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EXISTENCE OF INFINITELY MANY ANTI-PERIODIC SOLUTIONS FOR SECOND-ORDER IMPULSIVE DIFFERENTIAL INCLUSIONS

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Abstract. In this article, we establish the existence of infinitely many anti-periodic solutions for a second-order impulsive differential inclusion with a perturbed nonlinearity and two parameters. The technical approach is mainly based on a critical point theorem for non-smooth functionals.

1. Introduction

The aim of this article is to show the existence of infinitely many solutions for the following two parameter second-order impulsive differential inclusion subject to anti-periodic boundary conditions

$$\begin{gathered} -(\phi_p(u'(x)))' + M\phi_p(u(x)) \in \lambda F(u(x)) + \mu G(x,u(x)) \quad \text{in } [0,T]\setminus Q, \\ -\Delta\phi_p(u'(x_k)) = I_k(u(x_k)), \quad k=1,2,\dots,m, \\ u(0) = -u(T), \quad u'(0) = -u'(T), \end{gathered} \tag{1.1}$$

where $Q = \{x_1, x_2, \dots, x_m\}$, $p > 1$, $T > 0$, $M \geq 0$, $\phi_p(x) := |x|^{p-2}x$, $0 = x_0 < x_1 < \dots < x_m < x_{m+1} = T$, $\Delta\phi_p(u'(x_k)) := \phi_p(u'(x_k^+)) - \phi_p(u'(x_k^-))$, with $u'(x_k^+)$ and $u'(x_k^-)$ denoting the right and left limits, respectively, of $u'(x)$ at $x = x_k$, $I_k \in C(\mathbb{R},\mathbb{R})$, $k = 1,2,\dots,m$, λ is a positive parameter, μ is a nonnegative parameter, and F is a multifunction defined on $\mathbb{R}$, satisfying

- (F1) $F : \mathbb{R} \to 2^{\mathbb{R}}$ is upper semicontinuous with compact convex values;
- (F2) $\min F$, $\max F : \mathbb{R} \to \mathbb{R}$ are Borel measurable;
- (F3) $|\xi| \leq a(1 + |s|^{r-1})$ for all $s \in \mathbb{R}$, $\xi \in F(s)$, $r > 1$ $(a > 0)$.

Also, G is a multifunction defined on $[0,T] \times \mathbb{R}$, satisfying

- (G1) $G(x,\cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is upper semicontinuous with compact convex values for a.e. $x \in [0,T]\setminus Q$;
- (G2) $\min G$, $\max G : ([0,T]\setminus Q) \times \mathbb{R} \to \mathbb{R}$ are Borel measurable;
- (G3) $|\xi| \leq a(1 + |s|^{r-1})$ for a.e. $x \in [0,T]$, $s \in \mathbb{R}$, $\xi \in G(x,s)$, $r > 1$ $(a > 0)$.

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Impulsive differential equations are used to describe various models of real-world processes that are subject to a sudden change. These models are studied in physics, population dynamics, ecology, industrial robotics, biotechnology, economics, optimal control, and so forth. Associated with this development, a theory of impulsive differential equations has been given extensive attention. Differential inclusions arise in models for control systems, mechanical systems, economical systems, game theory, and biological systems to name a few. It is very important to study anti-periodic boundary value problems because they can be applied to interpolation problems [5], antiperiodic wavelets [3], the Hill differential operator [6], and so on. It is natural from both a physical standpoint as well as a theoretical view to give considerable attention to a synthesis involving problems for impulsive differential inclusion with anti-periodic boundary conditions.

Recently, multiplicity of solutions for differential inclusions via non-smooth variational methods and critical point theory has been considered and here we cite the papers [9, 10, 11, 12, 16]. For instance, in [11], the author, employing a non-smooth Ricceri-type variational principle [15], developed by Marano and Motreanu [13], has established the existence of infinitely many, radially symmetric solutions for a differential inclusion problem in $\mathbb{R}^N$. Also, in [12], the authors extended a recent result of Ricceri concerning the existence of three critical points of certain non-smooth functionals. Two applications have been given, both in the theory of differential inclusions; the first one concerns a non-homogeneous Neumann boundary value problem, the second one treats a quasilinear elliptic inclusion problem in the whole $\mathbb{R}^N$. In [9], the author, under convenient assumptions, has investigated the existence of at least three positive solutions for a differential inclusion involving the p-Laplacian operator on a bounded domain, with homogeneous Dirichlet boundary conditions and a perturbed nonlinearity depending on two positive parameters; his result also ensured an estimate on the norms of the solutions independent of both the perturbation and the parameters. Very recently, Tian and Henderson in [16], based on a non-smooth version of critical point theory of Ricceri due to Iannizzotto [9], have established the existence of at least three solutions for the problem (1.1) whenever λ is large enough and μ is small enough.

In the present paper, motivated by [16], employing an abstract critical point result (see Theorem 2.6 below), we are interested in ensuring the existence of infinitely many anti-periodic solutions for the problem (1.1); see Theorem 3.1 below. We refer to [2], in which related variational methods are used for non-homogeneous problems.

To the best of our knowledge, no investigation has been devoted to establishing the existence of infinitely many solutions to such a problem as (1.1). For a couple of references on impulsive differential inclusions, we refer to [7] and [8].

A special case of our main result is the following theorem.

