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ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS OF A SEMILINEAR DIRICHLET PROBLEM OUTSIDE THE UNIT BALL

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Abstract. In this article, we are concerned with the existence, uniqueness and asymptotic behavior of a positive classical solution to the semilinear boundary-value problem

$$-\Delta u = a(x)u^\sigma \quad \text{in } D,$$
$$\lim_{|x|\to 1} u(x) = \lim_{|x|\to\infty} u(x) = 0.$$

Here D is the complement of the closed unit ball of $\mathbb{R}^n$ $(n \geq 3)$, $\sigma < 1$ and the function a is a nonnegative function in $C^\gamma_{\rm loc}(D)$, $0 < \gamma < 1$, satisfying some appropriate assumptions related to Karamata regular variation theory.

1. Introduction

The semilinear elliptic equation

$$-\Delta u = a(x)u^\sigma, \quad \sigma < 1, \quad x \in \Omega \subset \mathbb{R}^n,$$

has been extensively studied for both bounded and unbounded domains Ω in $\mathbb{R}^n$ $(n \geq 2)$. We refer to [2, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 24, 26] and the references therein, for various existence and uniqueness results related to solutions for the above equation with homogeneous Dirichlet boundary conditions. The asymptotic behavior of such solutions interested many authors who developed the Karamata regular variation theory (see [3, 7, 9, 15, 21, 28]).

Mâagli [21] considered the problem

$$\begin{gathered} -\Delta u = a(x)u^\sigma \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded $C^{1,1}$-domain, $\sigma < 1$ and a satisfies an assumption related to $\mathcal{K}_0$ the set of Karamata functions regularly varying at zero (see Definition 1.1 below).

Thanks to the sub-supersolution method and using some potential theory tools, Mâagli [21] showed that (1.1) has a unique positive classical solution and gave sharp estimates on the solution. These estimates improve and extend those stated

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in [10, 15, 20, 22, 28]. Before stating the result proved in [21], we shall recall the definition of the Karamata class $\mathcal{K}_0$.

Definition 1.1. A measurable function L is in $\mathcal{K}_0$ if there exist $\eta > 0$ such that L is a positive function in $C^1((0,\eta])$ satisfying

$$\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0.$$

As a typical example of function $L \in \mathcal{K}_0$, we have

$$L(t) = \prod_{i=1}^{m} (\log_i(\frac{2\eta}{t}))^{-\mu_i},$$

where $\log_i x = \log \circ \log \circ \dots \circ \log x$ (i times), $\mu_i \in \mathbb{R}$. Throughout this paper, for two nonnegative functions f and g defined on a set S, the notation $f(x) \approx g(x)$, $x \in S$, means that there exists a constant $c > 0$ such that for each $x \in S$, $\frac{1}{c}\, g(x) \le f(x) \le c\, g(x)$. Further, we denote by $\delta(x) = dist(x, \partial\Omega)$. Also for $\mu \le 2$, $\sigma < 1$ and $L \in \mathcal{K}_0$ defined on $(0,\eta]$, $\eta > 0$ such that $\int_0^\eta t^{1-\mu} L(t)dt < \infty$, we put $\Phi_{L,\mu,\sigma}$ the function defined on $(0,\eta)$ by

$$\Phi_{L,\mu,\sigma}(t) := \begin{cases} 1, & \text{if } \mu < 1+\sigma, \\ \big(\int_t^\eta \frac{L(s)}{s} ds\big)^{1/(1-\sigma)}, & \text{if } \mu = 1+\sigma, \\ (L(t))^{1/(1-\sigma)}, & \text{if } 1+\sigma < \mu < 2, \\ \big(\int_0^t \frac{L(s)}{s} ds\big)^{1/(1-\sigma)}, & \text{if } \mu = 2. \end{cases}$$

Now, let us present the result by Mâagli in [21].

Theorem 1.2. *Let $a \in C^\gamma_{\rm loc}(\Omega)$, $0 < \gamma < 1$, satisfying*

$$a(x) \approx \delta(x)^{-\mu} L(\delta(x)) \quad \textit{for } x \in \Omega,$$

where $\mu \le 2$, $L \in \mathcal{K}_0$ defined on $(0,\eta]$, $(\eta > \operatorname{diam}(\Omega))$ such that $\int_0^\eta t^{1-\mu} L(t)dt < \infty$. Then problem (1.1) has a unique positive classical solution u satisfying

$$u(x) \approx \delta(x)^{\min(1,\frac{2-\mu}{1-\sigma})} \Phi_{L,\mu,\sigma}(\delta(x)) \quad \textit{for each } x \in \Omega.$$

Chemmam et al [7] were concerned with $\mathcal{K}_\infty$ the set of Karamta functions regularly varying at infinity (see Definition 1.4 below). More precisely, by using properties of functions in $\mathcal{K}_\infty$, the authors studied the asymptotic behavior of the unique classical solution of the problem

$$\begin{gathered} -\Delta u = a(x)u^\sigma \quad \text{in } \mathbb{R}^n, \\ u > 0 \quad \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = 0, \end{gathered} \tag{1.2}$$

where $n \ge 3$ and $\sigma < 1$. The existence of a unique classical solution of (1.2) has been proved in [4, 19]. Namely, Chemmam et al in [7] proved the following result.

