

NONAUTONOMOUS ILL-POSED EVOLUTION PROBLEMS WITH STRONGLY ELLIPTIC DIFFERENTIAL OPERATORS

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ABSTRACT. In this article, we consider the nonautonomous evolution problem $du/dt = a(t)Au(t)$, $0 \leq s \leq t < T$ with initial condition $u(s) = \chi$ where $-A$ generates a holomorphic semigroup of angle $\theta \in (0, \pi/2]$ on a Banach space X and $a \in C([0, T] : \mathbb{R}^+)$. The problem is generally ill-posed under such conditions, and so we employ methods to approximate known solutions of the problem. In particular, we prove the existence of a family of regularizing operators for the problem which stems from the solution of an approximate well-posed problem. In fact, depending on whether $\theta \in (0, \pi/4]$ or $\theta \in (\pi/4, \pi/2]$, we provide two separate approximations each yielding a regularizing family. The theory has applications to ill-posed partial differential equations in $L^p(\Omega)$, $1 < p < \infty$ where A is a strongly elliptic differential operator and Ω is a fixed domain in \mathbb{R}^n .

1. INTRODUCTION

Due to the unstable nature of a given ill-posed problem, whose solutions (if they exist) may not depend continuously on initial data, many approximation techniques have been applied to study known solutions of the problem. Consider the abstract Cauchy problem

$$\begin{aligned} \frac{du}{dt} &= Au(t) \quad 0 \leq t < T \\ u(0) &= \chi \end{aligned} \tag{1.1}$$

in a Banach space X , which under many different circumstances, depending on the operator A , may be ill-posed. For instance, letting $A = -\Delta$, (1.1) becomes the prototypical ill-posed problem, the backwards heat equation. More generally, (1.1) is ill-posed in the parabolic case when $-A$ generates a holomorphic semigroup on X . In this case, one approach recently applied by Mel'nikova [11] and Huang and Zheng [7, 8] is to regularize the ill-posed problem; that is, to approximate a known solution of (1.1) by the solution of an approximate well-posed problem (see also [2, 12, 23, 24]).

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In this paper, we extend these ideas to the study of the *nonautonomous* parabolic evolution problem

$$\begin{aligned} \frac{du}{dt} &= a(t)Au(t) \quad 0 \leq s \leq t < T \\ u(s) &= \chi \end{aligned} \quad (1.2)$$

in a Banach space X where $-A$ generates a holomorphic semigroup of angle $\theta \in (0, \pi/2]$ on X and $a \in C([0, T] : \mathbb{R}^+)$, so that the governing operators $a(t)A, 0 \leq t \leq T$ of the problem are nonconstant. We prove the existence of a family of regularizing operators for the problem (so that the problem is “regularized”) which refers specifically to the following.

Definition 1.1 ([8, Definition 3.1]). A family $\{R_\beta(t) : \beta > 0, t \in [s, T]\}$ of bounded linear operators on X is called a *family of regularizing operators for the problem* (1.2) if for each solution $u(t)$ of (1.2) with initial data $\chi \in X$, and for any $\delta > 0$, there exists $\beta(\delta) > 0$ such that

- (i) $\beta(\delta) \rightarrow 0$ as $\delta \rightarrow 0$,
- (ii) $\|u(t) - R_{\beta(\delta)}(t)\chi_\delta\| \rightarrow 0$ as $\delta \rightarrow 0$ for $s \leq t \leq T$ whenever $\|\chi - \chi_\delta\| \leq \delta$.

As in the case of regularization for the autonomous problem (1.1), we will show that a family of regularizing operators for (1.2) stems from the solution of an approximate well-posed problem

$$\begin{aligned} \frac{dv}{dt} &= f_\beta(t, A)v(t) \quad 0 \leq s \leq t < T \\ v(s) &= \chi \end{aligned} \quad (1.3)$$

where, for $\beta > 0$, the operators $f_\beta(t, A), 0 \leq t \leq T$ are defined by two different approximations of the operators $a(t)A$ depending on where θ lies in the interval $(0, \pi/2]$:

$$f_\beta(t, A) = \begin{cases} a(t)A - \beta A^\sigma & \text{if } \theta \in (0, \pi/4] \\ a(t)A(I + \beta A)^{-1} & \text{if } \theta \in (\pi/4, \pi/2] \end{cases} \quad (1.4)$$

where $\sigma > 1$ when $\theta \in (0, \pi/4]$.

Each approximation in (1.4) yields a well-posed problem (1.3), and also continuous dependence on modeling for the ill-posed problem (1.2) in the sense that as $\beta \rightarrow 0$, the operators $f_\beta(t, A)$ approach the operators $a(t)A$, and given solutions $u(t)$ and $v_\beta(t)$ of (1.2) and (1.3) respectively, we have

$$\|u(t) - v_\beta(t)\| \rightarrow 0 \quad \text{as } \beta \rightarrow 0 \quad (1.5)$$

for each $t \in [s, T]$. We use (1.5) to establish the main result of the paper, that the family $\{V_\beta(t, s) : \beta > 0, t \in [s, T]\}$ is a family of regularizing operators for the ill-posed problem (1.2) where $V_\beta(t, s), 0 \leq s \leq t \leq T$ is an evolution system associated with the well-posed problem (1.3) satisfying $V_\beta(t, s)\chi = v_\beta(t)$. In other words, given a small change in the initial data $\|\chi - \chi_\delta\| \leq \delta$ (which, since (1.2) is ill-posed, could yield a very large difference in solutions), there exists $\beta > 0$ so that $\beta \rightarrow 0$ as $\delta \rightarrow 0$, and $\|u(t) - V_\beta(t, s)\chi_\delta\| \rightarrow 0$ as $\delta \rightarrow 0$ for $s \leq t \leq T$. Hence, although $u(t)$ may not be “close” to the solution of (1.2) with initial data χ_δ , we can still approximate $u(t)$ by utilizing the well-posed problem (1.3) with regularization parameter $\beta > 0$.

The use of the two approximations in (1.4) extends results from previous works in which the approximations $A - \beta A^\sigma$ and $A(I + \beta A)^{-1}$ are used to obtain regularization for the autonomous problem (1.1) where $-A$ generates a holomorphic semigroup of angle θ on X (cf. [2, 7, 8, 11, 12]). For instance, in [7], Huang and Zheng obtain regularization for (1.1) using the quasi-reversibility method, first introduced by Lattes and Lions [10], which involves the approximation $A - \beta A^\sigma$ of the operator A . Here, the requirements that $\sigma > 1$ and $\sigma(\pi/2 - \theta) < \pi/2$ are crucial in order for $A - \beta A^\sigma$ to generate a semigroup (so as to yield an approximate well-posed problem). Hence, if $\theta \in (0, \pi/4]$, these requirements force $1 < \sigma < 2$ whence the use of the fractional power A^σ is in order. In light of definition (1.4), we will adopt the same requirements in the current paper for the extension $a(t)A - \beta A^\sigma$. The second approximation $A(I + \beta A)^{-1}$, introduced by Showalter [20], is applied by Ames and Hughes [2] and Huang and Zheng [8] but only in the case where $\theta \in (\pi/4, \pi/2]$ because the perturbation methods used to establish regularization in these papers (and in the current paper) are not applicable when $\theta \in (0, \pi/4]$ (cf. [8, pp. 3011–3012]).

Note, if $\theta \in (\pi/2, \pi/4]$, the approximation $a(t)A - \beta A^\sigma$ may still be used, but it is standard and easier in this case to let $\sigma = 2$ (cf. [2, 4, 10, 11, 12, 13, 15, 16]). In this regard, the current paper also furthers results from [4] where the author uses the approximation $\sum_{j=1}^k a_j(t)A^j - \beta A^{k+1}$ to obtain regularization for the nonautonomous problem

$$\begin{aligned} \frac{du}{dt} &= \sum_{j=1}^k a_j(t)A^j u(t) \quad 0 \leq s \leq t < T \\ u(s) &= \chi, \end{aligned}$$

but only in the case that $\theta \in (\pi/4, \pi/2]$.

This article is organized as follows. In Section 2, we adapt methods of Huang and Zheng [7, 8] to show that problem (1.3) is well-posed under definition (1.4) with the existence of an evolution system $V_\beta(t, s), 0 \leq s \leq t \leq T$ generating solutions of (1.3). The calculations here are quite similar to those in [7], but we provide the details to demonstrate the differences in treating nonautonomous equations. After introducing several lemmas in Section 3, we prove in Section 4, a Hölder-continuous dependence on modeling inequality which provides an estimate for the difference between the solutions $u(t)$ and $v_\beta(t)$ yielding (1.5). In Section 5, we use results from Section 4 to prove the existence of a family of regularizing operators for the ill-posed problem (1.2) and finally in Section 6, we apply the theory to partial differential equations in the Banach space $L^p(\Omega)$, $1 < p < \infty$ where A is a strongly elliptic differential operator and Ω is a fixed domain in \mathbb{R}^n .

Below, $B(X)$ will denote the space of bounded linear operators on X . For a linear operator A in X , $\rho(A)$ will denote the resolvent set of A consisting of all $w \in \mathbb{C}$ such that $(w - A)^{-1} \in B(X)$. Also, we will be concerned with *classical solutions* of (1.2) which are functions $u : [s, T] \rightarrow X$ such that $u(t) \in \text{Dom}(A)$ for all $t \in (s, T)$, $u \in C[s, T] \cap C^1(s, T)$, and u satisfies (1.2) in X (cf. [17, Chapter 5.1, p. 126]).

2. TWO APPROXIMATE WELL-POSED PROBLEMS

In this section, we show that the approximate problem (1.3), where the operators $f_\beta(t, A), 0 \leq t \leq T$ are defined by (1.4), is well-posed, meaning that a unique solution exists for each χ in a dense subset of X and solutions depend continuously on the initial data (cf. [6, Chapter 2.13, p. 140]). Much of the content here will rely on the assumption that $-A$ generates a holomorphic semigroup and so we first gather relevant properties.