Theorem 1.1. *Assume that* (F1)–(F3) *hold, and* $I_i(0) = 0$*,* $I_i(s)s < 0$*,* $s \in \mathbb{R}$*,* $i = 1, 2, \dots, m$*. Furthermore, suppose that*

$$\liminf_{\xi\to+\infty} \frac{\sup_{|t|\le\xi} \min \int_0^t F(s)ds}{\xi^p} = 0,$$

$$\limsup_{\xi\to+\infty} \frac{\int_0^T \min \int_0^{\xi(\frac{T}{2}-x)} F(s)\,ds\,dx}{\frac{1}{p}\xi^p\left(T + \frac{2M}{p+1}(\frac{T}{2})^{p+1}\right) - \sum_{i=1}^m \int_0^{\xi(\frac{T}{2}-x_i)} I_i(s)ds} = +\infty.$$

Then, the problem (1.1)*, for* $\lambda = 1$ *and* $\mu = 0$*, admits a sequence of pairwise distinct solutions.*

2. Basic definitions and preliminary results

Let $(X, \|\cdot\|_X)$ be a real Banach space. We denote by X^* the dual space of X, while $\langle \cdot, \cdot \rangle$ stands for the duality pairing between X^* and X. A function $\varphi : X \to \mathbb{R}$ is called locally Lipschitz if, for all $u \in X$, there exist a neighborhood U of u and a real number $L > 0$ such that

$$|\varphi(v) - \varphi(w)| \le L\|v - w\|_X \quad \text{for all } v, w \in U.$$

If φ is locally Lipschitz and $u \in X$, the generalized directional derivative of φ at u along the direction $v \in X$ is

$$\varphi^\circ(u; v) := \limsup_{w \to u,\, \tau \to 0^+} \frac{\varphi(w + \tau v) - \varphi(w)}{\tau}.$$

The generalized gradient of φ at u is the set

$$\partial\varphi(u) := \{u^* \in X^* : \langle u^*, v\rangle \le \varphi^\circ(u; v) \text{ for all } v \in X\}.$$

So $\partial\varphi : X \to 2^{X^*}$ is a multifunction. We say that φ has compact gradient if $\partial\varphi$ maps bounded subsets of X into relatively compact subsets of X^*.

Lemma 2.1 ([14, Proposition 1.1])**.** *Let* $\varphi \in C^1(X)$ *be a functional. Then* φ *is locally Lipschitz and*

$$\begin{gathered} \varphi^\circ(u; v) = \langle \varphi'(u), v\rangle \quad \textit{for all } u, v \in X; \\ \partial\varphi(u) = \{\varphi'(u)\} \quad \textit{for all } u \in X. \end{gathered}$$

Lemma 2.2 ([14, Proposition 1.3])**.** *Let* $\varphi : X \to \mathbb{R}$ *be a locally Lipschitz functional. Then* $\varphi^\circ(u; \cdot)$ *is subadditive and positively homogeneous for all* $u \in X$*, and*

$$\varphi^\circ(u; v) \le L\|v\| \quad \textit{for all } u, v \in X,$$

with $L > 0$ *being a Lipschitz constant for* φ *around* u*.*

Lemma 2.3 ([4])**.** *Let* $\varphi : X \to \mathbb{R}$ *be a locally Lipschitz functional. Then* $\varphi^\circ : X \times X \to \mathbb{R}$ *is upper semicontinuous and for all* $\lambda \ge 0$*,* $u, v \in X$*,*

$$(\lambda\varphi)^\circ(u; v) = \lambda\varphi^\circ(u; v).$$

Moreover, if $\varphi, \psi : X \to \mathbb{R}$ *are locally Lipschitz functionals, then*

$$(\varphi + \psi)^\circ(u; v) \le \varphi^\circ(u; v) + \psi^\circ(u; v) \quad \textit{for all } u, v \in X.$$

Lemma 2.4 ([14, Proposition 1.6])**.** *Let* $\varphi, \psi : X \to \mathbb{R}$ *be locally Lipschitz functionals. Then*

$$\begin{gathered} \partial(\lambda\varphi)(u) = \lambda\partial\varphi(u) \quad \textit{for all } u \in X,\ \lambda \in \mathbb{R}, \\ \partial(\varphi + \psi)(u) \subseteq \partial\varphi(u) + \partial\psi(u) \quad \textit{for all } u \in X. \end{gathered}$$

Lemma 2.5 ([9, Proposition 1.6])**.** *Let* $\varphi : X \to \mathbb{R}$ *be a locally Lipschitz functional with a compact gradient. Then* φ *is sequentially weakly continuous.*

We say that $u \in X$ is a (generalized) critical point of a locally Lipschitz functional φ if $0 \in \partial\varphi(u)$; i.e.,

$$\varphi^\circ(u;v) \geq 0 \quad \text{for all } v \in X.$$

When a non-smooth functional, $g : X \to (-\infty, +\infty)$, is expressed as a sum of a locally Lipschitz function, $\varphi : X \to \mathbb{R}$, and a convex, proper, and lower semicontinuous function, $j : X \to (-\infty, +\infty)$; that is, $g := \varphi + j$, a (generalized) critical point of g is every $u \in X$ such that

$$\varphi^\circ(u; v - u) + j(v) - j(u) \geq 0$$

for all $v \in X$ (see [14, Chapter 3]).

Hereafter, we assume that X is a reflexive real Banach space, $\mathcal{N} : X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous functional, $\Upsilon : X \to \mathbb{R}$ is a sequentially weakly upper semicontinuous functional, λ is a positive parameter, $j : X \to (-\infty, +\infty)$ is a convex, proper, and lower semicontinuous functional, and $D(j)$ is the effective domain of j. Write

$$\mathcal{M} := \Upsilon - j, \quad I_\lambda := \mathcal{N} - \lambda\mathcal{M} = (\mathcal{N} - \lambda\Upsilon) + \lambda j.$$

We also assume that $\mathcal{N}$ is coercive and

$$D(j) \cap \mathcal{N}^{-1}((-\infty, r)) \neq \emptyset \tag{2.1}$$

for all $r > \inf_X \mathcal{N}$. Moreover, owing to (2.1) and provided $r > \inf_X \mathcal{N}$, we can define

$$\varphi(r) := \inf_{u \in \mathcal{N}^{-1}((-\infty,r))} \frac{\big(\sup_{v \in \mathcal{N}^{-1}((-\infty,r))} \mathcal{M}(v)\big) - \mathcal{M}(u)}{r - \mathcal{N}(u)},$$

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf_X \mathcal{N})^+} \varphi(r).$$

If $\mathcal{N}$ and Υ are locally Lipschitz functionals, in [1, Theorem 2.1] the following result is proved; it is a more precise version of [13, Theorem 1.1] (see also [15]).

