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EXISTENCE OF SOLUTIONS FOR ITERATIVE DIFFERENTIAL EQUATIONS

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Abstract. The presence of self-mapping increases the difficulty in proving the existence of solutions for general iterative differential equation. In this article we provide conditions for the existence of solutions for the initial value problem, in which the conditions are natural and easily verifiable. We generalize the relevant results and point out the mistake in some references.

1. Introduction

Differential equations with state-dependent delays attract interests of specialists since they widely arise from application models, such as two-body problem of classical electrodynamics [9, 10], position control [6, 7], mechanical models [15], infection disease transmission [23], population models [3, 18], the dynamics of economical systems [4], etc. As special type of state-dependent delay-differential equations, iterative differential equations have distinctive characteristics and have been investigated in recent years, e.g. smoothness [8, 19], equivariance [25], analyticity [21], [26]-[27], monotonicity [11, 22]), convexity [20] as well as numerical solution [17]. In the theory of differential equations, one of the fundamental and important problems is the initial value problem, there are many existence results [1, 2], [5], [11]-[16], [24] on special iterative differential equations. In 1984 Eder [11] proved the existence of the unique monotone solution for the 2-th iterative differential equation

$$x'(t) = x(x(t)) \tag{1.1}$$

associated with $x(t_0) = t_0$ ($t_0 \in [-1, 1]$) by Contraction Principle. Later, M. Fečkan ([12]) investigated the generally 2-th iterative differential equation

$$x'(t) = f(x(x(t))) \tag{1.2}$$

with the initial value $x(0) = 0$ and obtained the local solution applying Contraction Principle. By using Schauder's fixed point theorem, Wang [24] obtained the strong solutions of equation (1.2) associated with $x(a) = a$, where a is an endpoint of

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the well-defined interval. Consequently, Ge and Mo [13] provided the sufficient conditions for the initial value problem of (1.2) associated with

$$x(t_0) = x_0 \tag{1.3}$$

on a given compact interval, where the endpoints of the interval are two adjacent null points of f. The 2-th nonautonomous equation

$$x'(t) = f(t, x(t), x(x(t))), \tag{1.4}$$

together with initial value

$$x(0) = c \quad (c > 0)$$

was investigated by P. Andrzej ([1]) using Picard's successive approximation, where 0 is the left end point of the domain.

In 2010 Berinde [2] applied the nonexpansive operators to investigate (1.2) associated with (1.3) and extended the existence results in [5]. Subsequently, Lauran investigated the nonautonomous equation (1.4) together with (1.3) in [16]. We see that the existence of solutions for the general iterative differential equation

$$x'(t) = f(t, x^{[1]}(t), x^{[2]}(t), \dots, x^{[n]}(t)) \tag{1.5}$$

associated with (1.3) is still open, $x^{[i]}(t) := x(x^{[i-1]}(t))$ indicates the i-th iterate of self-mapping x, where $i = 1, 2, \dots, n$. In this paper we provide two existence results for the initial value problem, in which the conditions are natural and easily verifiable. We generalize the relevant results and point out the mistake in [2] and [16]. As the application, we consider the smooth solutions of the equation discussed in [19] by Theorem 2.2 and give an example to verify Theorem 2.3.

2. Main results

For the continuous function $\varphi(x)$, we use the supremum norm

$$\|\varphi\|_P = \sup_{x \in P \subset \mathbb{R}^n} \|\varphi(x)\|$$

and need the following lemma (the statement is slightly different from the original one presented in [28] but perfectly equivalent):

Lemma 2.1 ([28]). *Let*

$$\Phi_M = \{x \in C^0([t_0 - h, t_0 + h]) : |x(t) - x(s)| \le M|t - s|, \forall t, s \in [t_0 - h, t_0 + h]\},$$

where $M < 1$. If $f, g \in \Phi_M$, then

$$\|f^{[j]} - g^{[j]}\|_{[t_0-h,t_0+h]} \le \frac{1 - M^j}{1 - M} \|f - g\|_{[t_0-h,t_0+h]}, \quad j = 1, 2, \dots. \tag{2.1}$$