Definition 2.1 ([18, Section X.8, p. 248, 252]). Let $\theta \in (0, \pi/2]$. A strongly continuous bounded semigroup $T(t), t > 0$ on a Banach space X is called a *bounded holomorphic semigroup of angle θ* if the following conditions are satisfied:

- (i) $T(t)$ is the restriction to the positive real axis of an analytic family of operators $T(z)$ in the open sector $S_\theta = \{re^{i\theta'} : r > 0, |\theta'| < \theta\}$ satisfying $T(z+w) = T(z)T(w)$ for all $z, w \in S_\theta$.
- (ii) For each $\theta_1 < \theta$, $T(z)x \rightarrow x$ as $z \rightarrow 0$ in S_{θ_1} for all $x \in X$.
- (iii) For each $\theta_1 < \theta$, $T(z)$ is uniformly bounded in the sector S_{θ_1} .

More generally, a strongly continuous semigroup $T(t)$ on X is called a *holomorphic semigroup of angle θ* if $T(t)$ satisfies all the properties of a bounded holomorphic semigroup of angle θ with the exception of (iii).

Theorem 2.2 ([18, Theorem X.52]). *Let A be a closed operator on a Banach space X . Then $-A$ is the infinitesimal generator of a bounded holomorphic semigroup of angle θ if and only if for each $\theta_1 < \theta$ there exists a constant $M_1 > 0$ such that if $w \notin \bar{S}_{\pi/2-\theta_1}$, then $w \in \rho(A)$ and*

$$\|(w - A)^{-1}\| \leq \frac{M_1}{\text{dist}(w, \bar{S}_{\pi/2-\theta_1})}. \quad (2.1)$$

For this paper, we first assume that $-A$ generates a *bounded* holomorphic semigroup of angle θ . In fact, for most of the paper, we will make this assumption for convenience, but then generalize our results at the end for holomorphic semigroups for which only conditions (i) and (ii) of Definition 2.1 hold.

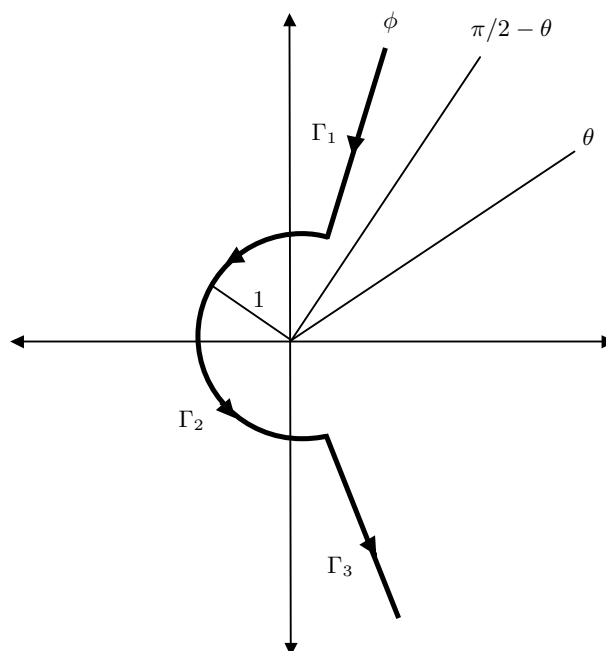
Since $-A$ generates a bounded holomorphic semigroup of angle θ , by Theorem 2.2, it follows that the spectrum $\sigma(A)$ of A is contained in $\bar{S}_{\pi/2-\theta} = \{re^{i\theta'} : r \geq 0, |\theta'| \leq \pi/2 - \theta\}$. Further, for $t > 0$, $T(t)$ is given by the Cauchy integral formula

$$T(t) = \frac{1}{2\pi i} \int_{\Gamma_\phi} e^{-tw} (w - A)^{-1} dw \quad (2.2)$$

where $\pi/2 > \phi > \pi/2 - \theta$ and Γ_ϕ is a curve in $\rho(A)$ consisting of three pieces: $\Gamma_1 = \{re^{i\phi} : r \geq 1\}$, $\Gamma_2 = \{e^{i\theta'} : \phi \leq \theta' \leq 2\pi - \phi\}$, and $\Gamma_3 = \{re^{-i\phi} : r \geq 1\}$; Γ_ϕ is oriented so that it runs from $\infty e^{i\phi}$ to $\infty e^{-i\phi}$ (see Figure 1). Similarly, for $z \in S_\theta$,

$$T(z) = \frac{1}{2\pi i} \int_{\Gamma_\phi} e^{-zw} (w - A)^{-1} dw.$$

We will first prove that the approximate problem (1.3) is well-posed in the case that $\theta \in (0, \pi/4]$ and $f_\beta(t, A), 0 \leq t \leq T$ is defined by $f_\beta(t, A) = a(t)A - \beta A^\sigma$ (Proposition 2.5 below). The idea in this case is to construct an evolution system $V_\beta(t, s)$ which will be defined similarly as in (2.2). For this, we will need to choose an appropriate value for ϕ in a contour similar to Γ_ϕ . In particular, we will require that $\sigma > 1$ and $\sigma(\pi/2 - \theta) < \pi/2$ in order to allow $\pi/2\sigma > \phi > \pi/2 - \theta$. As noted

FIGURE 1. Γ_ϕ

in the introduction, since $\theta \in (0, \pi/4]$, these requirements force $1 < \sigma < 2$ and so we will need to make sense of the operator A^σ which is defined by the fractional power. To this end, we will require the assumption that $0 \in \rho(A)$ (see Definition 2.3 below).

Definition 2.3 ([7, Definition 2.4]). Let $-A$ be the infinitesimal generator of a bounded holomorphic semigroup of angle θ , and let $0 \in \rho(A)$. For $\sigma > 0$, the *fractional power of A* is defined as follows:

$$A^{-\sigma} = \frac{1}{2\pi i} \int_{\Gamma} w^{-\sigma} (w - A)^{-1} dw, \quad (2.3)$$

where $w^{-\sigma}$ is defined by the principal branch, and Γ is a path running from $\infty e^{i\phi}$ to $\infty e^{-i\phi}$ with $\pi > \phi > \pi/2 - \theta$ while avoiding the negative real axis and the origin. Define $A^\sigma = (A^{-\sigma})^{-1}$ (see Lemma 2.4 (i) below) and $A^0 = I$.

Note, in Definition 2.3, the definition of A^σ relies on the fact that the operator in (2.3) is invertible which follows from the following properties of the fractional power.

Lemma 2.4 ([7, Lemma 2.5], [17, Lemma 2.6.6, Theorem 2.6.8]). *Let $-A$ be the infinitesimal generator of a bounded holomorphic semigroup of angle θ , and let $0 \in \rho(A)$. Then*

- (i) $A^{-\sigma}$ is a bounded, injective operator for $\sigma > 0$.
- (ii) A^σ is a closed operator, and $\text{Dom}(A^\sigma) \subseteq \text{Dom}(A^{\sigma'})$ for $\sigma > \sigma' > 0$.
- (iii) $\text{Dom}(A^\sigma)$ is dense in X for every $\sigma \geq 0$.

- (iv) $A^{\sigma_1+\sigma_2}x = A^{\sigma_1}A^{\sigma_2}x$ for every $\sigma_1, \sigma_2 \in \mathbb{R}$ and $x \in \text{Dom}(A^\sigma)$ where $\sigma = \max\{\sigma_1, \sigma_2, \sigma_1 + \sigma_2\}$.

Proposition 2.5. *Let $-A$ be the infinitesimal generator of a bounded holomorphic semigroup of angle $\theta \in (0, \pi/4]$, and let $0 \in \rho(A)$. Let $0 < \beta < 1$ and assume σ satisfies $\sigma > 1$ and $\sigma(\pi/2 - \theta) < \pi/2$. Define the family of operators $f_\beta(t, A), 0 \leq t \leq T$ by*

$$f_\beta(t, A) = a(t)A - \beta A^\sigma.$$

Then (1.3) is well-posed with unique classical solution $v_\beta(t) = V_\beta(t, s)\chi$ for each $\chi \in X$ where

$$V_\beta(t, s) = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_\phi} e^{\int_s^t f_\beta(\tau, w) d\tau} (w - A)^{-1} dw & 0 \leq s < t \leq T \\ I & t = s, \end{cases}$$

and Γ_ϕ is a contour lying in $\rho(A)$ that is similar to that in Figure 1, with $\pi/2\sigma > \phi > \pi/2 - \theta$ but avoids the negative real axis and the origin.

Proof. Notice our choice for ϕ is valid by the assumption $\sigma(\pi/2 - \theta) < \pi/2$. We first show that $V_\beta(t, s)$ is uniformly bounded for $0 \leq s \leq t \leq T$. Following [7, Proof of Theorem 3.1], we will show this in two cases. Let $0 \leq s < t \leq T$. Since $0 \in \rho(A)$ and the resolvent set is an open set in the complex plane, there exists a closed disk of radius $d \in (0, 1)$ centered at the origin that is fully contained in $\rho(A)$. In the first case, if $(t-s)^{-1/\sigma} \leq d$, using Cauchy's Theorem, we may shift Γ_ϕ within $\rho(A)$ to the contour (see Figure 2) consisting of the three pieces

$$\begin{aligned} \Gamma^1 &= \{re^{i\phi} : r \geq (t-s)^{-1/\sigma}\}, \\ \Gamma^2 &= \{(t-s)^{-1/\sigma}e^{-i\theta'} : -\phi \leq \theta' \leq \phi\}, \\ \Gamma^3 &= \{re^{-i\phi} : r \geq (t-s)^{-1/\sigma}\}. \end{aligned}$$