Theorem 1.3. *Let $a \in C^\gamma_{\rm loc}(\mathbb{R}^n)$, $0 < \gamma < 1$, satisfying*

$$a(x) \approx (1+|x|)^{-\lambda} L(1+|x|) \quad \textit{for } x \in \mathbb{R}^n,$$

where $\lambda \geq 2$, $L \in \mathcal{K}_\infty$ such that $\int_1^\infty t^{1-\lambda} L(t)dt < \infty$. Then the solution u of problem (1.2) *satisfies*

$$u(x) \approx \frac{1}{(1+|x|)^{\min(\frac{\lambda-2}{1-\sigma}, n-2)}} \Psi_{L,\lambda,\sigma}(1+|x|) \quad \text{for each } x \in \mathbb{R}^n.$$

In this article, for $\lambda \geq 2$, $\sigma < 1$ and $L \in \mathcal{K}_\infty$ such that $\int_1^\infty t^{1-\lambda} L(t)dt < \infty$, the function $\Psi_{L,\lambda,\sigma}$ is defined on $[1,\infty)$ by

$$\Psi_{L,\lambda,\sigma}(t) := \begin{cases} \left(\int_t^\infty \frac{L(s)}{s} ds\right)^{1/(1-\sigma)}, & \text{if } \lambda = 2, \\ (L(t))^{1/(1-\sigma)}, & \text{if } 2 < \lambda < n - \sigma(n-2), \\ \left(\int_1^{t+1} \frac{L(s)}{s} ds\right)^{1/(1-\sigma)}, & \text{if } \lambda = n - \sigma(n-2), \\ 1, & \text{if } \lambda > n - \sigma(n-2). \end{cases}$$

For the convenience of the readers, we recall the definition of the Karamata class $\mathcal{K}_\infty$.

Definition 1.4. A measurable function L defined on $[1,\infty)$ is in $\mathcal{K}_\infty$ if L is a positive function in $C^1([1,\infty))$ such that

$$\lim_{t\to\infty} \frac{tL'(t)}{L(t)} = 0.$$

As a typical example of function $L \in \mathcal{K}_\infty$, we have

$$L(t) = \prod_{i=1}^m (log_i(\omega t))^{-\lambda_i},$$

where ω is a positive real number sufficiently large and $\lambda_i \in \mathbb{R}$.

In this paper, we are concerned with the existence, uniqueness and estimates of positive classical solutions to the semilinear Dirichlet problem

$$\begin{gathered} -\Delta u = a(x)u^\sigma \quad \text{in } D, \\ u > 0 \quad \text{in } D, \\ \lim_{|x|\to 1} u(x) = \lim_{|x|\to\infty} u(x) = 0, \end{gathered} \tag{1.3}$$

where $D = \{x \in \mathbb{R}^n : |x| > 1\}$ is the complementary of the closed unit ball of $\mathbb{R}^n$ ($n \geq 3$) and $\sigma < 1$. The importance of the sublinear case ($0 \leq \sigma < 1$) in applications has been widely recognized for many years, see for example Wong [27] for an extensive bibliography and the significance of the case $\sigma < 0$ has been noticed in studies of boundary layer phenomena for viscous fluids [5, 6]. The main feature of this paper is the presence of homogeneous Dirichlet boundary conditions which combines those of [7] and [21]. For this reason the condition imposed on the function a in problem (1.3) is an appropriate assumption related to $\mathcal{K}_0$ and $\mathcal{K}_\infty$. To simplify our statements, we call $\mathcal{K}$ the set of functions L defined on $(0,\infty)$ by

$$L(t) := M\left(\frac{t}{1+t}\right)N(1+t),$$

where $M \in \mathcal{K}_0$ defined on $(0,\eta]$ for some $\eta > 1$ and $N \in \mathcal{K}_\infty$. Also, for $x \in D$, we denote by $\rho(x) = 1 - \frac{1}{|x|}$. Let us consider the following hypothesis.

(H1) a is a positive function in $C^\gamma_{\rm loc}(D)$, $0 < \gamma < 1$, satisfying for $x \in D$,

$$a(x) \approx \frac{L(|x|-1)}{|x|^{\lambda-\mu}(|x|-1)^\mu},$$

where $\mu \leq 2 \leq \lambda$, $L \in \mathcal{K}$ such that $\int_0^\infty t^{1-\mu}\,(1+t)^{\mu-\lambda} L(t)dt < \infty$.

To illustrate (H1), we give an example. Let a be the positive function defined on D by

$$a(x) = \frac{(\log(\frac{4|x|}{|x|-1})\,\log(2|x|))^{-\alpha}}{|x|^{\lambda-\mu}\,(|x|-1)^{\mu}},$$

where the real numbers μ, λ and α satisfy one of the following conditions

- $\mu < 2 < \lambda$ and $\alpha \in \mathbb{R}$;
- $\mu = 2$, $\lambda > 2$ and $\alpha > 1$;
- $\mu \leq 2$, $\lambda = 2$ and $\alpha > 1$.

Then the function a satisfies (H1). Our main result in this paper is the following.

Theorem 1.5. *Assume* (H1)*, then problem* (1.3) *has a unique classical solution* u *satisfying*

$$u(x) \approx \theta(x), \quad x \in D, \tag{1.4}$$

where

$$\theta(x) := \frac{(\rho(x))^{\min(\frac{2-\mu}{1-\sigma},1)}}{|x|^{\min(\frac{\lambda-2}{1-\sigma},n-2)}}\,\Phi_{M,\mu,\sigma}(\rho(x))\,\Psi_{N,\lambda,\sigma}(|x|). \tag{1.5}$$

The techniques used for proving Theorem 1.5 are based on the sub-supersolution method. For the convenience of the readers, we shall recall the following definitions. A positive function $v \in C^{2,\gamma}(D)$, $0 < \gamma < 1$, is called a subsolution of problem (1.3) if

$$\begin{gathered} -\Delta v \leq a(x)\,v^{\sigma} \quad \text{in } D, \\ \lim_{|x|\to 1} v(x) = \lim_{|x|\to\infty} v(x) = 0. \end{gathered}$$

As always, a supersolution is defined by reversing the inequality.