Theorem 2.2. *Suppose that $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is continuous. If there exists a positive r such that*

$$(1 - M_1)\, r \ge l_0, \tag{2.2}$$

where $M_1 = \|f\|_{\bar{B}(y_0,r)} \le 1$ and $l_0 = |x_0 - t_0|$ and $\bar{B}(y_0, r)$ denotes the closed ball centered at $y_0 = (t_0, x_0, \dots, x_0)$ with radius r. Then equation (1.5) associated with (1.3) has a solution defined on $[t_0 - l, t_0 + l]$ for any $l \in [\, l_0/(1 - M_1), r]$.

Proof. The existence of solutions of equation (1.5) associated with (1.3) is equivalent to find a continuous solution of the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x^{[1]}(s), x^{[2]}(s), \dots, x^{[n]}(s))ds. \tag{2.3}$$

Define

$$\begin{aligned}\Phi_{M_1} = \big\{ &x \in C^0([t_0 - l, t_0 + l]) : x(t_0) = x_0, |x(t) - x(s)| \le M_1|t - s|, \\ &\forall t, s \in [t_0 - l, t_0 + l] \big\}.\end{aligned}$$

for any $l \in [l_0/(1 - M_1), r]$. Then for $x \in \Phi_{M_1}$, we show that $x^{[i]}(t)$ $(i = 2, 3, \dots, n)$ are well defined on $[t_0 - l, t_0 + l]$. It is suffices to prove

$$|x^{[i]}(t) - t_0| \le l \tag{2.4}$$

for $i \in \mathbb{N}$ by induction. In fact

$$\begin{aligned}|x(t) - t_0| &\le |x(t) - x(t_0)| + |x(t_0) - t_0| \\ &\le M_1 l + |x_0 - t_0| \le l,\end{aligned}$$

we assume that $|x^{[i]}(t) - t_0| \le l$ for positive integer $i \ge 1$, then

$$\begin{aligned}|x^{[i+1]}(t) - t_0| &\le |x^{[i+1]}(t) - x(t_0)| + |x(t_0) - t_0| \\ &\le M_1|x^{[i]}(t) - t_0| + |x_0 - t_0| \\ &\le M_1 l + |x_0 - t_0| \le l.\end{aligned}$$

Hence it follows by induction that (2.4) holds and $x^{[i]}([t_0 - l, t_0 + l])$ are well defined for any $x \in \Phi_{M_1}$.

In the sequel we apply the Schauder's fixed point theorem to prove the existence of the continuous solution of (2.3). To this end, we define the integral operator $\mathcal{G} : \Phi_{M_1} \to C^0([t_0 - l, t_0 + l])$ by

$$\mathcal{G}x(t) := x_0 + \int_{t_0}^{t} f(s, x^{[1]}(s), x^{[2]}(s), \dots, x^{[n]}(s))ds. \tag{2.5}$$

Clearly

$$\mathcal{G}x(t_0) = x_0 + \int_{t_0}^{t_0} f(s, x^{[1]}(s), x^{[2]}(s), \dots, x^{[n]}(s))ds = x_0 \tag{2.6}$$

for any $x \in \Phi_{M_1}$. In view of

$$\begin{aligned}&\|(t, x^{[1]}(t), x^{[2]}(t), \dots, x^{[n]}(t)) - (t_0, x_0, x_0, \dots, x_0)\| \\ &= \max\{|t - t_0|, |x^{[1]}(t) - x_0|, |x^{[2]}(t) - x_0|, \dots, |x^{[n]}(t) - x_0|\} \\ &\le \max\{|t - t_0|, M_1|t - t_0|, M_1|x^{[1]}(t) - t_0|, \dots, M_1|x^{[n-1]}(t) - t_0|\} \\ &\le \max\{l, M_1 l, M_1 l, \dots, M_1 l\} \\ &\le l \le r,\end{aligned}$$