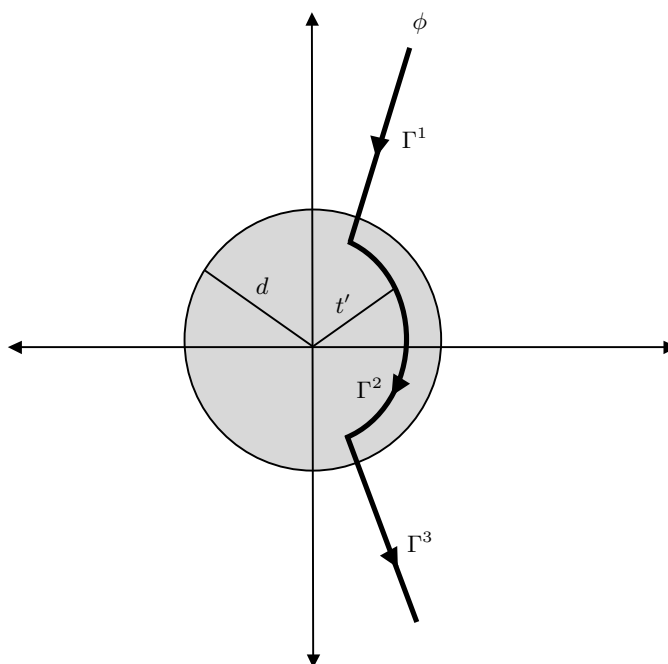
First consider $w \in \Gamma^1 \cup \Gamma^3$. Fix $\theta_1 < \theta$ so that $\phi > \pi/2 - \theta_1 > \pi/2 - \theta$. We have $\text{dist}(w, \bar{S}_{\pi/2-\theta_1}) = |w| \sin(\phi - (\pi/2 - \theta_1))$ (cf. [4, Figure 2]) so that by Theorem 2.2,

$$\|(w - A)^{-1}\| \leq \frac{M_1}{|w| \sin(\phi - (\pi/2 - \theta_1))}. \quad (2.4)$$

Set $M'_1 = M_1/\sin(\phi - (\pi/2 - \theta_1))$ and $B = \max_{t \in [0, T]} |a(t)|$. Then

$$\begin{aligned} \left\| \int_{\Gamma^1 \cup \Gamma^3} \right\| &\leq M'_1 \int_{\Gamma^1 \cup \Gamma^3} |e^{\int_s^t (a(\tau)w - \beta w^\sigma) d\tau}| |w|^{-1} |dw| \\ &= 2M'_1 \int_{(t-s)^{-1/\sigma}}^\infty e^{\int_s^t (a(\tau)r \cos \phi - \beta r^\sigma \cos \sigma \phi) d\tau} r^{-1} dr \\ &\leq 2M'_1 \int_{(t-s)^{-1/\sigma}}^\infty e^{B(t-s)r \cos \phi - \beta(t-s)r^\sigma \cos \sigma \phi} r^{-1} dr \\ &= 2M'_1 \int_1^\infty e^{B(t-s)^{1-1/\sigma} x \cos \phi - \beta x^\sigma \cos \sigma \phi} x^{-1} dx \\ &\leq 2M'_1 \int_1^\infty e^{BT^{1-1/\sigma} x \cos \phi - \beta x^\sigma \cos \sigma \phi} dx \leq K \end{aligned}$$

where K is a constant independent of t and s since $\sigma > 1$ and since $\pi/2\sigma > \phi > \pi/2 - \theta$ implies $0 < \phi < \sigma\phi < \pi/2$ so that $\cos \phi > 0$ and $\cos(\sigma\phi) > 0$.

FIGURE 2. $t' := (t - s)^{-1/\sigma} \leq d$

Also, for $w \in \Gamma^2$, we have

$$\begin{aligned}
 \left\| \int_{\Gamma^2} \right\| &\leq M_d \int_{\Gamma^2} |e^{\int_s^t (a(\tau)w - \beta w^\sigma) d\tau}| |dw| \\
 &= M_d \int_{-\phi}^{\phi} e^{\int_s^t (a(\tau)(t-s)^{-1/\sigma} \cos \theta' - \beta (t-s)^{-1} \cos \sigma \theta') d\tau} (t-s)^{-1/\sigma} d\theta' \\
 &\leq dM_d \int_{-\phi}^{\phi} e^{B(t-s)^{1-1/\sigma} \cos \theta' - \beta \cos \sigma \theta'} d\theta' \\
 &\leq dM_d \int_{-\phi}^{\phi} e^{BT^{1-1/\sigma} \cos \theta'} d\theta' \\
 &\leq dM_d e^{BT^{1-1/\sigma}} 2\phi
 \end{aligned}$$

where we have set $M_d = \max_{|w| \leq d} \|(w - A)^{-1}\|$ since $w \rightarrow (w - A)^{-1}$ is continuous on the interior of $\rho(A)$. Hence, $V_\beta(t, s)$ is bounded uniformly for $0 \leq s \leq t \leq T$ in the first case.

For the second case, if $(t - s)^{-1/\sigma} > d$, then we shift Γ_ϕ to the contour (see Figure 3) consisting of the seven pieces:

$$\begin{aligned}
 \Gamma_1 &= \{re^{i\phi} : r \geq (t - s)^{-1/\sigma}\}, & \Gamma_2 &= \{(t - s)^{-1/\sigma} e^{i\theta'} : \phi \leq \theta' \leq \pi\}, \\
 \Gamma_3 &= \{re^{i\pi} : d \leq r \leq (t - s)^{-1/\sigma}\} & \Gamma_4 &= \{de^{-i\theta'} : -\pi \leq \theta' \leq \pi\}, \\
 \Gamma_5 &= \{re^{-i\pi} : d \leq r \leq (t - s)^{-1/\sigma}\} & \Gamma_6 &= \{(t - s)^{-1/\sigma} e^{i\theta'} : -\pi \leq \theta' \leq -\phi\},
 \end{aligned}$$

$$\Gamma_7 = \{re^{-i\phi} : r \geq (t-s)^{-1/\sigma}\}.$$

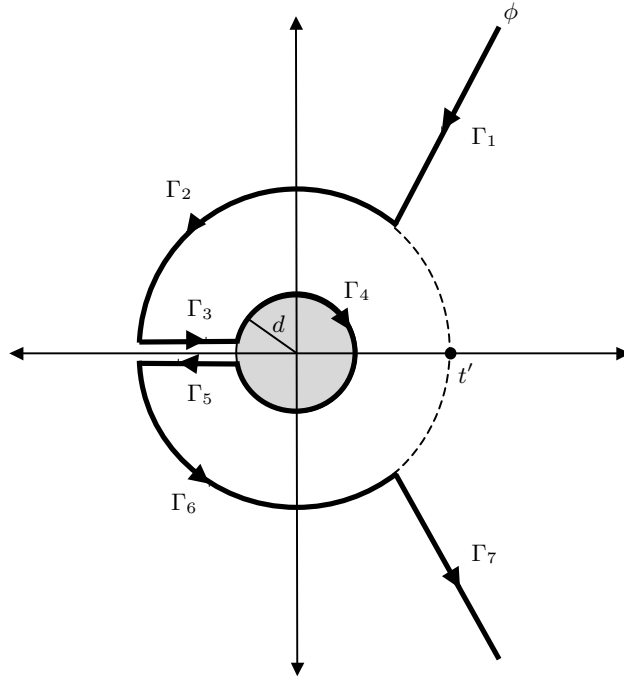


FIGURE 3. $t' := (t-s)^{-1/\sigma} > d$

First, since $\Gamma_1 = \Gamma^1$ and $\Gamma_7 = \Gamma^3$, we have $\|\int_{\Gamma_1 \cup \Gamma_7}\| = \|\int_{\Gamma^1 \cup \Gamma^3}\| \leq K$ as before. Next, note that (2.4) holds for $w \in \Gamma_2$ since these w satisfy the inequality $\text{dist}(w, \bar{S}_{\pi/2-\theta_1}) \geq \text{dist}((t-s)^{-1/\sigma} e^{i\phi}, \bar{S}_{\pi/2-\theta_1})$. Then

$$\begin{aligned} \left\| \int_{\Gamma_2} \right\| &\leq M'_1 \int_{\Gamma_2} |e^{\int_s^t (a(\tau)w - \beta w^\sigma) d\tau}| |w|^{-1} |dw| \\ &= M'_1 \int_{\phi}^{\pi} e^{\int_s^t (a(\tau)(t-s)^{-1/\sigma} \cos \theta' - \beta (t-s)^{-1} \cos \sigma \theta') d\tau} d\theta' \\ &\leq M'_1 \int_{\phi}^{\pi} e^{BT^{1-1/\sigma} \cos \phi - \beta \cos \sigma \theta'} d\theta' \\ &\leq M'_1 \int_{\phi}^{\pi} e^{1+BT^{1-1/\sigma} \cos \phi} d\theta' \\ &= M'_1 e^{1+BT^{1-1/\sigma} \cos \phi} (\pi - \phi) \end{aligned}$$

since $0 < \beta < 1$. The same estimate holds for $\|\int_{\Gamma_6}\|$.