Since our approach is based on potential theory tools, we lay out some basic arguments that we are mainly concerned with in this work. An explicit expression for the Green function G of the Laplace operator Δ in D with zero Dirichlet boundary conditions is given in [1] by

$$G(x,y) = \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}}(|x-y|^{2-n} - [x,y]^{2-n}), \quad x, y \in D,$$

where $[x,y]^2 = |x-y|^2 + (|x|^2-1)(|y|^2-1)$.

We refer in this paper to the potential of a nonnegative measurable function f defined on D by

$$Vf(x) = \int_D G(x,y)\,f(y)dy,\ x \in D.$$

Recall that for each nonnegative function f in $C^{\gamma}_{\rm loc}(D)$, $0 < \gamma < 1$, such that $Vf \in L^\infty(D)$, we have $Vf \in C^{2,\gamma}_{\rm loc}(D)$ and satisfies $-\Delta(Vf) = f$ in D; see [24, Theorem 6.6].

The rest of the paper is organized as follows. In Section 2, we state and prove some preliminary lemmas, involving some already known results on functions in $\mathcal{K}_0$ and $\mathcal{K}_\infty$. In Section 3, we give estimates on some potential functions. Section 4 is devoted to the proof of Theorem 1.5. The last Section is reserved for an application.

2. Karamata regular variation theory

2.1. **On the Karamata class $\mathcal{K}_0$.** In what follows, we recall some fundamental properties of Karamata functions regularly varying at zero.

Lemma 2.1 (Karamata's Theorem [25])**.** *Let $\gamma \in \mathbb{R}$ and $L \in \mathcal{K}_0$ defined on $(0,\eta]$, $\eta > 1$. Then we have the following assertions*

(i) *If $\gamma < 2$, then $\int_0^\eta t^{1-\gamma}L(t)dt$ converges and $\int_0^t s^{1-\gamma}L(s)ds \sim \frac{t^{2-\gamma}L(t)}{2-\gamma}$ as $t \to 0^+$;*
(ii) *If $\gamma > 2$, then $\int_0^\eta t^{1-\gamma}L(t)dt$ diverges and $\int_t^\eta s^{1-\gamma}L(s)ds \sim \frac{t^{2-\gamma}L(t)}{\gamma-2}$ as $t \to 0^+$.*

Lemma 2.2 ([8, 25])**.** *Let $L_1, L_2 \in \mathcal{K}_0$ defined on $(0,\eta]$, $\eta > 1$, $p \in \mathbb{R}$ and $\varepsilon > 0$. Then we have the following assertions:*

(i) *$L_1L_2 \in \mathcal{K}_0$ and $L_1^p \in \mathcal{K}_0$;*
(ii) *$\lim_{t\to 0^+} t^\varepsilon L_1(t) = 0$;*
(iii) *$\lim_{t\to 0^+} \frac{L_1(t)}{\int_t^\eta \frac{L_1(s)}{s}ds} = 0$ and $t \mapsto \int_t^\eta \frac{L_1(s)}{s}ds \in \mathcal{K}_0$;*
(iv) *If $\int_0^\eta \frac{L_1(s)}{s}ds$ converges, then $\lim_{t\to 0^+} \frac{L_1(t)}{\int_0^t \frac{L_1(s)}{s}ds} = 0$ and $t \mapsto \int_0^t \frac{L_1(s)}{s}ds \in \mathcal{K}_0$.*

Lemma 2.3. *If $L \in \mathcal{K}_0$ defined on $(0,\eta]$, $\eta > 1$, then there exists $m \geq 0$ such that for each $t \in (0,1]$, we have*

$$2^{-m}L(t) \leq L(\frac{t}{1+t}) \leq 2^m L(t).$$

Proof. Since $L \in \mathcal{K}_0$, then by the representation theorem [25], there exist $c > 0$ and $z \in C([0,\eta])$ such that $z(0) = 0$ and satisfying for each $r \in (0,\eta]$, $L(r) = c\exp(\int_r^\eta \frac{z(s)}{s}ds)$.

Put $m := \sup_{s\in[0,\eta]} |z(s)|$. Then for each $t \in (0,1]$, we have

$$-m\log(1+t) \leq \int_{\frac{t}{1+t}}^t \frac{z(s)}{s}ds \leq m\log(1+t).$$

This implies

$$-m\log(2) \leq \int_{\frac{t}{1+t}}^t \frac{z(s)}{s}ds \leq m\log(2).$$

That is,

$$2^{-m} \leq \exp\Big(\int_{\frac{t}{1+t}}^t \frac{z(s)}{s}ds\Big) \leq 2^m.$$

It follows that

$$2^{-m}L(t) \leq L(\frac{t}{1+t}) \leq 2^m L(t).$$

$\square$

Lemma 2.4. *Let $\mu \leq 2$, $L \in \mathcal{K}_0$ defined on $(0,\eta]$, $\eta > 1$, such that $\int_0^\eta t^{1-\mu}L(t)dt < \infty$ and let*

$$I(t) = t(1 + \int_t^1 s^{-\mu}L(s)ds), \quad \text{for } t \in (0,1].$$

Then we have

$$I(t) \approx \begin{cases} t, & \text{if } \mu < 1, \\ t\int_{\frac{t}{1+t}}^{1} \frac{L(s)}{s}ds, & \text{if } \mu = 1, \\ t^{2-\mu}L(\frac{t}{1+t}), & \text{if } 1 < \mu \leq 2. \end{cases}$$