we get

$$\begin{aligned}|\mathcal{G}x(t_1) - \mathcal{G}x(t_2)| &\le \Big|\int_{t_2}^{t_1} |f(s, x^{[1]}(s), x^{[2]}(s), \dots, x^{[n]}(s))|ds\Big| \\ &\le M_1|t_1 - t_2|\end{aligned} \tag{2.7}$$

for any $t_1, t_2 \in [t_0 - l, t_0 + l]$. Thus (2.5), (2.6) and (2.7) yield $\mathcal{G}x \in \Phi_{M_1}$; i.e., $\mathcal{G}$ is a self-mapping operator.

It remains to show that $\mathcal{G}$ is continuous. For this purpose, take any $x_1, x_2 \in \Phi_{M_1}$, we have

$$
\begin{aligned}
&|\mathcal{G}x_1(t) - \mathcal{G}x_2(t)| \\
&\le \Big| \int_{t_0}^{t} |f(s, x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s)) - f(s, x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s))| ds \Big|.
\end{aligned}
$$

By Lemma 2.1,

$$
\begin{aligned}
&\|(s, x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s)) - (s, x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s))\| \\
&= \max\{|x_1^{[1]}(s) - x_2^{[1]}(s)|, |x_1^{[2]}(s) - x_2^{[2]}(s)|, \dots, |x_1^{[n]}(s) - x_2^{[n]}(s)|\} \\
&\le \max\{\|x_1 - x_2\|_{[t_0-l, t_0+l]}, \frac{1 - {M_1}^2}{1 - M_1} \|x_1 - x_2\|_{[t_0-l, t_0+l]}, \\
&\quad \dots, \frac{1 - {M_1}^n}{1 - M_1} \|x_1 - x_2\|_{[t_0-l, t_0+l]}\} \\
&= \frac{1 - {M_1}^n}{1 - M_1} \|x_1 - x_2\|_{[t_0-l, t_0+l]} \\
&< \frac{1}{1 - M_1} \|x_1 - x_2\|_{[t_0-l, t_0+l]}.
\end{aligned}
$$

Because of the uniform continuity of f on $\bar{B}(y_0, r)$, for any $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$, the inequality

$$\|\mathcal{G}x_1 - \mathcal{G}x_2\| < \varepsilon l$$

holds for $\|x_1 - x_2\|_{[t_0-l, t_0+l]} < \delta$, which implies $\mathcal{G}$ is continuous.

Φ_{M_1} is a convex, compact subset of Banach space $C^0([t_0 - l, t_0 + l])$ and $\mathcal{G}$ is a continuous operator, which satisfy all conditions of the Schauder's fixed point theorem, so $\mathcal{G}$ has a fixed point $g \in \Phi_{M_1}$ and g is a solution for equation (1.5) associated with (1.3) on the interval $[t_0 - l, t_0 + l]$. This completes the proof. □

Theorem 2.3. *Suppose that $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is continuous and any compact interval $[a, b]$ includes t_0 and x_0. If*

$$M_2 A_{t_0} \le B_{x_0}, \tag{2.8}$$

where $A_{t_0} = \max\{t_0 - a, b - t_0\}$, $B_{x_0} = \min\{x_0 - a, b - x_0\}$, $M_2 = \|f\|_{[a,b]^{n+1}}$ and $[a, b]^{n+1}$ denotes the product of $n+1$ intervals $[a, b]$. Then equation (1.5) associated with (1.3) has a solution defined on $[a, b]$.