Next, using (2.4),

$$\left\| \int_{\Gamma_3} + \int_{\Gamma_5} \right\|$$

$$\begin{aligned}
 &= \left\| \int_d^{(t-s)^{-1/\sigma}} \left(e^{\int_s^t (-a(\tau)r - \beta r^\sigma e^{-i\pi\sigma})d\tau} - e^{\int_s^t (-a(\tau)r - \beta r^\sigma e^{i\pi\sigma})d\tau} \right) (-r - A)^{-1} dr \right\| \\
 &\leq M'_1 \int_d^{(t-s)^{-1/\sigma}} \left| e^{-(\int_s^t a(\tau)d\tau)r} \left(e^{-\beta(t-s)r^\sigma e^{-i\pi\sigma}} - e^{-\beta(t-s)r^\sigma e^{i\pi\sigma}} \right) \right| r^{-1} dr \\
 &= M'_1 \int_d^{(t-s)^{-1/\sigma}} e^{-(\int_s^t a(\tau)d\tau)r} \left| e^{-\beta(t-s)r^\sigma \cos \sigma\pi} 2i \sin(\beta(t-s)r^\sigma \sin \sigma\pi) \right| r^{-1} dr \\
 &\leq M'_1 \int_d^{(t-s)^{-1/\sigma}} e^{-\beta(t-s)r^\sigma \cos \sigma\pi} 2 |\sin(\beta(t-s)r^\sigma \sin \sigma\pi)| r^{-1} dr \\
 &= M'_1 \int_{(t-s)^{1/\sigma}d}^1 e^{-\beta x^\sigma \cos \sigma\pi} 2 |\sin(\beta x^\sigma \sin \sigma\pi)| x^{-1} dx \\
 &= M'_1 \int_{(t-s)^{1/\sigma}d}^1 x^{-1/2} e^{-\beta x^\sigma \cos \sigma\pi} \{4x^{-1} \sin^2(\beta x^\sigma \sin \sigma\pi)\}^{1/2} dx \\
 &= M'_1 \int_{(t-s)^{1/\sigma}d}^1 x^{-1/2} e^{-\beta x^\sigma \cos \sigma\pi} \{2x^{-1}(1 - \cos(2\beta x^\sigma \sin \sigma\pi))\}^{1/2} dx.
 \end{aligned}$$

It is easily shown by L'Hospital's Rule that

$$2x^{-1}(1 - \cos(2\beta x^\sigma \sin \sigma\pi)) \rightarrow 0 \quad \text{as } x \rightarrow 0.$$

Hence, we have for a possibly different constant M'_1 independent of β ,

$$\left\| \int_{\Gamma_3} + \int_{\Gamma_5} \right\| \leq M'_1 \int_0^1 x^{-1/2} e^{-\beta x^\sigma \cos \sigma\pi} dx \leq M'_1 e \int_0^1 x^{-1/2} dx = M'_1 2e$$

since $0 < \beta < 1$. Finally,

$$\begin{aligned}
 \left\| \int_{\Gamma_4} \right\| &\leq M_d \int_{\Gamma_4} \left| e^{\int_s^t (a(\tau)w - \beta w^\sigma)d\tau} \right| |dw| \\
 &= dM_d \int_{-\pi}^\pi e^{\int_s^t (a(\tau)d \cos \theta' - \beta d^\sigma \cos \sigma\theta')d\tau} d\theta' \\
 &\leq dM_d \int_{-\pi}^\pi e^{BTd} e^{-\beta(t-s)d^\sigma \cos \sigma\theta'} d\theta' \\
 &\leq dM_d e^{BTd} (1 + e^{Td^\sigma}) 2\pi
 \end{aligned}$$

where $M_d = \max_{|w| \leq d} \|(w - A)^{-1}\|$ as before. Thus we have shown that in both cases, each term may be bounded independently of t and s , and so $V_\beta(t, s)$ is uniformly bounded on $0 \leq s \leq t \leq T$.

Next, we show that $(t, s) \mapsto V_\beta(t, s)$ is strongly continuous for $0 \leq s \leq t \leq T$. It follows from (2.3) and by a standard argument using Cauchy's Integral Formula that

$$V_\beta(t, s)A^{-\sigma} = \frac{1}{2\pi i} \int_{\Gamma_\phi} w^{-\sigma} e^{\int_s^t f_\beta(\tau, w)d\tau} (w - A)^{-1} dw$$

(cf. [7, p. 46]). Then since $t \mapsto f_\beta(t, w)$ is continuous, using the above calculations for $\|V_\beta(t, s)\|$, it follows by a dominated convergence argument that $\|V_\beta(t, s)A^{-\sigma} - V_\beta(t_0, s_0)A^{-\sigma}\| \rightarrow 0$ as $(t, s) \rightarrow (t_0, s_0)$. Then, for $x \in \text{Dom}(A^\sigma)$, we have

$$\begin{aligned}
 \|V_\beta(t, s)x - V_\beta(t_0, s_0)x\| &\leq \|V_\beta(t, s)A^{-\sigma} - V_\beta(t_0, s_0)A^{-\sigma}\| \|A^\sigma x\| \\
 &\rightarrow 0 \quad \text{as } (t, s) \rightarrow (t_0, s_0).
 \end{aligned}$$

Strong continuity of $V_\beta(t, s)$ then follows since $\text{Dom}(A^\sigma)$ is dense in X (Lemma 2.4 (iii)) and $V_\beta(t, s)$ is uniformly bounded.

Now, we show that the mapping $[s, T] \rightarrow X$ given by $t \mapsto V_\beta(t, s)\chi$ is a classical solution of (1.3) for $\chi \in X$. We have already established that $t \mapsto V_\beta(t, s)\chi$ is continuous on $[s, T]$. Next, we show that $\frac{\partial}{\partial t} V_\beta(t, s)\chi = f_\beta(t, A)V_\beta(t, s)\chi$ for $t \in (s, T)$. We have

$$\frac{\partial}{\partial t} V_\beta(t, s)\chi = \frac{1}{2\pi i} \int_{\Gamma_\phi} \left(\frac{\partial}{\partial t} e^{\int_s^t f_\beta(\tau, w) d\tau} \right) (w - A)^{-1} \chi dw \quad (2.5)$$

$$= \frac{1}{2\pi i} \int_{\Gamma_\phi} e^{\int_s^t f_\beta(\tau, w) d\tau} f_\beta(t, w) (w - A)^{-1} \chi dw \quad (2.6)$$

$$= \frac{1}{2\pi i} \int_{\Gamma_\phi} e^{\int_s^t f_\beta(\tau, w) d\tau} a(t) w (w - A)^{-1} \chi dw \quad (2.7)$$

$$+ \frac{1}{2\pi i} \int_{\Gamma_\phi} e^{\int_s^t f_\beta(\tau, w) d\tau} (-\beta w^\sigma) (w - A)^{-1} \chi dw. \quad (2.8)$$

Now,

$$\begin{aligned} \text{Expression (2.7)} &= a(t) \frac{1}{2\pi i} \int_{\Gamma_\phi} e^{\int_s^t f_\beta(\tau, w) d\tau} ((w - A) + A) (w - A)^{-1} \chi dw \\ &= \left(a(t) \frac{1}{2\pi i} \int_{\Gamma_\phi} e^{\int_s^t f_\beta(\tau, w) d\tau} dw \right) \chi \\ &\quad + a(t) \frac{1}{2\pi i} \int_{\Gamma_\phi} e^{\int_s^t f_\beta(\tau, w) d\tau} A (w - A)^{-1} \chi dw \\ &= a(t) AV_\beta(t, s)\chi \end{aligned}$$

where we have used Cauchy's Theorem since $w \mapsto e^{\int_s^t f_\beta(\tau, w) d\tau}$ is analytic, and also the fact that A is a closed operator.

Next, fix $t \in (s, T)$ and set $G = \frac{1}{2\pi i} \int_{\Gamma_\phi} w^\sigma e^{\int_s^t f_\beta(\tau, w) d\tau} (w - A)^{-1} dw$. It is clear that G is a bounded operator on X by calculations similarly used to calculate $\|V_\beta(t, s)\|$. Also, by (2.3) and a standard argument using Cauchy's Integral Formula (cf. [9, Equation IX.1.52]), it follows that $A^{-\sigma}G = V_\beta(t, s)$. Hence, by the fact that $A^\sigma = (A^{-\sigma})^{-1}$, we have $\text{Ran}(V_\beta(t, s)) \subseteq \text{Ran}(A^{-\sigma}) = \text{Dom}(A^\sigma)$ and $G = A^\sigma V_\beta(t, s)$. Hence (2.8) $= -\beta G\chi = -\beta A^\sigma V_\beta(t, s)\chi$, and altogether we have shown $\frac{\partial}{\partial t} V_\beta(t, s) = a(t)AV_\beta(t, s)\chi - \beta A^\sigma V_\beta(t, s)\chi = f_\beta(t, A)V_\beta(t, s)\chi$ for $t \in (s, T)$. Also by definition, $V_\beta(s, s)\chi = \chi$. Thus, $t \mapsto V_\beta(t, s)\chi$ satisfies (1.3).

Finally, calculation (2.5)–(2.8) shows that $t \mapsto f_\beta(t, A)V_\beta(t, s)\chi$ is continuous on (s, T) since $t \mapsto e^{\int_s^t f_\beta(\tau, w) d\tau} f_\beta(t, w)$ is continuous. Therefore, we have that $t \mapsto V_\beta(t, s)\chi$ is continuously differentiable on (s, T) , and so we have shown altogether that $t \mapsto V_\beta(t, s)\chi$ is a classical solution of (1.3).

It follows that problem (1.3) is well-posed due to uniqueness of the solution $t \mapsto V_\beta(t, s)\chi$ and continuous dependence of solutions on initial data, both of which are proved by standard arguments (see e.g. [4, Proof of Proposition 2.3]). \square

Corollary 2.6. *Let $0 < \beta < 1$ and let the operators $f_\beta(t, A), 0 \leq t \leq T$ and $V_\beta(t, s), 0 \leq s \leq t \leq T$ be defined under the hypotheses of Proposition 2.5. Then for small β ,*

$$\|V_\beta(t, s)\| \leq K' e^{K\beta^{-1}/(\sigma-1)}$$

for all $0 \leq s \leq t \leq T$ where K and K' are constants independent of β , t , and s .