Theorem 2.6. *Under the above assumption on X, $\mathcal{N}$ and $\mathcal{M}$, one has*

- (a) *For every $r > \inf_X \mathcal{N}$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional $I_\lambda = \mathcal{N} - \lambda\mathcal{M}$ to $\mathcal{N}^{-1}((-\infty, r))$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*
- (b) *If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*
  - (b1) *I_λ possesses a global minimum, or*
  - (b2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to+\infty} \mathcal{N}(u_n) = +\infty$.*
- (c) *If $\delta < +\infty$, then for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either*
  - (c1) *there is a global minimum of $\mathcal{N}$ which is a local minimum of I_λ, or*
  - (c2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ, with $\lim_{n\to+\infty} \mathcal{N}(u_n) = \inf_X \mathcal{N}$, which converges weakly to a global minimum of $\mathcal{N}$.*

Now we recall some basic definitions and notation. On the reflexive Banach space $X := \{u \in W^{1,p}([0,T]) : u(0) = -u(T)\}$ we consider the norm

$$\|u\|_X := \Big(\int_0^T \big(|u'(x)|^p + M|u(x)|^p\big) dx \Big)^{1/p}$$

for all $u \in X$, which is equivalent to the usual norm (note that $M \geq 0$). We recall that X is compactly embedded into the space $C^0([0,T])$ endowed with the maximum norm $\|\cdot\|_{C^0}$.

Lemma 2.7 ([16, Lemma 3.3])**.** *Let $u \in X$. Then*

$$\|u\|_{C^0} \leq \frac{1}{2} T^{1/q} \|u\|_X, \tag{2.2}$$

where $1/p + 1/q = 1$.

Obviously, X is compactly embedded into $L^\gamma([0,T])$ endowed with the usual norm $\|\cdot\|_{L^\gamma}$, for all $\gamma \geq 1$.

Definition 2.8. A function $u \in X$ is a weak solution of the problem (1.1) if there exists $u^* \in L^\gamma([0,T])$ (for some $\gamma > 1$) such that

$$\int_0^T \Big[\phi_p(u'(x))v'(x) + M\phi_p(u(x))v(x) - u^*(x)v(x)\Big]dx - \sum_{i=1}^m I_i(u(x_i))v(x_i) = 0$$

for all $v \in X$ and $u^* \in \lambda F(u(x)) + \mu G(x,u(x))$ for a.e. $x \in [0,T]$.

Definition 2.9. *By a solution of the impulsive differential inclusion* (1.1) *we will understand a function $u : [0,T] \setminus Q \to \mathbb{R}$ is of class C^1 with $\phi_p(u')$ absolutely continuous, satisfying*

$$\begin{gathered}
-(\phi_p(u'(x)))' + M\phi_p(u(x)) = u^* \quad \text{in } [0,T] \setminus Q, \\
-\Delta\phi_p(u'(x_k)) = I_k(u(x_k)), \quad k = 1,2,\dots,m, \\
u(0) = -u(T), \quad u'(0) = -u'(T),
\end{gathered}$$

where $u^* \in \lambda F(u(x)) + \mu G(x,u(x))$ and $u^* \in L^\gamma([0,T])$ (for some $\gamma > 1$).

Lemma 2.10 ([16, Lemma 3.5])**.** *If a function $u \in X$ is a weak solution of* (1.1)*, then u is a classical solution of* (1.1)*.*

We introduce for a.e. $x \in [0,T]$ and all $s \in \mathbb{R}$, the Aumann-type set-valued integral

$$\int_0^s F(t)dt = \Big\{ \int_0^s f(t)dt \,:\, f : \mathbb{R} \to \mathbb{R} \text{ is a measurable selection of } F \Big\}$$

and set $\mathcal{F}(u) = \int_0^T \min \int_0^u F(s)\,ds\,dx$ for all $u \in L^p([0,T])$; the Aumann-type set-valued integral

$$\int_0^s G(x,t)dt = \Big\{ \int_0^s g(x,t)dt \,:\, g : [0,T] \times \mathbb{R} \to \mathbb{R} \text{ is a measurable selection of } G \Big\}$$

and set $\mathcal{G}(u) = \int_0^T \min \int_0^u G(x,s)\,ds\,dx$ for all $u \in L^p([0,T])$.

Lemma 2.11 ([10, Lemma 3.1])**.** *The functionals $\mathcal{F}, \mathcal{G} : L^p([0,T]) \to \mathbb{R}$ are well defined and Lipschitz on any bounded subset of $L^p([0,T])$. Moreover, for all $u \in L^p([0,T])$ and all $u^* \in \partial(\mathcal{F}(u) + \mathcal{G}(u))$,*

$$u^*(x) \in F(u(x)) + G(x,u(x)) \quad \text{for a.e. } x \in [0,T].$$

We define an energy functional for the problem (1.1) by setting

$$I_\lambda(u) = \frac{1}{p}\|u\|_X^p - \lambda\mathcal{F}(u) - \mu\mathcal{G}(u) - \sum_{i=1}^m \int_0^{u(x_i)} I_i(s)ds$$

for all $u \in X$.

Lemma 2.12 ([16, Lemma 4.4]). *The functional $I_\lambda : X \to \mathbb{R}$ is locally Lipschitz. Moreover, for each critical point $u \in X$ of I_λ, u is a weak solution of* (1.1).

3. Main results

We formulate our main result using the following assumptions:

(F4)
$$\begin{aligned}&\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F(s)ds}{\xi^p}\\&<\frac{1}{p}(\frac{2}{T})^p\limsup_{\xi\to+\infty}\frac{\int_0^T\min\int_0^{\xi(\frac{T}{2}-x)}F(s)\,ds\,dx}{\frac{1}{p}\xi^p\big(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\big)-\sum_{i=1}^m\int_0^{\xi(\frac{T}{2}-x_i)}I_i(s)ds};\end{aligned}$$

(I1) $I_i(0)=0$, $I_i(s)s<0$, $s\in\mathbb{R}$, $i=1,2,\dots,m$.