Proof. We distinguish three cases.
Case 1: If $\mu < 1$, then by applying Lemma 2.1 (i), we have $\int_0^\eta s^{-\mu}L(s)ds$ converges. So we obtain that $I(t) \approx t$.
Case 2: If $\mu = 1$, then since $\int_0^\eta \frac{L(s)}{s}ds < \infty$, we have

$$1 + \int_t^1 \frac{L(s)}{s}ds \approx \int_{\frac{t}{1+t}}^1 \frac{L(s)}{s}ds,$$

and hence

$$I(t) \approx t\int_{\frac{t}{1+t}}^1 \frac{L(s)}{s}ds.$$

Case 3: If $1 < \mu \leq 2$, then applying Lemma 2.1 (ii), the integral $\int_0^\eta s^{-\mu}L(s)ds$ diverges and $\int_t^\eta s^{-\mu}L(s)ds \approx t^{1-\mu}L(t)$. Combining this with the fact that $1 + \int_t^1 s^{-\mu}L(s)ds \approx \int_t^\eta s^{-\mu}L(s)ds$, we deduce by using Lemma 2.3 that

$$I(t) \approx t^{2-\mu}L(\frac{t}{1+t}).$$

□

2.2. **On the Karamata class $\mathcal{K}_\infty$.** We quote some properties of functions belonging to the Karamata class $\mathcal{K}_\infty$.

Lemma 2.5 ([7]). *Let $L \in \mathcal{K}_\infty$ and $\gamma \in \mathbb{R}$. Then we have the following*

(i) *If $\gamma > 2$, then $\int_1^\infty t^{1-\gamma}L(t)dt$ converges and $\int_t^\infty s^{1-\gamma}L(s)ds \sim \frac{t^{2-\gamma}L(t)}{\gamma-2}$ as $t \to \infty$;*
(ii) *If $\gamma < 2$, then $\int_1^\infty t^{1-\gamma}L(t)dt$ diverges and $\int_1^t s^{1-\gamma}L(s)ds \sim \frac{t^{2-\gamma}L(t)}{2-\gamma}$ as $t \to \infty$.*

Lemma 2.6 ([7]). *Let $L_1, L_2 \in \mathcal{K}_\infty$, $p \in \mathbb{R}$ and $\varepsilon > 0$. Then we have the following assertions:*

(i) *$L_1L_2 \in \mathcal{K}_\infty$ and $L_1^p \in \mathcal{K}_\infty$;*
(ii) *$\lim_{t\to\infty} t^{-\varepsilon}L_1(t) = 0$;*
(iii) *$\lim_{t\to\infty} \frac{L_1(t)}{\int_1^t \frac{L_1(s)}{s}ds} = 0$ and $t \mapsto \int_1^{t+1} \frac{L_1(s)}{s}ds \in \mathcal{K}_\infty$;*
(iv) *If $\int_1^\infty \frac{L_1(s)}{s}ds$ converges, then $\lim_{t\to\infty} \frac{L_1(t)}{\int_t^\infty \frac{L_1(s)}{s}ds} = 0$ and $t \mapsto \int_t^\infty \frac{L_1(s)}{s}ds \in \mathcal{K}_\infty$;*
(v) *There exists $m \geq 0$ such that for every $\alpha > 0$ and $t \geq 1$, we have*

$$(1+\alpha)^{-m}L_1(t) \leq L_1(\alpha + t) \leq (1+\alpha)^m L_1(t).$$

Lemma 2.7. *Let $\lambda \geq 2$ and $L \in \mathcal{K}_\infty$ be such that $\int_1^\infty t^{1-\lambda}L(t)dt < \infty$. Put*

$$J(t) = \frac{1}{(1+t)^{n-2}}(1 + \int_1^t s^{n-\lambda-1}L(s)ds), \quad \text{for } t \in [1,\infty).$$

Then we have

$$J(t) \approx \begin{cases} \frac{L(t)}{(1+t)^{\lambda-2}}, & \text{if } 2 \le \lambda < n, \\ \frac{1}{(1+t)^{n-2}} \int_1^{t+1} \frac{L(s)}{s} ds, & \text{if } \lambda = n, \\ \frac{1}{(1+t)^{n-2}}, & \text{if } \lambda > n. \end{cases}$$

Proof. We split the proof into three cases.
Case 1: If $2 \le \lambda < n$, then it follows by Lemma 2.5 (ii) that

$$J(t) \approx \frac{1}{(1+t)^{n-2}} (1 + t^{n-\lambda} L(t)).$$

Which implies from Lemma 2.6 (i) and (ii) that

$$J(t) \approx \frac{L(t)}{(1+t)^{\lambda-2}}.$$

Case 2: If $\lambda = n$, then we get

$$J(t) \approx \frac{1}{(1+t)^{n-2}} \int_1^{t+1} \frac{L(s)}{s} ds.$$

Case 3: If $\lambda > n$, then it follows from Lemma 2.5 (i) that $\int_1^\infty s^{n-\lambda-1} L(s) ds$ converges. So we reach

$$J(t) \approx \frac{1}{(1+t)^{n-2}}.$$

$\square$

3. Asymptotic behavior of some potential functions

In what follows, we are going to give estimates on the potential functions Va and $V(a\,\theta^\sigma)$, where a is a function satisfying $(H1)$ and θ is the function given in (1.5). These estimates will be useful in the proof of our main result.