Proof. Let $0 \leq s < t \leq T$. From our calculations for $\|V_\beta(t, s)\|$ in Proposition 2.5, all terms are bounded independently of β except $\|\int_{\Gamma^1 \cup \Gamma^3}\| = \|\int_{\Gamma^1 \cup \Gamma^7}\|$, and so we have

$$\|V_\beta(t, s)\| \leq K_1 + \frac{M'_1}{\pi} \int_1^\infty e^{BT^{1-1/\sigma} x \cos \phi - \beta x^\sigma \cos \sigma \phi} dx$$

where K_1 is a constant independent of β . It is a standard calculation to show that for small β , the function $q(x) = 2BT^{1-1/\sigma} x \cos \phi - \beta x^\sigma \cos(\sigma \phi)$ has a maximum value on $[1, \infty)$ at $x_0 = \left(\frac{2BT^{1-1/\sigma} \cos \phi}{\beta \sigma \cos(\sigma \phi)}\right)^{1/(\sigma-1)}$. Then on $[1, \infty)$,

$$\begin{aligned} 2BT^{1-1/\sigma} x \cos \phi - \beta x^\sigma \cos(\sigma \phi) &\leq q(x_0) \\ &= \beta^{-1/(\sigma-1)} \frac{(2BT^{1-1/\sigma} \cos \phi)^{\sigma/(\sigma-1)}}{\sigma^{\sigma/(\sigma-1)} \cos^{1/(\sigma-1)}(\sigma \phi)} (\sigma - 1) \\ &:= K_2 \beta^{-1/(\sigma-1)}, \end{aligned}$$

and so

$$\begin{aligned} \int_1^\infty e^{BT^{1-1/\sigma} x \cos \phi - \beta x^\sigma \cos \sigma \phi} dx &\leq e^{K_2 \beta^{-1/(\sigma-1)}} \int_1^\infty e^{-BT^{1-1/\sigma} x \cos \phi} dx \\ &= \frac{e^{K_2 \beta^{-1/(\sigma-1)}}}{BT^{1-1/\sigma} \cos \phi e^{BT^{1-1/\sigma} \cos \phi}}. \end{aligned}$$

Altogether we have $\|V_\beta(t, s)\| \leq K_1 + K_3 e^{K_2 \beta^{-1/(\sigma-1)}}$ for $0 \leq s \leq t \leq T$ where K_1 , K_2 , and K_3 are positive constants each independent of β , t , and s . It follows that for small β , $\|V_\beta(t, s)\| \leq K'_3 e^{K_2 \beta^{-1/(\sigma-1)}}$ for all $0 \leq s \leq t \leq T$ for a suitable constant K'_3 larger than K_3 . □

We now turn to the second approximate problem (1.3) motivated by the work of Showalter [20] where $\theta \in (\pi/4, \pi/2]$ and $f_\beta(t, A) = a(t)A(I + \beta A)^{-1}$ for $0 \leq t \leq T$.

Proposition 2.7. *Let $-A$ be the infinitesimal generator of a bounded holomorphic semigroup of angle $\theta \in (\pi/4, \pi/2]$, and let $0 \in \rho(A)$. Let $0 < \beta < 1$ and define the family of operators $f_\beta(t, A), 0 \leq t \leq T$ by*

$$f_\beta(t, A) = a(t)A(I + \beta A)^{-1}.$$

Then (1.3) is well-posed with unique classical solution $v_\beta(t) = V_\beta(t, s)\chi$ for each $\chi \in X$, where $V_\beta(t, s), 0 \leq s \leq t \leq T$ is an evolution system satisfying

$$\|V_\beta(t, s)\| \leq e^{CT/\beta} \quad \text{for } 0 \leq s \leq t \leq T$$

and C is a constant independent of β , t , and s .

Proof. Note by the Hille-Yosida Theorem, since $-A$ generates a bounded holomorphic semigroup, it follows that $1/\beta \in \rho(-A)$ and $\|(I + \beta A)^{-1}\| = (1/\beta)\|((1/\beta)I - (-A))^{-1}\| \leq (1/\beta) \times C\beta = C$ for some constant C independent of β (cf. [17, Theorem 1.5.3]). Now, $f_\beta(t, A)$ is a bounded operator on X for each $t \in [0, T]$ by the

following calculation:

$$\begin{aligned}
 \|f_\beta(t, A)\| &= \|a(t)A(I + \beta A)^{-1}\| \\
 &= \|a(t)\frac{1}{\beta}(I - (I + \beta A)^{-1})\| \\
 &\leq \frac{B}{\beta}(\|I\| + \|(I + \beta A)^{-1}\|) \\
 &\leq \frac{B(1 + C)}{\beta}
 \end{aligned} \tag{2.9}$$

where we have set $B = \max_{t \in [0, T]} |a(t)|$. Also, $t \rightarrow f_\beta(t, A)$ is continuous in the uniform operator topology since $A(I + \beta A)^{-1}$ is a bounded operator and $a(t)$ is a continuous function. By [17, Theorem 5.1.1], the evolution problem (1.3) is well-posed with a unique classical solution $v_\beta(t)$ for every $\chi \in X$. The solution $v_\beta(t)$ is generated by the solution operator $V_\beta(t, s)$ associated with the problem; that is $v_\beta(t) = V_\beta(t, s)\chi$. Furthermore, $V_\beta(t, s)$ is an evolution system satisfying $\|V_\beta(t, s)\| \leq e^{\int_s^t \|f_\beta(\tau, A)\| d\tau}$ (cf. [17, Theorem 5.1.2]). This together with calculation (2.9) establishes the desired result for a possibly different constant C independent of β , t , and s . \square

To summarize the results of Proposition 2.5 and Proposition 2.7, we provide the following.

Corollary 2.8. *Let $-A$ be the infinitesimal generator of a bounded holomorphic semigroup of angle θ , and let $0 \in \rho(A)$. Let $0 < \beta < 1$ and let the operators $f_\beta(t, A), 0 \leq t \leq T$ be defined by (1.4). Then (1.3) is well-posed and there exists an evolution system $V_\beta(t, s), 0 \leq s \leq t \leq T$ associated with the family $f_\beta(t, A), 0 \leq t \leq T$ such that for each $\chi \in X$, $v_\beta(t) = V_\beta(t, s)\chi$ is a unique classical solution of (1.3).*

3. PRELIMINARY LEMMAS

So far, we have shown that (1.3) is well-posed under the definition (1.4). In this case, as seen in Corollary 2.8, there is an evolution system $V_\beta(t, s)$ which generates solutions of (1.3). Since (1.2) is generally ill-posed, we may not construct an evolution system for the problem in the same way. However, we will make use of the assumption that $-A$ generates a bounded holomorphic semigroup in order to construct C -regularized evolution systems (cf. [21, 22], [5, Definition 2]) associated with problem (1.2).

Fix $\epsilon > 0$ and let $\alpha > 1$ satisfy $\alpha(\pi/2 - \theta) < \pi/2$. Then $e^{-\epsilon A^\alpha}, \epsilon > 0$ defined by

$$e^{-\epsilon A^\alpha} = \frac{1}{2\pi i} \int_{\Gamma_\phi} e^{-\epsilon w^\alpha} (w - A)^{-1} dw \tag{3.1}$$

is a strongly continuous holomorphic semigroup generated by the fractional power $-A^\alpha$ where Γ_ϕ is similar to the contour described in Proposition 2.5 but with $\pi/2\alpha > \phi > \pi/2 - \theta$ (cf. [3, Definition 3.4]). For $\epsilon > 0$, set $C_\epsilon = e^{-\epsilon A^\alpha}$. It follows that C_ϵ is injective for $\epsilon > 0$ (cf. [3, Lemma 3.1]). We construct C_ϵ -regularized evolution systems as follows.

Proposition 3.1. *Let $\epsilon > 0$ and let $\alpha > 1$ satisfy $\alpha(\pi/2 - \theta) < \pi/2$. For every $\chi \in X$, the evolution problem*

$$\begin{aligned} \frac{du}{dt} &= a(t)Au(t) \quad 0 \leq s \leq t < T \\ u(s) &= C_\epsilon \chi \end{aligned} \quad (3.2)$$

has a unique classical solution $u(t) = U_\epsilon(t, s)\chi$ where

$$U_\epsilon(t, s) = \frac{1}{2\pi i} \int_{\Gamma_\phi} e^{-\epsilon w^\alpha} e^{(\int_s^t a(\tau) d\tau)w} (w - A)^{-1} dw$$

for all $0 \leq s \leq t \leq T$ and Γ_ϕ is similar to the contour described in Proposition 2.5 with $\pi/2\alpha > \phi > \pi/2 - \theta$.

Proof. The proof is similar to that of Proposition 2.5. In particular, $U_\epsilon(t, s)$ is a uniformly bounded operator on X for $0 \leq s \leq t \leq T$ by the assumptions on α . Also, the function $t \mapsto U_\epsilon(t, s)\chi$ is a unique classical solution of (3.2) since $\frac{\partial}{\partial t} U_\epsilon(t, s)\chi = a(t)AU_\epsilon(t, s)\chi$ for $t \in (s, T)$, and by equation (3.1),

$$U_\epsilon(s, s)\chi = \frac{1}{2\pi i} \int_{\Gamma_\phi} e^{-\epsilon w^\alpha} (w - A)^{-1} \chi dw = e^{-\epsilon A^\alpha} \chi = C_\epsilon \chi.$$

□

Lemma 3.2. *Let $\chi \in X$. If $u(t)$ is a classical solution of problem (1.2), then*

$$C_\epsilon u(t) = U_\epsilon(t, s)\chi \quad \text{for all } t \in [s, T].$$

Proof. Since $C_\epsilon \in B(X)$ and C_ϵ commutes with A , it is easily shown that $C_\epsilon u(t)$ is a classical solution of (3.2). The uniqueness of solutions from Proposition 3.1 then yields the desired result. □

To establish regularization, we will make use of the nature in which the operators $f_\beta(t, A)$ approximate the operators $a(t)A$. Motivated by the approximation condition, Condition A of Ames and Hughes (cf. [2, Definition 1]), we demonstrate the following property.

Lemma 3.3. *Let $-A$ be the infinitesimal generator of a bounded holomorphic semigroup of angle θ , and let $0 \in \rho(A)$. Let $0 < \beta < 1$ and let the family of operators $f_\beta(t, A), 0 \leq t \leq T$ be defined by (1.4). Then there exist positive constants R and κ each independent of β and t such that $\text{Dom}(A^{1+\kappa}) \subseteq \text{Dom}(f_\beta(t, A))$ and*

$$\|(-a(t)A + f_\beta(t, A))\psi\| \leq R\beta\|A^{1+\kappa}\psi\| \quad (3.3)$$

for all $t \in [0, T]$ and for all $\psi \in \text{Dom}(A^{1+\kappa})$.

Note that in the statement of the lemma we use implicitly that $\text{Dom}(A^{1+\kappa}) \subseteq \text{Dom}(A)$ which follows from Lemma 2.4 (ii).