Theorem 3.1. *Assume that* (F1)–(F4), (I1) *hold. Let*
$$\lambda_1 := 1\Big/\limsup_{\xi\to+\infty}\frac{\int_0^T\min\int_0^{\xi(\frac{T}{2}-x)}F(s)\,ds\,dx}{\frac{1}{p}\xi^p\Big(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\Big)-\sum_{i=1}^m\int_0^{\xi(\frac{T}{2}-x_i)}I_i(s)ds},$$
$$\lambda_2 := 1\Big/\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F(s)ds}{\frac{1}{p}\Big(\frac{2\xi}{T}\Big)^p}.$$
Then, for every $\lambda\in(\lambda_1,\lambda_2)$, and every multifunction G satisfying

(G4) $\int_0^T\min\int_0^t G(x,s)\,ds\,dx\ge 0$ *for all $t\in\mathbb{R}$, and*

(G5) $G_\infty := \lim_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t G(x,s)ds}{\xi^p}<+\infty$,

if we put
$$\mu_{G,\lambda} := \frac{1}{pG_\infty}\frac{2^p}{T^p}\big(1-\lambda\frac{pT^p}{2^p}\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F(s)ds}{\xi^p}\big),$$
where $\mu_{G,\lambda}=+\infty$ when $G_\infty=0$, problem (1.1) *admits an unbounded sequence of solutions for every $\mu\in[0,\mu_{G,\lambda})$ in X.*

Proof. Our aim is to apply Theorem 2.6(b) to (1.1). To this end, we fix $\overline{\lambda}\in(\lambda_1,\lambda_2)$ and let G be a multifunction satisfying (G1)–(G5). Since $\overline{\lambda}<\lambda_2$, we have
$$\mu_{G,\overline{\lambda}} = \frac{1}{pG_\infty}\frac{2^p}{T^p}\big(1-\overline{\lambda}\frac{pT^p}{2^p}\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F(s)ds}{\xi^p}\big)>0.$$
Now fix $\overline{\mu}\in(0,\mu_{g,\overline{\lambda}})$, put $\nu_1:=\lambda_1$, and
$$\nu_2 := \frac{\lambda_2}{1+\frac{pT^p}{2^p}\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2\,G_\infty}.$$
If $G_\infty=0$, then $\nu_1=\lambda_1$, $\nu_2=\lambda_2$ and $\overline{\lambda}\in(\nu_1,\nu_2)$. If $G_\infty\neq 0$, since $\overline{\mu}<\mu_{G,\overline{\lambda}}$, we have
$$\frac{\overline{\lambda}}{\lambda_2}+\frac{pT^p}{2^p}\overline{\mu}\,G_\infty<1,$$
and so
$$\frac{\lambda_2}{1+\frac{pT^p}{2^p}\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2\,G_\infty}>\overline{\lambda},$$

namely, $\overline{\lambda} < \nu_2$. Hence, taking into account that $\overline{\lambda} > \lambda_1 = \nu_1$, one has $\overline{\lambda} \in (\nu_1, \nu_2)$. Now, set

$$J(x,s) := F(s) + \frac{\overline{\mu}}{\overline{\lambda}} G(x,s)$$

for all $(x,s) \in [0,T] \times \mathbb{R}$. Assume j identically zero in X and for each $u \in X$ put

$$\mathcal{N}(u) := \frac{1}{p}\|u\|_X^p - \sum_{i=1}^{m} \int_0^{u(x_i)} I_i(s)ds, \quad \Upsilon(u) := \int_0^T \min \int_0^u J(x,s)\,ds\,dx,$$
$$\mathcal{M}(u) := \Upsilon(u) - j(u) = \Upsilon(u),$$
$$I_{\overline{\lambda}}(u) := \mathcal{N}(u) - \overline{\lambda}\mathcal{M}(u) = \mathcal{N}(u) - \overline{\lambda}\Upsilon(u).$$

It is a simple matter to verify that $\mathcal{N}$ is sequentially weakly lower semicontinuous on X. Clearly, $\mathcal{N} \in C^1(X)$. By Lemma 2.1, $\mathcal{N}$ is locally Lipschitz on X. By Lemma 2.11, $\mathcal{F}$ and $\mathcal{G}$ are locally Lipschitz on $L^p([0,T])$. So, Υ is locally Lipschitz on $L^p([0,T])$. Moreover, X is compactly embedded into $L^p([0,T])$. So Υ is locally Lipschitz on X. Furthermore, Υ is sequentially weakly upper semicontinuous. For all $u \in X$, by (I_1),

$$\int_0^{u(x_i)} I_i(s)ds < 0, \quad i = 1, 2, \dots, m.$$

So, we have

$$\mathcal{N}(u) = \frac{1}{p}\|u\|_X^p - \sum_{i=1}^{m} \int_0^{u(x_i)} I_i(s)ds > \frac{1}{p}\|u\|_X^p$$

for all $u \in X$. Hence, $\mathcal{N}$ is coercive and $\inf_X \mathcal{N} = \mathcal{N}(0) = 0$. We want to prove that, under our hypotheses, there exists a sequence $\{\overline{u}_n\} \subset X$ of critical points for the functional $I_{\overline{\lambda}}$, that is, every element $\overline{u}_n$ satisfies

$$I^{\circ}_{\overline{\lambda}}(\overline{u}_n, v - \overline{u}_n) \geq 0, \quad \text{for every } v \in X.$$

Now, we claim that $\gamma < +\infty$. To see this, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \xi_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\sup_{|t|\leq\xi_n} \min \int_0^t J(x,s)ds}{\xi_n^p} = \liminf_{\xi\to+\infty} \frac{\sup_{|t|\leq\xi} \min \int_0^t J(x,s)ds}{\xi^p}. \tag{3.1}$$