Remark 3.1. Let $L \in \mathcal{K}$. Then there exist $M \in \mathcal{K}_0$ defined on $(0, \eta]$ for some $\eta > 1$ and $N \in \mathcal{K}_\infty$ such that

$$L(t) = M(\frac{t}{1+t}) N(1+t).$$

Since $M \in C^1((0, \eta]$ and $N \in C^1([1, \infty))$, we obtain by virtue of Lemmas 2.3 and 2.6 that

$$L(t) \approx M(t), \text{ for } 0 < t \le 1 \quad \text{and} \quad L(t) \approx N(t), \text{ for } t \ge 1. \tag{3.1}$$

Which implies that

$$\begin{aligned} &\Big(\int_0^\infty t^{1-\mu}\,(1+t)^{\mu-\lambda} L(t) dt < \infty\Big) \\ &\Leftrightarrow \Big(\int_0^1 t^{1-\mu}\, M(t) dt < \infty \text{ and } \int_1^\infty t^{1-\lambda}\, N(t) dt < \infty\Big). \end{aligned} \tag{3.2}$$

According to Lemmas 2.1 and 2.5, we need to verify that $\int_0^\infty t^{1-\mu}\,(1+t)^{\mu-\lambda} L(t) dt < \infty$ in hypothesis (H1), only if $\lambda = 2$ or $\mu = 2$.

Proposition 3.2. *Let a be a function satisfying* (H1)*. Then for $x \in D$, we have*

$$Va(x) \approx \frac{(\rho(x))^{\min(2-\mu,1)}}{|x|^{\min(\lambda-2,n-2)}}\, \Phi_{M,\mu,0}(\rho(x))\, \Psi_{N,\lambda,0}(|x|).$$

Proof. Let a be a function satisfying $(H1)$. Let $\mu \leq 2 \leq \lambda$, $M \in \mathcal{K}_0$ and $N \in \mathcal{K}_\infty$ such that $L(t) = M(\frac{t}{1+t})N(1+t)$, for $t \in (0,\infty)$ and $\int_0^\infty t^{1-\mu}(1+t)^{\mu-\lambda}L(t)dt < \infty$ satisfying

$$a(x) \approx \frac{L(|x|-1)}{|x|^{\lambda-\mu}\,(|x|-1)^\mu}, \quad x \in D.$$

For $x \in D$, we have

$$Va(x) \approx \int_D G(x,y)\frac{L(|y|-1)}{|y|^{\lambda-\mu}\,(|y|-1)^\mu}dy.$$

Since the function

$$x \mapsto \int_D G(x,y)\frac{L(|y|-1)}{|y|^{\lambda-\mu}\,(|y|-1)^\mu}dy$$

is radial, then by elementary calculus, we obtain that

$$\int_D G(x,y)\frac{L(|y|-1)}{|y|^{\lambda-\mu}\,(|y|-1)^\mu}dy = c_n\int_1^\infty \frac{r^{n+\mu-\lambda-1}(1-(\min(|x|,r))^{2-n})L(r-1)}{(\max(|x|,r))^{n-2}(r-1)^\mu}dr,$$

where $c_n > 0$. That is,

$$\begin{aligned}\int_D G(x,y)\frac{L(|y|-1)}{|y|^{\lambda-\mu}\,(|y|-1)^\mu}dy = c_n\Big(&\int_1^{|x|} \frac{r^{n+\mu-\lambda-1}(1-r^{2-n})L(r-1)}{|x|^{n-2}(r-1)^\mu}dr \\ &+\int_{|x|}^\infty \frac{r^{\mu-\lambda+1}(1-|x|^{2-n})L(r-1)}{(r-1)^\mu}dr\Big).\end{aligned}$$

In what follows, we distinguish two cases.
Case 1: $1 < |x| \leq 2$. We have

$$\begin{aligned}Va(x) \approx{}& \frac{1}{|x|^{n-2}}\int_1^{|x|} \frac{r^{n+\mu-\lambda-1}(1-r^{2-n})L(r-1)}{(r-1)^\mu}dr \\ &+ (1-|x|^{2-n})(\int_{|x|}^2 \frac{r^{\mu-\lambda+1}L(r-1)}{(r-1)^\mu}dr + \int_2^\infty \frac{r^{\mu-\lambda+1}L(r-1)}{(r-1)^\mu}dr).\end{aligned}$$

Since for $|x| \in (1,2]$, $\frac{1}{|x|^{n-2}} \approx 1$ and $1-|x|^{2-n} \approx |x|-1$, it follows that

$$\begin{aligned}Va(x) \approx{}& \int_1^{|x|} \frac{r^{n+\mu-\lambda-1}(1-r^{2-n})L(r-1)}{(r-1)^\mu}dr \\ &+ (|x|-1)\Big(\int_{|x|}^2 \frac{r^{\mu-\lambda+1}L(r-1)}{(r-1)^\mu}dr + \int_2^\infty \frac{r^{\mu-\lambda+1}L(r-1)}{(r-1)^\mu}dr\Big).\end{aligned}$$

This implies

$$\begin{aligned}Va(x) \approx{}& \int_1^{|x|}(r-1)^{1-\mu}L(r-1)dr + (|x|-1)\Big(\int_{|x|}^2 (r-1)^{-\mu}L(r-1)dr \\ &+ \int_2^\infty (r-1)^{1-\lambda}L(r-1)dr\Big).\end{aligned}$$

That is,

$$Va(x) \approx \int_0^{|x|-1} s^{1-\mu}L(s)ds + (|x|-1)\Big(\int_{|x|-1}^1 s^{-\mu}L(s)ds + \int_1^\infty s^{1-\lambda}L(s)ds\Big).$$

Using (3.1), we obtain that

$$Va(x) \approx \int_0^{|x|-1} s^{1-\mu} M(s)ds + (|x|-1)\Big(\int_{|x|-1}^1 s^{-\mu} M(s)ds + \int_1^\infty s^{1-\lambda} N(s)ds\Big).$$