Proof. First, assume $\theta \in (0, \pi/4]$ so that $f_\beta(t, A)$ is defined as in Proposition 2.5 where σ satisfies $\sigma > 1$ and $\sigma(\pi/2 - \theta) < \pi/2$. Then for $\psi \in \text{Dom}(A^\sigma)$ and $t \in [0, T]$, we have $\psi \in \text{Dom}(f_\beta(t, A))$ and

$$\|(-a(t)A + f_\beta(t, A))\psi\| = \|(-a(t)A + (a(t)A - \beta A^\sigma))\psi\| = \beta\|A^\sigma\psi\|.$$

Hence, (3.3) is satisfied with $R = 1$ and $\kappa = \sigma - 1$.

Next, we assume that $\theta \in (\pi/4, \pi/2]$ in which case $f_\beta(t, A)$ is defined as in Proposition 2.7. Then $f_\beta(t, A)$ is a bounded, everywhere defined operator and so $\text{Dom}(f_\beta(t, A)) = X$ for each $t \in [0, T]$. Further, for $\psi \in \text{Dom}(A^2)$,

$$\begin{aligned} \|(-a(t)A + f_\beta(t, A))\psi\| &= \|(-a(t)A + a(t)A(I + \beta A)^{-1})\psi\| \\ &= \|-a(t)A(I - (I + \beta A)^{-1})\psi\| \\ &= \|-a(t)A(\beta A(I + \beta A)^{-1})\psi\| \\ &= \|-a(t)\beta(I + \beta A)^{-1}A^2\psi\| \\ &\leq B\beta\|(I + \beta A)^{-1}\| \|A^2\psi\| \\ &\leq BC\beta\|A^2\psi\|, \end{aligned}$$

where $B = \max_{t \in [0, T]} |a(t)|$ and C is as in the proof of Proposition 2.7. Hence, (3.3) is satisfied with $R = BC$ and $\kappa = 1$. \square

In light of Lemma 3.3, for each $t \in [0, T]$, we define the operator $g_\beta(t, A)$ in X by

$$g_\beta(t, A)x = -a(t)Ax + f_\beta(t, A)x \quad (3.4)$$

for $x \in \text{Dom}(A) \cap \text{Dom}(f_\beta(t, A))$. Properties of the operators $g_\beta(t, A)$, $0 \leq t \leq T$ and associated evolution systems will be used heavily in proving Hölder-continuous dependence on modeling, those of which we provide now in the following proposition.

Proposition 3.4. *Let $-A$ be the infinitesimal generator of a bounded holomorphic semigroup of angle θ , and let $0 \in \rho(A)$. For $0 < \beta < 1$, let the operators $f_\beta(t, A)$, $0 \leq t \leq T$ and $g_\beta(t, A)$, $0 \leq t \leq T$ be defined by (1.4) and (3.4) respectively. Then there exists an evolution system $W_\beta(t, s)$, $0 \leq s \leq t \leq T$ associated with the family $g_\beta(t, A)$, $0 \leq t \leq T$ satisfying the following properties:*

- (i) $\|W_\beta(t, s)\| \leq L$ for all $0 \leq s \leq t \leq T$ where L is a constant independent of t , s , and β .
- (ii) $\frac{\partial}{\partial t} W_\beta(t, s)\chi = g_\beta(t, A)W_\beta(t, s)\chi$ for $0 \leq s < t < T$ for every $\chi \in X$.
- (iii) $\frac{\partial}{\partial s} W_\beta(t, s)\chi = -W_\beta(t, s)g_\beta(s, A)\chi$ for $0 < s < t \leq T$ for every $\chi \in X$.

Proof. First, if $\theta \in (0, \pi/4]$, then $g_\beta(t, A) = -\beta A^\sigma$ by equation (3.4), and as in Proposition 2.5, we may define the two-parameter family of bounded operators $W_\beta(t, s)$, $0 \leq s \leq t \leq T$ on X by $W_\beta(t, s) = I$ when $t = s$ and

$$\begin{aligned} W_\beta(t, s) &= \frac{1}{2\pi i} \int_{\Gamma_\phi} e^{\int_s^t (-\beta w^\sigma) d\tau} (w - A)^{-1} dw \\ &= \frac{1}{2\pi i} \int_{\Gamma_\phi} e^{-\beta(t-s)w^\sigma} (w - A)^{-1} dw \end{aligned}$$

when $t \neq s$. It follows as in arguments in the proof of Proposition 2.5 that $W_\beta(t, s)$ is uniformly bounded for $0 \leq s \leq t \leq T$, say $\|W_\beta(t, s)\| \leq L$, and for every $\chi \in X$, $W_\beta(t, s)\chi$ satisfies (ii) and (iii).

In fact, it may be shown that $\|W_\beta(t, s)\| \leq L$ where L is independent of β in the following way. Similar to the proof of Proposition 2.5, the bound for $\|W_\beta(t, s)\|$ is calculated in two cases, the first when $\beta^{-1/\sigma}(t-s)^{-1/\sigma} \leq d$ and the second when $\beta^{-1/\sigma}(t-s)^{-1/\sigma} > d$ where d is the radius of the disk contained in $\rho(A)$ as in

Figure 2 and Figure 3. For the pieces

$$\begin{aligned} \Gamma^1 &= \Gamma_1 = \{re^{i\phi} : r \geq \beta^{-1/\sigma}(t-s)^{-1/\sigma}\}, \\ \Gamma^3 &= \Gamma_7 = \{re^{-i\phi} : r \geq \beta^{-1/\sigma}(t-s)^{-1/\sigma}\}, \end{aligned}$$

we have the calculation

$$\begin{aligned} \left\| \int_{\Gamma_1 \cup \Gamma_7} \right\| &= \left\| \int_{\Gamma^1 \cup \Gamma^3} \right\| \leq M'_1 \int_{\Gamma^1 \cup \Gamma^3} |e^{-\beta(t-s)w^\sigma}| |w|^{-1} |dw| \\ &= 2M'_1 \int_{\beta^{-1/\sigma}(t-s)^{-1/\sigma}}^\infty e^{-\beta(t-s)r^\sigma \cos \sigma\phi} r^{-1} dr \\ &= 2M'_1 \int_1^\infty e^{-x^\sigma \cos \sigma\phi} x^{-1} dx \\ &\leq 2M'_1 \int_1^\infty e^{-x^\sigma \cos \sigma\phi} dx \leq K \end{aligned}$$

where K is a constant independent of t, s , and β since $\sigma > 1$ and $0 < \sigma\phi < \pi/2$ because $\pi/2\sigma > \phi > \pi/2 - \theta$. Also, as in the proof of Proposition 2.5, in either of the two cases, the remaining pieces of the contour may be bounded independently of t, s , and β . Hence (i)–(iii) are satisfied and the proposition is proved when $\theta \in (0, \pi/4]$.

If, on the other hand, $\theta \in (\pi/4, \pi/2]$ as in Proposition 2.7, then $g_\beta(t, A) = -a(t)A + a(t)A(I + \beta A)^{-1}$ and in this case, we use perturbation theory to construct an evolution system $W_\beta(t, s), 0 \leq s \leq t \leq T$ satisfying (i)–(iii). We've seen so far that $A(I + \beta A)^{-1}$ is a bounded operator on X . Then since $-A$ generates a bounded holomorphic semigroup of angle θ , it follows that $-(A - A(I + \beta A)^{-1})$ is also the infinitesimal generator of a holomorphic semigroup of the same angle (cf. [17, Corollary 3.2.2]). Set $G_\beta = A - A(I + \beta A)^{-1}$. It is shown in [8] that $\mathbb{C} \setminus S_{\pi-2\theta} \subseteq \rho(G_\beta)$ where $S_{\pi-2\theta} = \{re^{i\theta'} : r > 0, |\theta'| < \pi - 2\theta\}$, and

$$\|(w - G_\beta)^{-1}\| \leq \frac{M}{|w|} \quad \text{for } w \in \mathbb{C} \setminus S_{\pi-2\theta}$$

where M is a constant independent of β (cf. [8, Theorem 2.1]). Hence for $0 \leq s \leq t \leq T$, the operator $W_\beta(t, s)$ defined by

$$W_\beta(t, s) = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_\phi} e^{-(\int_s^t a(\tau) d\tau)w} (w - G_\beta)^{-1} dw & 0 \leq s < t \leq T \\ I & t = s, \end{cases}$$

where Γ_ϕ is as in Figure 1 with $\pi/2 > \phi > \pi - 2\theta$, is a well-defined uniformly bounded operator satisfying $\|W_\beta(t, s)\| \leq L$ for $0 \leq s \leq t \leq T$ where L is a constant independent of β . Hence, (i) is satisfied. Also, similar to calculation (2.5)–(2.8), it is standard to show that for every $\chi \in X$, $\frac{\partial}{\partial t} W_\beta(t, s)\chi = -a(t)G_\beta W_\beta(t, s)\chi = g_\beta(t, A)W_\beta(t, s)\chi$ for $0 \leq s < t < T$ and $\frac{\partial}{\partial s} W_\beta(t, s)\chi = -W_\beta(t, s)(-a(s)G_\beta)\chi = -W_\beta(t, s)g_\beta(s, A)\chi$ for $0 < s < t \leq T$. Therefore (ii) and (iii) are satisfied as well. □

Corollary 3.5. *Let $\epsilon > 0$. Then*

$$U_\epsilon(t, s)W_\beta(t, s) = C_\epsilon V_\beta(t, s) = W_\beta(t, s)U_\epsilon(t, s)$$

for all $0 \leq s \leq t \leq T$.

Proof. The result follows from uniqueness of solutions as each term applied to $\chi \in X$ is a classical solution of the well-posed evolution problem (1.3) with initial data $C_\epsilon \chi$. \square

4. HÖLDER-CONTINUOUS DEPENDENCE ON MODELING

We now use the results of Section 2 and Section 3 to prove Hölder-continuous dependence on modeling for the problems (1.2) and (1.3), meaning a small change in the models from (1.2) to (1.3) implies a small change in the corresponding solutions. Again, as in Section 2 and Section 3, we assume that $-A$ generates a bounded holomorphic semigroup $T(t)$ of angle θ on X and $0 \in \rho(A)$. For $z \in S_\theta$, let us denote $T(z)$ by $T(z) = e^{-zA}$ and also define e^{-zA} to be the identity operator when $z = 0$.