Proof. As in the proof of Theorem 2.2, we apply the Schauder fixed point theorem to prove the result. Let

$$
\begin{aligned}
\Phi_{M_2} = \big\{ &x \in C^0([a, b], [a, b]) : x(t_0) = x_0, \\
&|x(t) - x(s)| \le M_2 |t - s|,\ \forall t, s \in [a, b] \big\},
\end{aligned} \tag{2.9}
$$

then Φ_{M_2} is a non-empty convex and compact subset of the Banach space $C^0([a, b])$. We consider the mapping $\mathcal{T} : \Phi_{M_2} \to C^0([a, b])$ defined by

$$\mathcal{T}x(t) := x_0 + \int_{t_0}^{t} f(s, x^{[1]}(s), x^{[2]}(s), \dots, x^{[n]}(s)) ds. \tag{2.10}$$

To prove $\mathcal{T}$ is a self-mapping, we note that

$$\begin{aligned}\mathcal{T}x(t) &\le x_0 + \Big|\int_{t_0}^{t} f(s, x^{[1]}(s), x^{[2]}(s), \dots, x^{[n]}(s))ds\Big| \\ &\le x_0 + M_2|t - t_0| \\ &\le x_0 + M_2 A_{t_0} \\ &\le x_0 + B_{t_0} \le b,\end{aligned} \tag{2.11}$$

$$\begin{aligned}\mathcal{T}x(t) &\ge x_0 - \Big|\int_{t_0}^{t} f(s, x^{[1]}(s), x^{[2]}(s), \dots, x^{[n]}(s))ds\Big| \\ &\ge x_0 - M_2|t - t_0| \\ &\ge x_0 - M_2 A_{t_0} \\ &\ge x_0 - B_{x_0} \ge a.\end{aligned} \tag{2.12}$$

Clearly,

$$\mathcal{T}x(t_0) = x_0. \tag{2.13}$$

Moreover, for any $t_1, t_2 \in [a, b]$, we have

$$\begin{aligned}|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)| &\le \Big|\int_{t_2}^{t_1} |f(s, x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s))|ds\Big| \\ &\le M_2|t_1 - t_2|.\end{aligned} \tag{2.14}$$

Thus (2.11), (2.12), (2.13) and (2.14) imply that $\mathcal{T}$ maps Φ_{M_2} into itself.

The definitions of A_{t_0} and B_{x_0} show that $M_2 \le 1$, then for any $x_1, x_2 \in \Phi_{M_2}$, according to Lemma 2.1, we have

$$\begin{aligned}&\|(s, x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s)) - (s, x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s))\| \\ &= \max\{|x_1^{[1]}(s) - x_2^{[1]}(s)|, |x_1^{[2]}(s) - x_2^{[2]}(s)|, \dots, |x_1^{[n]}(s) - x_2^{[n]}(s)|\} \\ &\le \max\{\|x_1 - x_2\|_{[a,b]}, \frac{1 - {M_2}^2}{1 - M_2}\|x_1 - x_2\|_{[a,b]}, \dots, \frac{1 - {M_2}^n}{1 - M_2}\|x_1 - x_2\|_{[a,b]}\} \\ &= \frac{1 - {M_2}^n}{1 - M_2}\|x_1 - x_2\|_{[a,b]} \\ &< \frac{1}{1 - M_2}\|x_1 - x_2\|_{[a,b]}.\end{aligned}$$

By the uniform continuity of f on $[a, b]^{n+1}$, for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, when $\|x_1 - x_2\|_{[a,b]} < \delta$ we have

$$|f(s, x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s)) - f(s, x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s))| < \varepsilon.$$

Consequently,

$$\begin{aligned}&|\mathcal{T}x_1(t) - \mathcal{T}x_2(t)| \\ &\le \Big|\int_{t_0}^{t} |f(s, x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s)) - f(s, x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s))|ds\Big| \\ &< \varepsilon(b - a),\end{aligned}$$

which means that $\mathcal{T}$ is a continuous operator.