Taking into account of the fact that $\int_0^\infty t^{1-\mu}(1+t)^{\mu-\lambda} L(t)dt < \infty$ and using (3.2), we reach

$$\begin{aligned} Va(x) &\approx \int_0^{|x|-1} s^{1-\mu} M(s)ds + (|x|-1)\Big(1 + \int_{|x|-1}^1 s^{-\mu} M(s)ds\Big) \\ &= \int_0^{|x|-1} s^{1-\mu} M(s)ds + I(|x|-1), \end{aligned}$$

where I is the function given in Lemma 2.4 by replacing L by M. Using Lemmas 2.1 and 2.2, we have

$$\int_0^{|x|-1} s^{1-\mu} M(s)ds \approx \begin{cases} (|x|-1)^{2-\mu} M(|x|-1), & \text{if } \mu < 2, \\ \int_0^{|x|-1} \frac{M(s)}{s} ds, & \text{if } \mu = 2. \end{cases}$$

Hence, we deduce from Lemma 2.4 that

$$Va(x) \approx \begin{cases} (|x|-1) + (|x|-1)^{2-\mu} M(|x|-1), & \text{if } \mu < 1, \\ (|x|-1)\Big(M(|x|-1) + \int_{|x|-1}^1 \frac{M(s)}{s} ds\Big), & \text{if } \mu = 1, \\ (|x|-1)^{2-\mu} M(|x|-1), & \text{if } 1 < \mu < 2, \\ \int_0^{|x|-1} \frac{M(s)}{s} ds + M(|x|-1), & \text{if } \mu = 2. \end{cases}$$

Now, applying Lemma 2.2, we obtain for $1 < |x| \le 2$ that

$$Va(x) \approx \begin{cases} |x|-1, & \text{if } \mu < 1, \\ (|x|-1)\int_{|x|-1}^{\eta} \frac{M(s)}{s} ds, & \text{if } \mu = 1, \\ (|x|-1)^{2-\mu} M(|x|-1), & \text{if } 1 < \mu < 2, \\ \int_0^{|x|-1} \frac{M(s)}{s} ds, & \text{if } \mu = 2. \end{cases}$$

Thus, we obtain that for $1 < |x| \le 2$,

$$Va(x) \approx (|x|-1)^{\min(2-\mu,1)} \Phi_{M,\mu,0}(|x|-1)$$

So, since $|x|-1 \approx \rho(x)$, it follows from Lemmas 2.2 and 2.3 that

$$Va(x) \approx (\rho(x))^{\min(2-\mu,1)} \Phi_{M,\mu,0}(\rho(x)), \quad \text{for } 1 < |x| \le 2. \tag{3.3}$$

Case 2: $|x| > 2$. We have

$$\begin{aligned} Va(x) \approx \frac{1}{|x|^{n-2}} \Big(&\int_1^2 \frac{r^{n+\mu-\lambda-1}(1-r^{2-n})L(r-1)}{(r-1)^\mu} dr \\ &+ \int_2^{|x|} \frac{r^{n+\mu-\lambda-1}(1-r^{2-n})L(r-1)}{(r-1)^\mu} dr\Big) \\ &+ (1-|x|^{2-n}) \int_{|x|}^\infty \frac{r^{\mu-\lambda+1} L(r-1)}{(r-1)^\mu} dr. \end{aligned}$$

Which implies

$$Va(x) \approx \frac{1}{|x|^{n-2}} \Big(\int_1^2 (r-1)^{1-\mu} L(r-1)dr + \int_2^{|x|} (r-1)^{n-\lambda-1} L(r-1)dr \Big)$$

$$+\int_{|x|}^{\infty}(r-1)^{1-\lambda}L(r-1)dr.$$

That is

$$Va(x)\approx\frac{1}{|x|^{n-2}}\Big(\int_0^1 s^{1-\mu}L(s)ds+\int_1^{|x|-1}s^{n-\lambda-1}L(s)ds\Big)+\int_{|x|-1}^{\infty}s^{1-\lambda}L(s)ds.$$

Using (3.1), we reach

$$Va(x)\approx\frac{1}{|x|^{n-2}}\Big(\int_0^1 s^{1-\mu}M(s)ds+\int_1^{|x|-1}s^{n-\lambda-1}N(s)ds\Big)+\int_{|x|-1}^{\infty}s^{1-\lambda}N(s)ds.$$

We deduce from the fact that $\int_0^\infty t^{1-\mu}(1+t)^{\mu-\lambda}L(t)dt<\infty$ and (3.2) that

$$\begin{aligned}Va(x)&\approx\frac{1}{|x|^{n-2}}\Big(1+\int_1^{|x|-1}s^{n-\lambda-1}N(s)ds\Big)+\int_{|x|-1}^{\infty}s^{1-\lambda}N(s)ds\\&=J(|x|-1)+\int_{|x|-1}^{\infty}s^{1-\lambda}N(s)ds,\end{aligned}$$

where J is the function given in Lemma 2.7 by replacing L by N. By applying Lemma 2.5, we have

$$\int_{|x|-1}^{\infty}s^{1-\lambda}N(s)ds\approx\begin{cases}\int_{|x|-1}^{\infty}\frac{N(s)}{s}ds, & \text{if } \lambda=2,\\ \frac{N(|x|-1)}{|x|^{\lambda-2}}, & \text{if } \lambda>2.\end{cases}$$

Then, we deduce from Lemma 2.7 that

$$Va(x)\approx\begin{cases}N(|x|-1)+\int_{|x|-1}^{\infty}\frac{N(s)}{s}ds, & \text{if } \lambda=2,\\ \frac{N(|x|-1)}{|x|^{\lambda-2}}, & \text{if } 2<\lambda<n,\\ \frac{1}{|x|^{n-2}}(N(|x|-1)+\int_1^{|x|}\frac{N(s)}{s}ds), & \text{if } \lambda=n,\\ \frac{1}{|x|^{n-2}}+\frac{N(|x|-1)}{|x|^{\lambda-2}}, & \text{if } \lambda>n.\end{cases}$$