Put

$$r_n := \frac{1}{p}\left(\frac{2\xi_n}{T^{1/q}}\right)^p, \quad \text{for all } n \in \mathbb{N}.$$

Then, for all $v \in X$ with $\mathcal{N}(v) < r_n$, taking into account that $\|v\|_X^p < pr_n$ and $\|v\|_{C^0} \leq \frac{1}{2}T^{1/q}\|v\|_X$, one has $|v(x)| \leq \xi_n$ for every $x \in [0,T]$. Therefore, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\mathcal{N}^{-1}((-\infty,r))} \frac{\big(\sup_{v\in\mathcal{N}^{-1}((-\infty,r))} \mathcal{M}(v)\big) - \mathcal{M}(u)}{r - \mathcal{N}(u)} \\
&\leq \frac{\sup_{\|v\|_X^p < pr_n} \big(\mathcal{F}(v) + \frac{\overline{\mu}}{\overline{\lambda}}\mathcal{G}(v)\big)}{r_n} \\
&\leq \frac{\sup_{|t|\leq\xi_n} \big(\int_0^T \min \int_0^t F(s)\,ds\,dx + \frac{\overline{\mu}}{\overline{\lambda}} \int_0^T \min \int_0^t G(x,s)\,ds\,dx\big)}{r_n} \\
&\leq p\Big(\frac{T}{2}\Big)^p \Big[\frac{\sup_{|t|\leq\xi_n} \min \int_0^t F(s)ds}{\xi_n^p} + \frac{\overline{\mu}}{\overline{\lambda}} \frac{\sup_{|t|\leq\xi_n} \min \int_0^t G(x,s)ds}{\xi_n^p}\Big].
\end{aligned}$$

Moreover, from Assumptions (F4) and (G5), we have

$$\lim_{n\to+\infty}\frac{\sup_{|t|\le\xi_n}\min\int_0^t F(s)ds}{\xi_n^p}+\lim_{n\to+\infty}\frac{\overline{\mu}}{\overline{\lambda}}\frac{\sup_{|t|\le\xi_n}\min\int_0^t G(x,s)ds}{\xi_n^p}<+\infty,$$

which follows

$$\lim_{n\to+\infty}\frac{\sup_{|t|\le\xi_n}\min\int_0^t J(x,s)ds}{\xi_n^p}<+\infty.$$

Therefore,

$$\gamma\le\liminf_{n\to+\infty}\varphi(r_n)\le p\Big(\frac{T}{2}\Big)^p\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t J(x,s)ds}{\xi^p}<+\infty. \tag{3.2}$$

Since

$$\frac{\sup_{|t|\le\xi}\min\int_0^t J(x,s)ds}{\xi^p}\le\frac{\sup_{|t|\le\xi}\min\int_0^t F(s)ds}{\xi^p}+\frac{\overline{\mu}}{\overline{\lambda}}\frac{\sup_{|t|\le\xi}\min\int_0^t G(x,s)ds}{\xi^p},$$

and taking (G5) into account, we get

$$\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t J(x,s)ds}{\xi^p}\le\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F(s)ds}{\xi^p}+\frac{\overline{\mu}}{\overline{\lambda}}G_\infty. \tag{3.3}$$

Moreover, from Assumption (G4) we obtain

$$\begin{aligned}
&\limsup_{\xi\to+\infty}\frac{\int_0^T\min\int_0^{\xi(\frac{T}{2}-x)}J(x,s)\,ds\,dx}{\frac{1}{p}\xi^p\big(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\big)-\sum_{i=1}^m\int_0^{\xi(\frac{T}{2}-x_i)}I_i(s)ds}\\
&\ge\limsup_{\xi\to+\infty}\frac{\int_0^T\min\int_0^{\xi(\frac{T}{2}-x)}F(s)\,ds\,dx}{\frac{1}{p}\xi^p\Big(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\Big)-\sum_{i=1}^m\int_0^{\xi(\frac{T}{2}-x_i)}I_i(s)ds}.
\end{aligned} \tag{3.4}$$

Therefore, from (3.3) and (3.4), we observe that

$$\begin{aligned}
\overline{\lambda}\in(\nu_1,\nu_2)\subseteq\Big(&\frac{1}{\limsup_{\xi\to+\infty}\frac{\int_0^T\min\int_0^{\xi(\frac{T}{2}-x)}J(x,s)\,ds\,dx}{\frac{1}{p}\xi^p\big(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\big)-\sum_{i=1}^m\int_0^{\xi(\frac{T}{2}-x_i)}I_i(s)ds}},\\
&\frac{1}{\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t J(x,s)ds}{\frac{1}{p}(\frac{2\xi}{T})^p}}\Big)\\
&\subseteq(0,1/\gamma).
\end{aligned}$$

For the fixed $\overline{\lambda}$, the inequality (3.2) ensures that the condition (b) of Theorem 2.6 can be applied and either $I_{\overline{\lambda}}$ has a global minimum or there exists a sequence $\{u_n\}$ of weak solutions of the problem (1.1) such that $\lim_{n\to\infty}\|u_n\|=+\infty$, The other step is to show that for the fixed $\overline{\lambda}$ the functional $I_{\overline{\lambda}}$ has no global minimum. Let us verify that the functional $I_{\overline{\lambda}}$ is unbounded from below. Since

$$\begin{aligned}
\frac{1}{\overline{\lambda}}&<\limsup_{\xi\to+\infty}\frac{\int_0^T\min\int_0^{\xi(\frac{T}{2}-x)}F(s)\,ds\,dx}{\frac{1}{p}\xi^p\big(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\big)-\sum_{i=1}^m\int_0^{\xi(\frac{T}{2}-x_i)}I_i(s)ds}\\
&\le\limsup_{\xi\to+\infty}\frac{\int_0^T\min\int_0^{\xi(\frac{T}{2}-x)}J(x,s)\,ds\,dx}{\frac{1}{p}\xi^p\big(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\big)-\sum_{i=1}^m\int_0^{\xi(\frac{T}{2}-x_i)}I_i(s)ds},
\end{aligned}$$