It follows that Φ_{M_2} is a convex, compact subset of Banach space $C^0([a, b])$ and $\mathcal{T}$ is a continuous operator. By the Schauder's fixed point theorem, $\mathcal{T}$ has a fixed point $h \in \Phi_{M_2}$ and h is a solution of equation (1.5) associated with (1.3) on the interval $[a, b]$. This completes the proof. $\square$

3. Examples and remarks

In this section our theorems are demonstrated by the following two examples. Firstly, we prove the existence of smooth solutions of the equation, discussed in [19], together with the general initial value (1.3) by using Theorem 2.2. Here, smooth function $g \in C^n$ means the function g has a number of continuous derivatives and its n-th continuous derivative also is Lipschtzian. We need the following lemma introduced in [19].

Lemma 3.1 ([19]). *Let*

$$\begin{aligned}\Omega(N_1, \dots, N_{n+1}; I) = \big\{g \in\ & C^n(I, I) : |g^{(i)}(t)| \le\ N_i,\ i = 1, 2, \dots, n;\\ & |g^{(n)}(t) - g^{(n)}(s)| \le N_{n+1}|t - s|,\ t, s \in I\big\}.\end{aligned}$$

For any $x(t) \in \Omega(N_1, \dots, N_{n+1}; I)$, there is

$$x_{*jk}(t) = P_{jk}(x_{10}(t), \dots, x_{1,j-1}(t); \dots; x_{k0}(t), \dots, x_{k,j-1}(t))$$

and exist positive constants N^{jk}_{uv} such that

$$|P_{jk}(\bar\lambda_{10}, \dots, \bar\lambda_{k,j-1}) - P_{jk}(\tilde\lambda_{10}, \dots, \tilde\lambda_{k,j-1})| \le \sum_{u=1}^{k}\sum_{v=0}^{j-1} N^{jk}_{uv}|\bar\lambda_{uv} - \tilde\lambda_{uv}|$$

for $(\bar\lambda_{10}, \dots, \bar\lambda_{k,j-1}), (\tilde\lambda_{10}, \dots, \tilde\lambda_{k,j-1})$ belong to compact set $[0, N_1]^j \times [0, N_2]^j \times \dots \times [0, N_k]^j$, where $x_{ij}(t) = x^{(i)}(x^{[j]}(t))$, $x_{*jk}(t) = (x^{[j]}(t))^{(k)}$ and P_{jk} is a uniquely defined multivariate polynomial with nonnegative coefficients and $1 \le u \le k$, $0 \le v \le j - 1$.

Example 3.2. Consider the equation

$$x'(t) = \sum_{j=1}^{m} a_j(t)x^{[j]}(t) + F(t) \tag{3.1}$$

associated with (1.3), where $a_j(t), F(t) \in C^n$ are given smooth functions.

For $R > 0$, by the smoothness of the given functions, we have positive M_{a_j} and M_F such that

$$|a_j(t)| \le M_{a_j}, \quad |F(t)| \le M_F, \quad t \in [t_0 - R, t_0 + R],\ j = 1, 2, \dots, m.$$

Denote

$$M_a = \max_{1\le j\le m}\{M_{a_j}\}, \quad N_1 = mM_a(|t_0| + R) + M_F.$$

If $(1 - N_1)R \ge |x_0 - t_0|$, the equation (3.1) associated with (1.3) has a solution in the function set

$$\begin{aligned}\Phi_{N_1} = \big\{&x \in C^0([t_0 - l_1, t_0 + l_1]) : x(t_0) = x_0,\\ &|x(t) - x(s)| \le N_1|t - s|, \forall t, s \in [t_0 - l_1, t_0 + l_1]\big\}\end{aligned}$$

by Theorem 2.2, where arbitrary $l_1 \in [|x_0 - t_0|/(1 - N_1), R]$. In fact, for any $x \in \Phi_{N_1}$, we see that the function

$$f(t, x^{[1]}(t), x^{[2]}(t), \dots, x^{[m]}(t)) = \sum_{j=1}^{m} a_j(t)x^{[j]}(t) + F(t)$$

is continuous on $[t_0 - l_1, t_0 + l_1]$ and

$$\begin{aligned}|f(t, x^{[1]}(t), x^{[2]}(t), \dots, x^{[m]}(t))| &= |\sum_{j=1}^{m} a_j(t)x^{[j]}(t) + F(t)| \\ &\le \sum_{j=1}^{m} M_a(|t_0| + R) + M_F \\ &= mM_a(|t_0| + R) + M_F = N_1.\end{aligned}$$