Therefore, using Lemma 2.6, we obtain that for $|x|>2$,

$$Va(x)\approx\begin{cases}\int_{|x|-1}^{\infty}\frac{N(s)}{s}ds, & \text{if } \lambda=2,\\ \frac{N(|x|-1)}{|x|^{\lambda-2}}, & \text{if } 2<\lambda<n,\\ \frac{1}{|x|^{n-2}}\int_1^{|x|}\frac{N(s)}{s}ds, & \text{if } \lambda=n,\\ \frac{1}{|x|^{n-2}}, & \text{if } \lambda>n.\end{cases}$$

Hence, we get that for $|x|>2$,

$$Va(x)\approx\frac{\Psi_{N,\lambda,0}(|x|-1)}{|x|^{\min(\lambda-2,n-2)}},$$

which implies by Lemma 2.6 that

$$Va(x)\approx\frac{\Psi_{N,\lambda,0}(|x|)}{|x|^{\min(\lambda-2,n-2)}}. \tag{3.4}$$

Combining (3.3) and (3.4), we deduce that for $x\in D$,

$$Va(x)\approx\frac{(\rho(x))^{\min(2-\mu,1)}}{|x|^{\min(\lambda-2,n-2)}}\Phi_{M,\mu,0}(\rho(x))\Psi_{N,\lambda,0}(|x|).$$

This completes the proof. $\square$

The following proposition plays a crucial role in the proof of Theorem 1.5.

Proposition 3.3. *Let a be a function satisfying* (H1) *and let θ be the function given by (1.5). Then for $x \in D$, we have*

$$V(a\,\theta^{\sigma})(x) \approx \theta(x).$$

Proof. Let a be a function satisfying (H1). Let $\mu \leq 2 \leq \lambda$ and $L \in \mathcal{K}$ satisfying the condition $\int_0^{\infty} t^{1-\mu}\,(1+t)^{\mu-\lambda} L(t)dt < \infty$ and such that for $x \in D$, we have

$$a(x) \approx \frac{L(|x|-1)}{|x|^{\lambda-\mu}(|x|-1)^{\mu}} := \frac{M(\rho(x))N(|x|)}{|x|^{\lambda-\mu}(|x|-1)^{\mu}}, \tag{3.5}$$

where $M \in \mathcal{K}_0$ and $N \in \mathcal{K}_{\infty}$. Put

$$\lambda_1 = \lambda + \sigma \min(\frac{\lambda-2}{1-\sigma}, n-2) \quad \text{and} \quad \mu_1 = \mu - \sigma \min(\frac{2-\mu}{1-\sigma}, 1).$$

We verify that $\mu_1 \leq 2 \leq \lambda_1$ and by using (1.5) and (3.5), we obtain

$$a(x)\theta^{\sigma}(x) \approx \frac{(M\Phi^{\sigma}_{M,\mu,\sigma})(\rho(x))(N\Psi^{\sigma}_{N,\lambda,\sigma})(|x|)}{|x|^{\lambda_1-\mu_1}\,(|x|-1)^{\mu_1}} := \frac{\widetilde{L}(|x|-1)}{|x|^{\lambda_1-\mu_1}\,(|x|-1)^{\mu_1}}.$$

Since $\widetilde{M} := M\Phi^{\sigma}_{M,\mu,\sigma} \in \mathcal{K}_0$ and $\widetilde{N} := N\Psi^{\sigma}_{N,\lambda,\sigma} \in \mathcal{K}_{\infty}$, we have that the function $\widetilde{L}$ defined on $(0,\infty)$ by $\widetilde{L}(t) = \widetilde{M}(\frac{t}{1+t})\widetilde{N}(1+t)$ is in $\mathcal{K}$ and by Lemmas 2.1 and 2.5, we obtain that the integral $\int_0^{\infty} t^{1-\mu_1}\,(1+t)^{\mu_1-\lambda_1}\widetilde{L}(t)dt < \infty$. Hence, it follows from Proposition 3.2 that

$$V(a\theta^{\sigma})(x) \approx \frac{(\rho(x))^{\min(2-\mu_1,1)}}{|x|^{\min(\lambda_1-2,n-2)}}\,\Phi_{\widetilde{M},\mu_1,0}(\rho(x))\Psi_{\widetilde{N},\lambda_1,0}(|x|), \quad x \in D.$$

Now, using

$$\min(2-\mu_1,1) = \min(\frac{2-\mu}{1-\sigma},1), \quad \min(\lambda_1-2,n-2) = \min(\frac{\lambda-2}{1-\sigma},n-2),$$

we obtain by elementary calculus that for $x \in D$,

$$\Phi_{\widetilde{M},\mu_1,0}(\rho(x)) = \Phi_{M,\mu,\sigma}(\rho(x)), \quad \Psi_{\widetilde{N},\lambda_1,0}(|x|) = \Psi_{N,\lambda,\sigma}(|x|).$$

This completes the proof. □

4. Proof of Theorem 1.5

4.1. **Existence and asymptotic behavior.** Let a be a function satisfying $(H1)$. We look now at the existence of positive solution of problem (1.3) satisfying (1.4). The main idea is to find a subsolution and a supersolution to problem (1.3) of the form $cV(a\omega^{\sigma})$, where $c > 0$ and $\omega(x) = \frac{L_0(|x|-1)}{|x|^{\beta-\alpha}(|x|-1)^{\alpha}}$, which will satisfy

$$V(a\omega^{\sigma}) \approx \omega. \tag{4.1}$$

So the choice of the real numbers α, β and the function L_0 in $\mathcal{K}$ is such that (4.1) is satisfied. Setting $\omega(x) = \theta(x)$, where θ is the function given by (1.5), we have by Proposition 3.3, that the function θ satisfies (4.1). Hence, let $v = V(a\theta^{\sigma})$ and let $m \geq 1$ be such that

$$\frac{1}{m}\theta \leq v \leq m\theta. \tag{4.2}$$

This implies that for $\sigma < 1$, we have

$$\frac{1}{m^{|\sigma|}}\theta^{\sigma} \leq v^{\sigma} \leq m^{|\sigma|}\theta^{\sigma}.$$

Put $c = m^{|\sigma|/(1-\sigma)}$, then it is easy to show that $\underline{u} = \frac{1}{c}v$ and $\overline{u} = cv$ are respectively a subsolution and a supersolution of problem (1.3).