Assume $u(t)$ and $v_\beta(t)$ are classical solutions of (1.2) and (1.3) respectively where $\chi \in X$ and let $\epsilon > 0$ be arbitrary. Then since C_ϵ is bounded and since e^{-zA} is uniformly bounded in each sector S_{θ_1} , $\theta_1 < \theta$ (Definition 2.1 (iii)), we may define for $\theta_1 \in (0, \theta)$ and for $\zeta = t + re^{\pm i\theta_1}$ in the bent strip $S = \{\zeta = t + re^{\pm i\theta_1} : s \leq t \leq T, r \geq 0\}$,

$$\phi_\epsilon(\zeta) = e^{-(re^{\pm i\theta_1})A} C_\epsilon(u(t) - v_\beta(t)).$$

Ultimately, we will apply Carleman's Inequality (cf. [14]) to a function related to $\phi_\epsilon(\zeta)$ on the bent strip S . Our methods are motivated by Agmon and Nirenberg [1].

Lemma 4.1. *Let $\epsilon > 0$. Then*

$$\phi_\epsilon(\zeta) = e^{-(re^{\pm i\theta_1})A} (U_\epsilon(t, s)\chi - C_\epsilon V_\beta(t, s)\chi)$$

for all $\zeta = t + re^{\pm i\theta} \in S$.

The above lemma follows immediately from Lemma 3.2 and Corollary 2.8.

Lemma 4.2 ([1, p. 148]). *Let $\phi(z)$ be a continuous and bounded complex function on the bent strip $S = \{z = x + \eta e^{\pm i\theta} : s \leq x \leq T, \eta \geq 0\}$. For $\zeta = t + re^{\pm i\theta} \in S$, define*

$$\Phi(\zeta) = -\frac{1}{\pi} \int \int_S \phi(z) \left(\frac{1}{z - \zeta} + \frac{1}{\bar{z} + 1 + \zeta} \right) dx d\eta.$$

Then $\Phi(\zeta)$ is absolutely convergent, $\bar{\partial}\Phi(\zeta) = \phi(\zeta)$ where $\bar{\partial}$ denotes the Cauchy-Riemann operator, and there exists a constant \tilde{K} such that

$$\int_{-\infty}^{\infty} \left| \frac{1}{z - \zeta} + \frac{1}{\bar{z} + 1 + \zeta} \right| d\eta \leq \tilde{K} \left(1 + \log \frac{1}{|x - t|} \right)$$

if $x \neq t$.

We prove now the following theorem establishing Hölder-continuous dependence on modeling for problems (1.2) and (1.3). We will use the results of this theorem to aid us in proving regularization in Section 5.

Theorem 4.3. *Let $-A$ be the infinitesimal generator of a bounded holomorphic semigroup of angle θ on a Banach space X and let $0 \in \rho(A)$. For $0 < \beta < 1$, let the family of operators $f_\beta(t, A)$, $0 \leq t \leq T$ be defined by (1.4). Let $u(t)$ and $v_\beta(t)$ be classical solutions of (1.2) and (1.3) respectively with $\chi \in X$, and assume that there exists a constant $M' \geq 0$ such that $\|A^{2+\kappa}u(t)\| \leq M'$ for all $t \in [s, T]$ where*

κ is defined by Lemma 3.3. Then there exist constants \tilde{C} and M independent of β such that for $0 \leq s \leq t < T$,

$$\|u(t) - v_\beta(t)\| \leq \tilde{C}\beta^{1-h(t)}M^{h(t)}$$

where $h(\zeta)$ is a harmonic function which is bounded and continuous on the bent strip $S = \{\zeta = t + re^{\pm i\theta_1} : s \leq t \leq T, r \geq 0\}$, $\theta_1 \in (0, \theta)$, and assumes the values 0 and 1 respectively on the left and right hand boundary curves of S .

Proof. Let $\epsilon > 0$, $\chi \in X$, and define

$$\phi_\epsilon(\zeta) = e^{-(re^{\pm i\theta_1})A}C_\epsilon(u(t) - v_\beta(t))$$

for $\zeta = t + re^{\pm i\theta_1} \in S$ as in the discussion preceding Lemma 4.1. Intending to apply Lemma 4.2, we determine $\bar{\partial}\phi_\epsilon(\zeta)$. Since $e^{-(re^{\pm i\theta_1})A}$ is bounded for every $r \geq 0$ and since C_ϵ commutes with A , we have by Lemma 4.1,

$$\begin{aligned} \frac{\partial}{\partial t}\phi_\epsilon(\zeta) &= \frac{\partial}{\partial t}e^{-(re^{\pm i\theta_1})A}(U_\epsilon(t, s)\chi - C_\epsilon V_\beta(t, s)\chi) \\ &= e^{-(re^{\pm i\theta_1})A}\left(\frac{\partial}{\partial t}U_\epsilon(t, s)\chi - C_\epsilon\frac{\partial}{\partial t}V_\beta(t, s)\chi\right) \\ &= e^{-(re^{\pm i\theta_1})A}(a(t)AU_\epsilon(t, s)\chi - f_\beta(t, A)C_\epsilon V_\beta(t, s)\chi). \end{aligned}$$

Also, since $-A$ generates e^{-zA} and since both $U_\epsilon(t, s)\chi$ and $C_\epsilon V_\beta(t, s)$ are in $\text{Dom}(A)$, we have

$$\begin{aligned} \frac{\partial}{\partial r}\phi_\epsilon(\zeta) &= \frac{\partial}{\partial r}e^{-(re^{\pm i\theta_1})A}(U_\epsilon(t, s)\chi - C_\epsilon V_\beta(t, s)\chi) \\ &= e^{-(re^{\pm i\theta_1})A}(-e^{\pm i\theta_1}A)(U_\epsilon(t, s)\chi - C_\epsilon V_\beta(t, s)\chi). \end{aligned}$$

Therefore, by definition of the Cauchy-Riemann operator $\bar{\partial}$,

$$\begin{aligned} \bar{\partial}\phi_\epsilon(\zeta) &= \frac{1}{2i \sin(\pm\theta_1)}\left(e^{\pm i\theta_1}\frac{\partial}{\partial t}\phi_\epsilon(\zeta) - \frac{\partial}{\partial r}\phi_\epsilon(\zeta)\right) \\ &= \frac{e^{\pm i\theta_1}}{2i \sin(\pm\theta_1)}\left[e^{-(re^{\pm i\theta_1})A}(a(t)AU_\epsilon(t, s)\chi - f_\beta(t, A)C_\epsilon V_\beta(t, s)\chi) \right. \\ &\quad \left. + e^{-(re^{\pm i\theta_1})A}(AU_\epsilon(t, s)\chi - AC_\epsilon V_\beta(t, s)\chi)\right]. \end{aligned} \tag{4.1}$$

Following [1], define

$$\Phi_\epsilon(\zeta) = -\frac{1}{\pi} \iint_S \bar{\partial}\phi_\epsilon(z) \left(\frac{1}{z - \zeta} + \frac{1}{\bar{z} + 1 + \zeta}\right) dx d\eta,$$

where $z = x + \eta e^{\pm i\theta_1}$ and $\zeta = t + re^{\pm i\theta_1}$ are in S . In order to apply Lemma 4.2, we show that $\bar{\partial}\phi_\epsilon(z)$ is continuous and bounded on S . We first show that it is bounded on S . Let $z = x + \eta e^{\pm i\theta_1} \in S$ be arbitrary. We have from (4.1),

$$\begin{aligned} \|\bar{\partial}\phi_\epsilon(z)\| &\leq \frac{1}{2|\sin \theta_1|} \|e^{-(\eta e^{\pm i\theta_1})A}\left(\|a(x)AU_\epsilon(x, s)\chi - f_\beta(x, A)C_\epsilon V_\beta(x, s)\chi\| \right. \\ &\quad \left. + \|AU_\epsilon(x, s)\chi - AC_\epsilon V_\beta(x, s)\chi\|\right) \\ &\leq \frac{\Theta}{2|\sin \theta_1|} \left(\|a(x)AU_\epsilon(x, s)\chi - a(x)AC_\epsilon V_\beta(x, s)\chi\| \right. \\ &\quad \left. + \|a(x)AC_\epsilon V_\beta(x, s)\chi - f_\beta(x, A)C_\epsilon V_\beta(x, s)\chi\| \right) \end{aligned}$$

$$\begin{aligned}
 & + \|AU_\epsilon(x, s)\chi - AC_\epsilon V_\beta(x, s)\chi\| \\
 \leq & \frac{\Theta}{2|\sin \theta_1|} \left((B + 1)\|AU_\epsilon(x, s)\chi - AC_\epsilon V_\beta(x, s)\chi\| \right. \\
 & \left. + \|a(x)AC_\epsilon V_\beta(x, s)\chi - f_\beta(x, A)C_\epsilon V_\beta(x, s)\chi\| \right)
 \end{aligned}$$

where we have set $\Theta = \max_{r \geq 0} \|e^{-(re^{\pm i\theta_1})A}\|$ and $B = \max_{t \in [0, T]} |a(t)|$. Since $U_\epsilon(x, s)\chi \in C_\epsilon(X) \subseteq \text{Dom}(A^j)$ for every $j \in \mathbb{N}$ (cf. [3, Proposition 2.10]), it follows that $AU_\epsilon(x, s)\chi \in \text{Dom}(A^j)$ for every j as well. Therefore, we have $AU_\epsilon(x, s)\chi \in \text{Dom}(A^{1+\kappa})$ by Lemma 2.4 (ii). Hence, by Corollary 3.5, Proposition 3.4, and Lemma 3.3,

$$\begin{aligned}
 \|AU_\epsilon(x, s)\chi - AC_\epsilon V_\beta(x, s)\chi\| &= \|AU_\epsilon(x, s)\chi - AW_\beta(x, s)U_\epsilon(x, s)\chi\| \\
 &= \|(I - W_\beta(x, s))AU_\epsilon(x, s)\chi\| \\
 &= \left\| \int_s^x \frac{\partial}{\partial \tau} (W_\beta(x, \tau)AU_\epsilon(x, s)\chi) d\tau \right\| \\
 &= \left\| \int_s^x -W_\beta(x, \tau)g_\beta(\tau, A)AU_\epsilon(x, s)\chi d\tau \right\| \tag{4.2} \\
 &\leq \int_s^x L \|g_\beta(\tau, A)AU_\epsilon(x, s)\chi\| d\tau \\
 &\leq TLR\beta \|A^{1+\kappa}AU_\epsilon(x, s)\chi\|.
 \end{aligned}$$