there exists a sequence $\{\eta_n\}$ of positive numbers and a constant τ such that $\lim_{n\to+\infty}\eta_n=+\infty$ and

$$\frac{1}{\overline{\lambda}}<\tau<\frac{\int_0^T \min\int_0^{\eta_n(\frac{T}{2}-x)} J(x,s)\,ds\,dx}{\frac{1}{p}{\eta_n}^p\big(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\big)-\sum_{i=1}^m\int_0^{\eta_n(\frac{T}{2}-x_i)} I_i(s)ds} \tag{3.5}$$

for each $n\in\mathbb{N}$ large enough. For all $n\in\mathbb{N}$, set

$$w_n(x):=\eta_n\big(\frac{T}{2}-x\big).$$

For any fixed $n\in\mathbb{N}$, it is easy to see that $w_n\in X$ and, in particular, one has

$$\|w_n\|_X^p=\eta_n^p\Big(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\Big),$$

and so

$$\mathcal{N}(w_n)=\frac{1}{p}\eta_n^p\Big(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\Big)-\sum_{i=1}^m\int_0^{\eta_n(\frac{T}{2}-x_i)} I_i(s)ds. \tag{3.6}$$

By (3.5) and (3.6), we see that

$$\begin{aligned}
I_{\overline{\lambda}}(w_n)&=\mathcal{N}(w_n)-\overline{\lambda}\mathcal{M}(w_n)\\
&=\frac{1}{p}\eta_n^p\Big(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\Big)-\sum_{i=1}^m\int_0^{\eta_n(\frac{T}{2}-x_i)} I_i(s)ds\\
&\quad-\overline{\lambda}\int_0^T\min\int_0^{\eta_n(\frac{T}{2}-x)} J(x,s)\,ds\,dx\\
&<\Big(\frac{1}{p}\eta_n^p\Big(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\Big)-\sum_{i=1}^m\int_0^{\eta_n(\frac{T}{2}-x_i)} I_i(s)ds\Big)(1-\overline{\lambda}\tau)
\end{aligned}$$

for every $n\in\mathbb{N}$ large enough. Since $\overline{\lambda}\tau>1$ and $\lim_{n\to+\infty}\eta_n=+\infty$, we have

$$\lim_{n\to+\infty} I_{\overline{\lambda}}(w_n)=-\infty.$$

Then, the functional $I_{\overline{\lambda}}$ is unbounded from below, and it follows that $I_{\overline{\lambda}}$ has no global minimum. Therefore, from part (b) of Theorem 2.6, the functional $I_{\overline{\lambda}}$ admits a sequence of critical points $\{\overline{u}_n\}\subset X$ such that $\lim_{n\to+\infty}\mathcal{N}(\overline{u}_n)=+\infty$. Since $\mathcal{N}$ is bounded on bounded sets, and taking into account that $\lim_{n\to+\infty}\mathcal{N}(\overline{u}_n)=+\infty$, then $\{\overline{u}_n\}$ has to be unbounded, i.e.,

$$\lim_{n\to+\infty}\|\overline{u}_n\|_X=+\infty.$$

Moreover, if $\overline{u}_n\in X$ is a critical point of $I_{\overline{\lambda}}$, clearly, by definition, one has

$$I_{\overline{\lambda}}^\circ(\overline{u}_n,v-\overline{u}_n)\geq 0,\quad \text{for every } v\in X.$$

Finally, by Lemma 2.12, the critical points of $I_{\overline{\lambda}}$ are weak solutions for the problem (1.1), and by Lemma 2.10, every weak solution of (1.1) is a solution of (1.1). Hence, the assertion follows. □

Remark 3.2. Under the conditions

$$\liminf_{\xi\to+\infty}\frac{\sup_{|t|\leq\xi}\min\int_0^t F(s)ds}{\xi^p}=0,$$

$$\limsup_{\xi\to+\infty}\frac{\int_0^T \min\int_0^{\xi(\frac{T}{2}-x)} F(s)\,ds\,dx}{\frac{1}{p}\xi^p\big(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\big)-\sum_{i=1}^m\int_0^{\xi(\frac{T}{2}-x_i)} I_i(s)ds}=+\infty,$$

from Theorem 3.1, we see that for every $\lambda > 0$ and for each $\mu \in \big[0, \frac{2^p}{pT^pG_\infty}\big)$, problem (1.1) admits a sequence of solutions which is unbounded in X. Moreover, if $G_\infty = 0$, the result holds for every $\lambda > 0$ and $\mu \geq 0$.

The following result is a special case of Theorem 3.1 with $\mu = 0$.

Theorem 3.3. *Assume that* (F1)–(F4), (I1) *hold. Then, for each*

$$\lambda\in\Bigg(\frac{1}{\limsup_{\xi\to+\infty}\frac{\int_0^T \min\int_0^{\xi(\frac{T}{2}-x)} F(s)\,ds\,dx}{\frac{1}{p}\xi^p\big(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\big)-\sum_{i=1}^m\int_0^{\xi(\frac{T}{2}-x_i)} I_i(s)ds}},\quad \frac{1}{\liminf_{\xi\to+\infty}\frac{\sup_{|t|\leq\xi}\min\int_0^t F(s)ds}{\frac{1}{p}(\frac{2\xi}{T})^p}}\Bigg),$$

the problem

$$\begin{gathered}
-(\phi_p(u'(x)))'+M\phi_p(u(x))\in\lambda F(u(x))\quad \text{in } [0,T]\setminus Q,\\
-\Delta\phi_p(u'(x_k))=I_k(u(x_k)),\quad k=1,2,\dots,m,\\
u(0)=-u(T),\quad u'(0)=-u'(T)
\end{gathered}$$

has an unbounded sequence of solutions in X.

Now, we present the following example to illustrate our results.

