int_\Omega |\nabla\varphi_1|^2dx + \frac{bt^4}{4}\left(\int_\Omega |\nabla\varphi_1|^2dx\right)^2 - \frac{t^{q+1}}{q+1}\int_\Omega h(x)\varphi_1^{q+1}dx\\
&\quad - \int_\Omega F(x,t\varphi_1)dx\\
&\leq \frac{at^2}{2}\|\varphi_1\|^2 + \frac{bt^4}{4}\|\varphi_1\|^4 - \frac{t^{q+1}}{q+1}\int_\Omega h(x)\varphi_1^{q+1}dx - \frac{at^2}{2}(\beta-\varepsilon)\lambda_1\|\varphi_1\|_2^2\\
&\quad - \frac{bt^4}{4}(\beta-\varepsilon)\mu_1\|\varphi_1\|_4^4 + B|\Omega|\\
&\leq \frac{at^2}{2}\|\varphi_1\|^2 + \frac{bt^4}{4}\|\varphi_1\|^4 - \frac{t^{q+1}}{q+1}\int_\Omega h(x)\varphi_1^{q+1}dx - \frac{bt^4}{4}(\beta-\varepsilon)\|\varphi_1\|^4 + B|\Omega|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{at^2}{2}\|\varphi_1\|^2 - \frac{bt^4}{4}(\beta-\varepsilon-1)\|\varphi_1\|^4 - \frac{t^{q+1}}{q+1}\int_\Omega h(x)\varphi_1^{q+1}dx + B|\Omega| \\
&\to -\infty
\end{aligned}
$$

as $t\to\infty$. So we can choose $t^0>0$ large enough and $e=t\varphi_1$ so that $I(e)<0$ and $\|e\|>\rho$. □

Proof of Theorem 1.1. Set ρ as in Lemma 2.1(a), define

$$
\overline{B}_\rho = \{u\in H_0^1(\Omega) : \|u\|\le\rho\}, \quad \partial B_\rho = \{u\in H_0^1(\Omega) :\ \|u\| = \rho\}
$$

and $\overline{B}_\rho$ is a complete metric space with the distance

$$
\operatorname{dist}(u,v) = \|u-v\| \text{ for } u,\ v\in\overline{B}_\rho.
$$

By Lemma 2.1,

$$
I(u)|_{\partial B_\rho} \ge \gamma > 0. \tag{2.4}
$$

Clearly, $I\in C^1(\overline{B}_\rho,\mathbb{R})$, hence I is lower semicontinuous and bounded from below on $\overline{B}_\rho$. Let

$$
c_1 = \inf\{I(u) :\ u\in\overline{B}_\rho\}. \tag{2.5}
$$

We claim that

$$
c_1 < 0. \tag{2.6}
$$

Indeed, let $v\in H_0^1(\Omega)$ be given by (H2), that is, $\int_\Omega h(x)(v^+)^{q+1}dx>0$, then for $t>0$ small enough such that for any $\varepsilon>0$, we have $|tv|<\varepsilon$. Therefore, together (F2) and $\alpha>1$ imply

$$
\begin{aligned}
I(tv) &= \frac{at^2}{2}\int_\Omega|\nabla v|^2dx + \frac{bt^4}{4}\Big(\int_\Omega|\nabla v|^2dx\Big)^2 - \frac{t^{q+1}}{q+1}\int_\Omega h(x)(v^+)^{q+1}dx \\
&\quad - \int_\Omega F(x,tv^+)dx \\
&\le \frac{at^2}{2}\|v\|^2 + \frac{bt^4}{4}\|v\|^4 - \frac{t^{q+1}}{q+1}\int_\Omega h(x)(v^+)^{q+1}dx \\
&\quad - \frac{at^2}{2}(\alpha+\varepsilon)\lambda_1\|v\|_2^2 - \frac{bt^4}{4}(\alpha+\varepsilon)\mu_1\|v\|_4^4 < 0,
\end{aligned}
$$

if $t>0$ small enough, because $0<q<1$. So (2.6) is proved.

By the Ekeland's variational principle [6, Theorem 1.1] in $\overline{B}_\rho$ and (2.5), there is a minimizing sequence $\{u_n\}\subset\overline{B}_\rho$ such that

(i) $c_1 < I(u_n) < c_1 + \frac{1}{n}$,
(ii) $I(w) \ge I(u_n) - \frac{1}{n}\|w-u_n\|$ for all $w\in\overline{B}_\rho$.

So, $I'(u_n)\to 0$ in $H_0^{-1}(\Omega)$ as $n\to\infty$. Moreover, by (i) and (ii), we obtain $I(u_n)\to c_1<0$ as $n\to\infty$.