Since $(1 - N_1)R \ge |x_0 - t_0|$, the condition of Theorem 2.2 is satisfied, there exists a solution $x = \varphi(t)$ of equation (3.1) together with (1.3) in the functional set Φ_{N_1}.

The form of equation (3.1) and $a_j(t), F(t) \in C^n([t_0 - l_1, t_0 + l_1])$ show that $\varphi(t) \in C^{(n+1)}([t_0 - l_1, t_0 + l_1])$. In the sequel, we prove $\varphi^{(n+1)}(t)$ also is Lipschtzian on the compact interval $[t_0 - l_1, t_0 + l_1]$. From Lemma 3.1, we have

$$\begin{aligned}x_{*jk}(t) &= P_{jk}(x_{10}(t), \dots, x_{1,j-1}(t); \dots; x_{k0}(t), \dots, x_{k,j-1}(t)) \\ &= P_{jk}(x'(t), x'(x_1), \dots, x'(x_{j-1}); \dots; x^{(k)}(t), x^{(k)}(x_1), \dots, x^{(k)}(x_{j-1})),\end{aligned}$$

where $x_m = x^{[m]}(t),\ m = 1, 2, \dots, j - 1$. Denote

$$\begin{gathered}H_{jk} = P_{jk}(\overbrace{N_1, \dots, N_1}^{j\ terms}; \overbrace{N_2, \dots, N_2}^{j\ terms}; \dots; \overbrace{N_k, \dots, N_k}^{j\ terms}), \\ a_j(t) \in \Omega(L_{j1}, \dots, L_{j(n+1)}; [t_0 - l_1, t_0 + l_1]), \\ F(t) \in \Omega(M_1, \dots, M_{n+1}; [t_0 - l_1, t_0 + l_1]).\end{gathered}$$

Then for any $t_1, t_2 \in [t_0 - l_1, t_0 + l_1]$, we get

$$\begin{aligned}&|\varphi^{(n+1)}(t_1) - \varphi^{(n+1)}(t_2)| \\ &\le \sum_{j=1}^{m}\sum_{s=0}^{n} C_n^s |a_j^{(n-s)}(t_1)(\varphi^{[j]}(t_1))^{(s)} - a_j^{(n-s)}(t_2)(\varphi^{[j]}(t_2))^{(s)}| \\ &\quad + |F^{(n)}(t_1) - F^{(n)}(t_2)| \\ &\le \sum_{j=1}^{m}\{|a_j^{(n)}(t_1) - a_j^{(n)}(t_2)| \cdot |\varphi^{[j]}(t_1)| + |a_j^{(n)}(t_2)| \cdot |\varphi^{[j]}(t_1) - \varphi^{[j]}(t_2)|\} \\ &\quad + \sum_{j=1}^{m}\sum_{s=1}^{n} C_n^s (|a_j^{(n-s)}(t_1) - a_j^{(n-s)}(t_2)| \cdot |(\varphi^{[j]}(t_1))^{(s)}| \\ &\quad + |a_j^{(n-s)}(t_2)| \cdot |p_{js}(\varphi_{10}(t_1), \dots, \varphi_{s,j-1}(t_1)) - p_{js}(\varphi_{10}(t_2), \dots, \varphi_{s,j-1}(t_2))|) \\ &\quad + M_{n+1}|t_1 - t_2| \\ &\le \sum_{j=1}^{m}(L_{j(n+1)}(|t_0| + l_1) + L_{jn}{N_1}^j)|t_1 - t_2| \\ &\quad + \sum_{j=1}^{m}\sum_{s=1}^{n} C_n^s (L_{j(n+1-s)} H_{js}|t_1 - t_2| + L_{j(n-s)} \sum_{u=1}^{s}\sum_{v=0}^{j-1} N_{uv}^{js}|\varphi_{uv}(t_1) - \varphi_{uv}(t_2)|) \\ &\quad + M_{n+1}|t_1 - t_2|.\end{aligned}$$