Now, since $c \geq 1$, we get $\underline{u} \leq \overline{u}$ on D and thanks to the method of sub-supersolution (see [23]), it follows that problem (1.3) has a classical solution u such that $\underline{u} \leq u \leq \overline{u}$ in D. Using (4.2), we deduce that u satisfies (1.4). This completes the proof.

4.2. **Uniqueness.** Let a be a function satisfying (H1) and θ be the function defined in (1.5). We aim to show that problem (1.3) has a unique positive solution in the cone

$$\Gamma = \{u \in C^{2,\gamma}(D) : u(x) \approx \theta(x)\}.$$

To this end, we consider the following cases.

Case $\sigma < 0$ Let u and v be two solutions of (1.3) in Γ and put $w = u - v$. Then the function w satisfies

$$w + V(hw) = 0 \text{ in } D,$$

where h is the nonnegative measurable function defined in D by

$$h(x) = \begin{cases} a(x)\frac{(v(x))^\sigma - (u(x))^\sigma}{u(x)-v(x)} & \text{if } u(x) \neq v(x), \\ 0 & \text{if } u(x) = v(x). \end{cases}$$

Furthermore, it is clear to see that $V(h|w|) < \infty$. Then, we get by [2, Lemma 4.1] that $w = 0$. This proves the uniqueness.

Case $0 \leq \sigma < 1$ Let us now assume that u and v are arbitrary solutions of problem (1.3) in Γ. Since $u, v \in \Gamma$, then there exists a constant $m \geq 1$ such that

$$\frac{1}{m} \leq \frac{u}{v} \leq m \quad \text{in } D.$$

This implies that the set $J := \{t \in (0,1] : tu \leq v\}$ is not empty. Now put $c := \sup J$. It is easy to see that $0 < c \leq 1$. On the other hand, we have

$$\begin{gathered} -\Delta(v - c^\sigma u) = a(x)(v^\sigma - c^\sigma u^\sigma) \geq 0 \quad \text{in } D, \\ \lim_{|x|\to 1}(v - c^\sigma u)(x) = \lim_{|x|\to\infty}(v - c^\sigma u)(x) = 0. \end{gathered}$$

Then, by the maximum principle, we deduce that $c^\sigma u \leq v$. Which implies that $c^\sigma \leq c$. Using the fact that $\sigma < 1$, we get that $c \geq 1$. Hence, we arrive at $u \leq v$ and by symmetry, we obtain that $u = v$. This completes the proof.

5. Applications

Let σ, $\beta < 1$ and let a be a function satisfying (H1). Let $\mu \leq 2 \leq \lambda$ and $L \in \mathcal{K}$ satisfying the condition $\int_0^\infty t^{1-\mu}(1+t)^{\mu-\lambda}L(t)dt < \infty$ and such that for $x \in D$, we have

$$a(x) \approx \frac{L(|x|-1)}{|x|^{\lambda-\mu}(|x|-1)^\mu} := \frac{M(\rho(x))N(|x|)}{|x|^{\lambda-\mu}(|x|-1)^\mu},$$

where $M \in \mathcal{K}_0$ and $N \in \mathcal{K}_\infty$. We are interested in the problem

$$\begin{gathered} -\Delta u + \frac{\beta}{u}|\nabla u|^2 = a(x)u^\sigma \quad \text{in } D, \\ u > 0 \quad \text{in } D, \\ \lim_{|x|\to 1} u(x) = \lim_{|x|\to\infty} u(x) = 0. \end{gathered} \tag{5.1}$$

Put $v = u^{1-\beta}$, then by calculus, we obtain that v satisfies

$$\begin{gathered}
-\Delta v = (1-\beta)a(x)v^{\frac{\sigma-\beta}{1-\beta}} \quad \text{in } D,\\
v > 0 \quad \text{in } D,\\
\lim_{|x|\to 1} v(x) = \lim_{|x|\to\infty} v(x) = 0.
\end{gathered} \tag{5.2}$$

Since $\sigma_1 := \frac{\sigma-\beta}{1-\beta} < 1$, we obtain by applying Theorem 1.5 that problem (5.2) has a unique solution v such that, for each $x \in D$,

$$v(x) \approx \frac{(\rho(x))^{\min(\frac{(2-\mu)(1-\beta)}{1-\sigma},1)}}{|x|^{\min(\frac{(\lambda-2)(1-\beta)}{1-\sigma},n-2)}}\,\Phi_{M,\mu,\sigma_1}(\rho(x))\,\Psi_{N,\lambda,\sigma_1}(|x|).$$

Consequently, we deduce that problem (5.1) has a unique positive solution u satisfying for each $x \in D$,

$$u(x) \approx \frac{(\rho(x))^{\min(\frac{2-\mu}{1-\sigma},\frac{1}{1-\beta})}}{|x|^{\min(\frac{\lambda-2}{1-\sigma},\frac{n-2}{1-\beta})}}\,\Phi^{\frac{1}{1-\beta}}_{M,\mu,\sigma_1}(\rho(x))\,\Psi^{\frac{1}{1-\beta}}_{N,\lambda,\sigma_1}(|x|).$$

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