From the above discussion, we know that $\{u_n\}$ is a bounded (PS) sequence, there exist a subsequence (still denoted by $\{u_n\}$) and $u_1\in H_0^1(\Omega)$ such that

$$
\begin{gathered}
u_n \rightharpoonup u_1 \quad \text{weakly in } H_0^1(\Omega), \\
u_n \to u_1 \quad \text{a.e. in } \Omega, \\
u_n \to u_1 \quad \text{strongly in } L^r(\Omega)
\end{gathered} \tag{2.7}
$$

as $n\to\infty$, where $r\in[1,6]$ if $N=3$ and $r\in(1,+\infty)$ if $N=1,2$. Thus, we have $\lim_{n\to\infty}\langle I'(u_n),v\rangle = \langle I'(u_1),v\rangle = 0$ for all $v\in H_0^1(\Omega)$ and $\lim_{n\to\infty} I(u_n) = c_1 < 0$. Moreover, it follows from $\langle I'(u_1),u_1^-\rangle = (a+b\|u_1\|^2)\|u_1^-\|^2 = 0$ that $u_1 = u_1^+ \ge$

0, where $u_1^- = \max\{-u_1, 0\}$. Therefore, u_1 is a nonnegative critical point of I. Furthermore, if $h(x) \geq 0$, the strong maximum principle [8] implies that u_1 is a positive solution of problem (1.1). □

3. Existence of a mountain pass solution

In this section, we use a variant version of mountain pass theorem to obtain a nonzero critical point of functional I; this theorem is used also in [10] and its proof can be found in [7], let us recall first this theorem.

Lemma 3.1 (Mountain Pass Theorem). *Let E be a real Banach space with its dual space E^* and suppose that $I \in C^1(E, R)$ satisfy the condition*

$$\max\{I(0), I(e)\} \leq \kappa < \gamma \leq \inf_{\|u\|=\rho}\{I(u)\}$$

for some $\kappa < \gamma$, $\rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \geq \gamma$ be characterized by

$$c = \inf_{h\in\Gamma} \max_{t\in[0,1]} I(h(t)),$$

where $\Gamma = \{h \in ([0,1], E) | h(0) = 0, h(1) = e\}$ is the set of continuous paths joining 0 and e. Then, there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c \geq \gamma \quad \text{and} \quad (1 + \|u_n\|)\|I'(u_n)\|_{E^{-1}} \to 0$$

as $n \to \infty$.

Proof of Theorem 1.2. Let ρ, γ and e be given in Lemma 2.1, applying Lemma 3.1 with $\kappa = 0$, $E = H_0^1(\Omega)$, and for c defined as in Lemma 3.1, then there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that

$$I(u_n) \to c \geq \gamma \quad \text{and} \quad (1 + \|u_n\|)\|I'(u_n)\|_{E^{-1}} \to 0$$

as $n \to \infty$. This implies that

$$\begin{aligned} &\frac{a}{2}\int_\Omega |\nabla u_n|^2 dx + \frac{b}{4}\Big(\int_\Omega |\nabla u_n|^2 dx\Big)^2 - \frac{1}{q+1}\int_\Omega h(x)(u_n^+)^{q+1} dx \\ &- \int_\Omega F(x, u_n^+) dx = c + o(1), \end{aligned} \tag{3.1}$$

$$\begin{aligned} &a\int_\Omega \nabla u_n \cdot \nabla\varphi dx + b\int_\Omega |\nabla u_n|^2 dx \int_\Omega \nabla u_n \cdot \nabla\varphi dx - \frac{1}{q+1}\int_\Omega h(x)(u_n^+)^q \varphi \\ &- \int_\Omega f(x, u_n^+)\varphi dx = o(1), \quad \text{for } \varphi \in H_0^1(\Omega), \end{aligned} \tag{3.2}$$

$$a\int_\Omega |\nabla u_n|^2 dx + b\Big(\int_\Omega |\nabla u_n|^2 dx\Big)^2 - \int_\Omega h(x)(u_n^+)^{q+1} dx - \int_\Omega f(x, u_n^+)u_n^+ dx = o(1). \tag{3.3}$$

By the compactness of Sobolev embedding and the standard procedures, we know that, if $\{u_n\}$ is bounded in $H_0^1(\Omega)$, there exists $u_2 \in H_0^1(\Omega)$ such that $I'(u_2) = 0$ and $I(u_2) = c > 0$ and u_2 is a nonnegative weak solution of problem (1.1), which is positive if $h(x) \geq 0$ by the strong maximum principle. Moreover, u_2 is different from the solution u_1 obtained in Theorem 1.1 since $I(u_1) = c_1 < 0$. So, to prove Theorem 1.2, we only need to prove that $\{u_n\}$ given by (3.1)–(3.3) is bounded in $H_0^1(\Omega)$.