Example 3.4. Consider the problem

$$\begin{gathered}
-(\phi_3(u'(x)))'+\phi_3(u(x))\in\lambda F(u(x))\quad \text{in } [0,2]\setminus\{1\},\\
-\Delta\phi_3(u'(x_1))=I_1(u(x_1)),\quad x_1=1,\\
u(0)=-u(2),\quad u'(0)=-u'(2),
\end{gathered}\tag{3.7}$$

where, for $s \in \mathbb{R}$,

$$F(s)=\begin{cases}\{0\}, & \text{if } |s|<2^{-1/3},\\ [0,1], & \text{if } |s|=2^{-1/3},\\ \{s-2^{-1/3}+1\}, & \text{if } s>2^{-1/3},\\ \{s+2^{-1/3}+1\}, & \text{if } s<-2^{-1/3}.\end{cases}$$

Simple calculations show that

$$\sup_{|t|\leq 2^{-1/3}}\min\int_0^t F(s)ds=0$$

and

$$\begin{aligned}
&\frac{\int_0^2\min\int_0^{\xi(1-x)}F(s)\,ds\,dx}{\frac{5}{6}\xi^3-\int_0^{\xi(1-x_1)}I_1(s)ds}\\
&=\frac{6}{5}\frac{1}{\xi^3}\int_{-1}^1\min\int_0^{\xi x}F(s)\,ds\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{6}{5}\frac{1}{\xi^3}\Big(\int_{-1}^{-2^{-1/3}}\int_0^{\xi x}\max F(s)\,ds\,dx + \int_{-2^{-1/3}}^{0}\int_0^{\xi x}\max F(s)\,ds\,dx \\
&\quad + \int_0^{2^{-1/3}}\int_0^{\xi x}\max F(s)\,ds\,dx + \int_{2^{-1/3}}^{1}\int_0^{\xi x}\max F(s)\,ds\,dx\Big) > 0
\end{aligned}
$$

for some $\xi \in \mathbb{R}$. So,

$$
\begin{gathered}
\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F(s)ds}{\frac{1}{3}\xi^3} = 0, \\
\limsup_{\xi\to+\infty}\frac{\int_0^2\min\int_0^{\xi(1-x)}F(s)\,ds\,dx}{\frac{5}{6}\xi^3 - \int_0^{\xi(1-x_1)}I_1(s)ds} > 0.
\end{gathered}
$$

Hence, using Theorem 3.3, problem (3.7), for λ lying in a convenient interval, has an unbounded sequence of solutions in $X := \{u \in W^{1,3}([0,2]) : u(0) = -u(2)\}$.

Here we point out the following consequences of Theorem 3.3, using the assumptions

- (F5) $\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F(s)ds}{\xi^p} < \frac{1}{p}\left(\frac{2}{T}\right)^p$;
- (F6) $\limsup_{\xi\to+\infty}\frac{\int_0^T\min\int_0^{\xi(\frac{T}{2}-x)}F(s)\,ds\,dx}{\frac{1}{p}\xi^p\left(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\right)-\sum_{i=1}^m\int_0^{\xi(\frac{T}{2}-x_i)}I_i(s)ds} > 1$.

Corollary 3.5. *Assume that* (F1)–(F3), (F5)–(F6), (I1) *hold. Then, the problem*

$$
\begin{gathered}
-(\phi_p(u'(x)))' + M\phi_p(u(x)) \in F(u(x)) \quad \text{in } [0,T]\setminus Q, \\
-\Delta\phi_p(u'(x_k)) = I_k(u(x_k)), \quad k = 1,2,\dots,m, \\
u(0) = -u(T), \quad u'(0) = -u'(T)
\end{gathered}
$$

has an unbounded sequence of solutions in X.

Remark 3.6. *Theorem 1.1 in the Introduction is an immediate consequence of Corollary 3.5.*

Now, we give the following consequence of the main result.

Corollary 3.7. *Let $F_1 : \mathbb{R} \to 2^{\mathbb{R}}$ be an upper semicontinuous multifunction with compact convex values, such that* $\min F_1, \max F_1 : \mathbb{R} \to \mathbb{R}$ *are Borel measurable and $|\xi| \le a(1+|s|^{r_1-1})$ for all $s \in \mathbb{R}$, $\xi \in F_1(s)$, $r_1 > 1\,(a > 0)$. Furthermore, suppose that*

- (C1) $\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}\min\int_0^t F_1(s)ds}{\xi^p} < +\infty$;
- (C2) $\limsup_{\xi\to+\infty}\frac{\int_0^T\min\int_0^{\xi(\frac{T}{2}-x)}F_1(s)\,ds\,dx}{\frac{1}{p}\xi^p\left(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\right)-\sum_{i=1}^m\int_0^{\xi(\frac{T}{2}-x_i)}I_i(s)ds} = +\infty$.

Then, for every multifunction $F_2 : \mathbb{R} \to 2^{\mathbb{R}}$ which is upper semicontinuous with compact convex values, $\min F_2, \max F_2 : \mathbb{R} \to \mathbb{R}$ *are Borel measurable and $|\xi| \le b(1+|s|^{r_2-1})$ for all $s \in \mathbb{R}$, $\xi \in F_2(s)$, $r_2 > 1\,(b > 0)$, and satisfies the conditions*

$$
\sup_{t\in\mathbb{R}}\min\int_0^t F_2(s)ds \le 0
$$

and

$$
\liminf_{\xi\to+\infty}\frac{\int_0^T\min\int_0^{\xi(\frac{T}{2}-x)}F_2(s)\,ds\,dx}{\frac{1}{p}\xi^p\left(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\right)-\sum_{i=1}^m\int_0^{\xi(\frac{T}{2}-x_i)}I_i(s)ds} > -\infty,
$$

for each

$$\lambda \in \Big(0, \ \frac{1}{\liminf_{\xi\to+\infty} \frac{\sup_{|t|\le\xi} \min\int_0^t F_1(s)ds}{\frac{1}{p}(\frac{2\xi}{T})^p}}\Big),$$

and the problem

$$\begin{gathered} -(\phi_p(u'(x)))' + M\phi_p(u(x)) \in \lambda(F_1(u(x)) + F_2(u(x))) \quad \text{in } [0,T]\setminus Q, \\ -\Delta\phi_p(u'(x_k)) = I_k(u(x_k)), \quad k = 1,2,\dots,m, \\ u(0) = -u(T), \quad u'(0) = -u'(T) \end{gathered}$$

has an unbounded sequence of solutions in X.