Since

$$|\varphi_{uv}(t_1) - \varphi_{uv}(t_2)| \le N_{u+1}|\varphi^{[v]}(t_1) - \varphi^{[v]}(t_2)| \le N_{u+1}{N_1}^v|t_1 - t_2|,$$

we have

$$
\begin{aligned}
&|\varphi^{(n+1)}(t_1) - \varphi^{(n+1)}(t_2)| \\
&\le \sum_{j=1}^{m} (L_{j(n+1)}(|t_0| + l_1) + L_{jn}{N_1}^j)|t_1 - t_2| \\
&\quad + \sum_{j=1}^{m}\sum_{s=1}^{n} C_n^s (L_{j(n+1-s)} H_{js}|t_1 - t_2| + L_{j(n-s)} \sum_{u=1}^{s}\sum_{v=0}^{j-1} N_{uv}^{js}|\varphi_{uv}(t_1) - \varphi_{uv}(t_2)|) \\
&\quad + M_{n+1}|t_1 - t_2| \\
&= \{(\sum_{j=1}^{m} L_{j(n+1)}(|t_0| + l_1) + L_{jn}{N_1}^j) \\
&\quad + (\sum_{j=1}^{m}\sum_{s=1}^{n} C_n^s (L_{j(n+1-s)} H_{js} + L_{j(n-s)} \sum_{u=1}^{s}\sum_{v=0}^{j-1} N_{uv}^{js} N_{u+1}{N_1}^v)) + M_{n+1}\}|t_1 - t_2|.
\end{aligned}
$$

That is, $\varphi^{(n+1)}(t)$ is Lipschtzian.

Remark 3.3. The existence and uniqueness of smooth solutions through (t_0, t_0), with $|t_0| < 1$, for (3.1) was studied in [19]. According to Theorem 2.2, we have the similar conclusion for (3.1) through general point (t_0, x_0) even for $|t_0| \ge 1$ provided M_a and M_F are small enough, which generalizes the results in [19]. The similar discussion can be applied for the equation in [8].

Example 3.4. Consider the equation

$$x'(t) = \frac{1}{5}x(x(t)) - \frac{1}{4} \tag{3.2}$$

associated with

$$x(-1) = -\frac{1}{2}. \tag{3.3}$$

For the compact interval $[-1, 0]$ including $t_0 = -1$ and $x_0 = -1/2$, it is clear that $M_2 = 1/5 + 1/4 = 9/20,\ A_{t_0} = 1,\ B_{x_0} = 1/2$, which satisfy the conditions of Theorem 2.3. Then the equation (3.2) associated with (3.3) has a solution.

Remark 3.5. In the proof of invariant set in [2] and [16], they require the inequalities

$$|(Fy)(t)| \le |y_0| + |\int_{x_0}^{t} f(s, y(s), y(y(s)))ds| \le |y_0| + M \cdot |t - x_0| \le b, \tag{3.4}$$

$$|(Fy)(t)| \ge |y_0| - |\int_{x_0}^{t} f(s, y(s), y(y(s)))ds| \ge y_0 - C_{y_0} \ge a. \tag{3.5}$$

The right-most inequality of (3.5) contradicts the definition of C_{y_0}. We overcome this difficulty by defining B_{x_0}. Furthermore, (3.4) implies that b is a nonnegative number, which is given up in Theorem 2.3 such as Example 3.4.

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