Next, we shall show that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. By contradiction, we suppose that $\|u_n\| \to \infty$ as $n \to \infty$, and set $w_n = \frac{u_n}{\|u_n\|}$. Clearly, $\{w_n\}$ is bounded in $H_0^1(\Omega)$. Thus, there exist a subsequence, still denoted by $\{w_n\}$, and $w \in H_0^1(\Omega)$, such that

$$\begin{gathered} w_n \rightharpoonup w \quad \text{weakly in } H_0^1(\Omega), \\ w_n \to w \quad \text{a.e. in } \Omega, \\ w_n \to w \quad \text{strongly in } L^r(\Omega) \end{gathered}$$

as $n \to \infty$, where $r \in [1,6]$ if $N = 3$ and $r \in (1, +\infty)$ if $N = 1, 2$.

Similarly, $w_n^+ = \frac{u_n^+}{\|u_n\|}$ also satisfies

$$\begin{gathered} w_n^+ \rightharpoonup w^+ \quad \text{weakly in } H_0^1(\Omega), \\ w_n^+ \to w^+ \quad \text{a.e. in } \Omega, \\ w_n^+ \to w^+ \quad \text{strongly in } L^r(\Omega) \end{gathered}$$

as $n \to \infty$. We first claim that $w \not\equiv 0$. Indeed, if $w \equiv 0$, then by (H1), we have

$$\lim_{n\to\infty} \int_\Omega h(x)(w_n^+)^{q+1} dx = 0. \tag{3.4}$$

Moreover, by (F1)-(F2), for any $\varepsilon > 0$, if $s > 0$ large enough, we obtain

$$(\beta - \varepsilon) a\lambda_1 s + (\beta - \varepsilon) b\mu_1 s^3 < f(x, s) < (\beta + \varepsilon) a\lambda_1 s + (\beta + \varepsilon) b\mu_1 s^3.$$

Therefore, we deduce

$$(\beta - \varepsilon) a\lambda_1 s - \varepsilon b\mu_1 s^3 < f(x, s) - \beta b\mu_1 s^3 < (\beta + \varepsilon) a\lambda_1 s + \varepsilon b\mu_1 s^3.$$

It implies that

$$\begin{aligned} &\frac{(\beta - \varepsilon)\lambda_1}{\|u_n\|^2} \int_\Omega w_n^+ \varphi dx - \varepsilon b\mu_1 \int_\Omega (w_n^+)^3 \varphi dx \\ &< \int_\Omega \frac{f(x, u_n^+) - b\beta\mu_1 (u_n^+)^3}{\|u_n\|^3} \varphi dx \\ &< \frac{(\beta + \varepsilon)\lambda_1}{\|u_n\|^2} \int_\Omega w_n^+ \varphi dx - \varepsilon b\mu_1 \int_\Omega (w_n^+)^3 \varphi dx \end{aligned}$$

for any $\varphi \in H_0^1(\Omega)$. By the arbitrariness of ε, we obtain

$$\lim_{n\to+\infty} \int_\Omega \frac{f(x, u_n^+) - b\beta\mu_1 (u_n^+)^3}{\|u_n\|^3} \varphi dx = 0. \tag{3.5}$$

Multiplying (3.2) by $\frac{1}{\|u_n\|^3}$, we have

$$\begin{aligned} &\frac{a}{\|u_n\|^2} \int_\Omega \nabla w_n \cdot \nabla \varphi dx + b \int_\Omega \nabla w_n \cdot \nabla \varphi dx - \frac{1}{\|u_n\|^{3-q}} \int_\Omega h(x)(w_n^+)^q \varphi dx \\ &- b\beta\mu_1 \int_\Omega (w_n^+)^3 \varphi dx - \int_\Omega \frac{f(x, u_n^+) - b\beta\mu_1 (u_n^+)^3}{\|u_n\|^3} \varphi dx = o(1). \end{aligned} \tag{3.6}$$

Letting $n \to \infty$ in (3.6), according to $\|u_n\| \to \infty$ as $n \to \infty$, (3.4), (3.5) and $b \neq 0$, we have

$$\int_\Omega \nabla w \cdot \nabla \varphi dx = \beta\mu_1 \int_\Omega (w^+)^3 \varphi dx$$

and $w \neq 0$. Hence, $\beta\mu_1$ is an eigenvalue of (1.3), which contradicts with the assumption. The proof is complete. $\square$

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