Proof. Set $F(t) = F_1(t) + F_2(t)$ for all $t \in \mathbb{R}$. Assumption (C2) along with the condition

$$\liminf_{\xi\to+\infty} \frac{\int_0^T \min\int_0^{\xi(\frac{T}{2}-x)} F_2(s)\,ds\,dx}{\frac{1}{p}\xi^p\Big(T + \frac{2M}{p+1}(\frac{T}{2})^{p+1}\Big) - \sum_{i=1}^m \int_0^{\xi(\frac{T}{2}-x_i)} I_i(s)ds} > -\infty$$

yield

$$\begin{aligned} &\limsup_{\xi\to+\infty} \frac{\int_0^T \min\int_0^{\xi(\frac{T}{2}-x)} F(s)\,ds\,dx}{\frac{1}{p}\xi^p\big(T + \frac{2M}{p+1}(\frac{T}{2})^{p+1}\big) - \sum_{i=1}^m \int_0^{\xi(\frac{T}{2}-x_i)} I_i(s)ds} \\ &= \limsup_{\xi\to+\infty} \frac{\int_0^T \min\int_0^{\xi(\frac{T}{2}-x)} F_1(s)\,ds\,dx + \int_0^T \min\int_0^{\xi(\frac{T}{2}-x)} F_2(s)\,ds\,dx}{\frac{1}{p}\xi^p\big(T + \frac{2M}{p+1}(\frac{T}{2})^{p+1}\big) - \sum_{i=1}^m \int_0^{\xi(\frac{T}{2}-x_i)} I_i(s)ds} = +\infty. \end{aligned}$$

Moreover, Assumption (C1) and the condition

$$\sup_{t\in\mathbb{R}} \min\int_0^t F_2(s)ds \le 0$$

ensure that

$$\liminf_{\xi\to+\infty} \frac{\sup_{|t|\le\xi} \min\int_0^t F(s)ds}{\xi^p} \le \liminf_{\xi\to+\infty} \frac{\sup_{|t|\le\xi} \min\int_0^t F_1(s)ds}{\xi^p} < +\infty.$$

Since

$$\frac{1}{\liminf_{\xi\to+\infty} \frac{\sup_{|t|\le\xi} \min\int_0^t F(s)ds}{\frac{1}{p}(\frac{2\xi}{T})^p}} \ge \frac{1}{\liminf_{\xi\to+\infty} \frac{\sup_{|t|\le\xi} \min\int_0^t F_1(s)ds}{\frac{1}{p}(\frac{2\xi}{T})^p}},$$

by applying Theorem 3.3, we have the desired conclusion. □

Remark 3.8. We observe that in Theorem 3.1 we can replace $\xi \to +\infty$ with $\xi \to 0^+$, and then by the same argument as in the proof of Theorem 3.1, but using conclusion (c) of Theorem 2.6 instead of (b), problem (1.1) has a sequence of solutions, which strongly converges to 0 in X.

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References

[1] G. Bonanno, G. Molica Bisci; Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, *Boundary Value Problems*, **2009** (2009), 1-20.

[2] G. Bonanno, G. Molica Bisci, V. Rădulescu; Existence of three solutions for a non-homogeneous Neumann problem through Orlicz-Sobolev spaces, *Nonlinear Anal.*, **74** (2011), 4785-4795.

[3] H. L. Chen; Antiperiodic wavelets, *J. Comput. Math.*, **14** (1996), 32-39.

[4] F. H. Clarke; *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[5] F. J. Delvos, L. Knoche; Lacunary interpolation by antiperiodic trigonometric polynomials, *BIT*, **39** (1999), 439-450.

[6] P. Djakov, B. Mityagin; Simple and double eigenvalues of the Hill operator with a two-term potential, *J. Approx. Theory*, **135** (2005), 70-104.

[7] L. H. Erbe, W. Krawcewicz; Existence of solutions to boundary value problems for impulsive second order differential inclusions, *Rocky Mountain J. Math.*, **22** (1992), 519-539.

[8] M. Frigon and D. O'Regan; First order impulsive initial and periodic problems with variable moments, *J. Math. Anal. Appl.*, **233** (1999), 730-739.

[9] A. Iannizzotto; Three critical points for perturbed nonsmooth functionals and applications, *Nonlinear Anal.*, **72** (2010), 1319-1338.

[10] A. Iannizzotto; Three periodic solutions for an ordinary differential inclusion with two parameters, *Ann. Polon. Math.*, **103** (2012), 89-100.

[11] A. Kristály; Infinitely many solutions for a differential inclusion problem in $\mathbb{R}^N$, *J. Differential Equations*, **220** (2006), 511-530.

[12] A. Kristály, W. Marzantowicz, C. Varga; A non-smooth three critical points theorem with applications in differential inclusions, *J. Glob. Optim.*, **46** (2010), 49-62.

[13] S. A. Marano, D. Motreanu; Infinitely many critical points of non-differentiable functions and applications to a Neumann-type problem involving the p-Laplacian, *J. Differential Equations*, **182** (2002), 108-120.

[14] D. Motreanu, P.D. Panagiotopoulos; *Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities*, Kluwer Academic Publishers, Dordrecht, 1999.

[15] B. Ricceri; A general variational principle and some of its applications, *J. Comput. Appl. Math.*, **113** (2000), 401-410.

[16] Y. Tian, J. Henderson; Three anti-periodic solutions for second-order impulsive differential inclusions via nonsmooth critical point theory, *Nonlinear Anal.*, **75** (2012), 6496-6505.

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