Also, by Lemma 3.3,

$$\begin{aligned}
 \|a(x)AC_\epsilon V_\beta(x, s)\chi - f_\beta(x, A)C_\epsilon V_\beta(x, s)\chi\| &= \|(-a(x)A + f_\beta(x, A))C_\epsilon V_\beta(x, s)\chi\| \\
 &\leq R\beta \|A^{1+\kappa}C_\epsilon V_\beta(x, s)\chi\| \\
 &= R\beta \|A^{1+\kappa}W_\beta(x, s)U_\epsilon(x, s)\chi\| \\
 &= R\beta \|W_\beta(x, s)A^{1+\kappa}U_\epsilon(x, s)\chi\| \\
 &\leq LR\beta \|A^{1+\kappa}U_\epsilon(x, s)\chi\|.
 \end{aligned}$$

Thus we have shown that

$$\|\bar{\partial}\phi_\epsilon(z)\| \leq \frac{\Theta(T + 1)LR\beta}{2|\sin \theta_1|} \left((B + 1)\|A^{1+\kappa}AU_\epsilon(x, s)\chi\| + \|A^{1+\kappa}U_\epsilon(x, s)\chi\| \right).$$

Now, by the assumption that $\|A^{2+\kappa}u(t)\| \leq M'$ for all $t \in [s, T]$ and by Lemma 2.4 (iv), we have $\|A^{1+\kappa}u(t)\| = \|A^{-1}A^{2+\kappa}u(t)\| \leq M''$ for all $t \in [s, T]$ for some constant $M'' \geq 0$, where we have used the fact that $0 \in \rho(A)$. By the fact that $C_\epsilon = e^{-\epsilon A^\alpha}$, $\epsilon > 0$ is a holomorphic semigroup, set $J = \sup_{0 < \epsilon < 1} \|C_\epsilon\|$. Then for small $\epsilon > 0$, since C_ϵ commutes with A , we have from Lemma 3.2,

$$\|A^{1+\kappa}U_\epsilon(x, s)\chi\| = \|A^{1+\kappa}C_\epsilon u(x)\| = \|C_\epsilon A^{1+\kappa}u(x)\| \leq JM'' \tag{4.3}$$

and similarly $\|A^{1+\kappa}AU_\epsilon(x, s)\chi\| = \|A^{2+\kappa}U_\epsilon(x, s)\chi\| \leq JM'$. Therefore, we have shown that

$$\|\bar{\partial}\phi_\epsilon(z)\| \leq \beta C', \tag{4.4}$$

where C' is a constant independent of ϵ and also of β since L is independent of β (Proposition 3.4 (i)).

We have shown that $\bar{\partial}\phi_\epsilon(z)$ is bounded on S . It follows easily that $\bar{\partial}\phi_\epsilon(z)$ is also continuous on S . Having satisfied the hypotheses of Lemma 4.2, it follows that

$\Phi_\epsilon(\zeta)$ is absolutely convergent, $\bar{\partial}\Phi_\epsilon(\zeta) = \bar{\partial}\phi_\epsilon(\zeta)$, and there exists a constant \tilde{K} such that, for $x \neq t$,

$$\int_{-\infty}^{\infty} \left| \frac{1}{z - \zeta} + \frac{1}{\bar{z} + 1 + \zeta} \right| d\eta \leq \tilde{K} \left(1 + \log \frac{1}{|x - t|} \right).$$

We now construct a candidate to satisfy Carleman’s Inequality. Define $\Psi_\epsilon : S \rightarrow \mathbb{C}$ by

$$\Psi_\epsilon(\zeta) = x^*(\phi_\epsilon(\zeta) - \Phi_\epsilon(\zeta))$$

where $x^* \in X^*$, the dual space of X , is arbitrary. For ζ in the interior of S , using the results from Lemma 4.2,

$$\bar{\partial}\Psi_\epsilon(\zeta) = x^*(\bar{\partial}\phi_\epsilon(\zeta) - \bar{\partial}\Phi_\epsilon(\zeta)) = x^*(0) = 0.$$

Therefore, Ψ_ϵ is analytic on the interior of S (cf. [19, Theorem 11.2]).

Next, we show that Ψ_ϵ is bounded on S . Similar to the calculation in (4.2), and using (4.3), we have

$$\begin{aligned} \|\phi_\epsilon(\zeta)\| &= \|e^{-(re^{\pm i\theta_1})A}(U_\epsilon(t, s)\chi - C_\epsilon V_\beta(t, s)\chi)\| \\ &\leq \Theta \|U_\epsilon(t, s)\chi - C_\epsilon V_\beta(t, s)\chi\| \\ &\leq \Theta TLR\beta \|A^{1+\kappa}U_\epsilon(t, s)\chi\| \leq \beta K' \end{aligned} \tag{4.5}$$

where K' is a constant independent of β , ϵ , and ζ . Next, from (4.4) and Lemma 4.2,

$$\begin{aligned} \|\Phi_\epsilon(\zeta)\| &= \left\| -\frac{1}{\pi} \int \int_S \bar{\partial}\phi_\epsilon(z) \left(\frac{1}{z - \zeta} + \frac{1}{\bar{z} + 1 + \zeta} \right) dx d\eta \right\| \\ &\leq \frac{1}{\pi} \beta C' \int_s^T \left(\int_{-\infty}^{\infty} \left| \frac{1}{z - \zeta} + \frac{1}{\bar{z} + 1 + \zeta} \right| d\eta \right) dx \\ &\leq \beta \frac{\tilde{K}}{\pi} C' \int_s^T \left(1 + \log \frac{1}{|x - t|} \right) dx \leq \beta C' \end{aligned} \tag{4.6}$$

for a possibly different constant C' independent of β , ϵ , and ζ . Then from (4.5) and (4.6), we have for $\zeta = t + re^{\pm i\theta_1} \in S$,

$$\begin{aligned} |\Psi_\epsilon(\zeta)| &= |x^*(\phi_\epsilon(\zeta) - \Phi_\epsilon(\zeta))| \\ &\leq \|x^*\| (\|\phi_\epsilon(\zeta)\| + \|\Phi_\epsilon(\zeta)\|) \\ &\leq \beta M \|x^*\| \end{aligned} \tag{4.7}$$

where M is a constant independent of β , ϵ , and ζ .

We have shown that Ψ_ϵ is bounded on S . It is easy to show that Ψ_ϵ is also continuous on S , and we have already seen that Ψ_ϵ is analytic on the interior of S . By Carleman’s Inequality (cf. [14]), we then obtain

$$|\Psi_\epsilon(t)| \leq M_\epsilon(s)^{1-h(t)} M_\epsilon(T)^{h(t)}, \tag{4.8}$$

for $s \leq t \leq T$, where $M_\epsilon(t) = \sup_{r \geq 0} |\Psi_\epsilon(t + re^{\pm i\theta_1})|$ and h is a harmonic function which is bounded and continuous on S and assumes the values 0 and 1 respectively on the left and right hand boundary curves of S . Note that

$$\begin{aligned} \|\phi_\epsilon(s + re^{\pm i\theta_1})\| &= \|e^{-(re^{\pm i\theta_1})A}(U_\epsilon(s, s)\chi - C_\epsilon V_\beta(s, s)\chi)\| \\ &= \|e^{-(re^{\pm i\theta_1})A}(C_\epsilon\chi - C_\epsilon\chi)\| = 0. \end{aligned}$$

Then from (4.6), we have

$$|\Psi_\epsilon(s + re^{\pm i\theta_1})| \leq \|x^*\| (\|\phi_\epsilon(s + re^{\pm i\theta_1})\| + \|\Phi_\epsilon(s + re^{\pm i\theta_1})\|) \leq \|x^*\| \beta C',$$

and so

$$M_\epsilon(s) = \sup_{r \geq 0} |\Psi_\epsilon(s + re^{\pm i\theta_1})| \leq \beta C' \|x^*\|. \quad (4.9)$$

Also, from (4.7) and the fact that $0 < \beta < 1$, we have

$$M_\epsilon(T) = \max_{r \geq 0} |\Psi_\epsilon(T + re^{\pm i\theta_1})| \leq M \|x^*\|. \quad (4.10)$$

From (4.8), (4.9), and (4.10), it follows that for $s \leq t < T$,

$$|\Psi_\epsilon(t)| \leq (\beta C')^{1-h(t)} M^{h(t)} \|x^*\|.$$

Taking the supremum over $x^* \in X^*$ with $\|x^*\| \leq 1$, we have $\|\phi_\epsilon(t) - \Phi_\epsilon(t)\| \leq \tilde{C} \beta^{1-h(t)} M^{h(t)}$ for $s \leq t < T$ where \tilde{C} and M are constants each independent of β and ϵ . Then by (4.6), for $s \leq t < T$,

$$\begin{aligned} \|C_\epsilon(u(t) - v_\beta(t))\| &= \|\phi_\epsilon(t)\| \\ &= \|(\phi_\epsilon(t) - \Phi_\epsilon(t)) + \Phi_\epsilon(t)\| \\ &\leq \tilde{C} \beta^{1-h(t)} M^{h(t)} + \beta C' \\ &= (\tilde{C} + \beta^{h(t)} M^{-h(t)} C') \beta^{1-h(t)} M^{h(t)} \\ &\leq \tilde{C} \beta^{1-h(t)} M^{h(t)} \end{aligned}$$

for a possibly different constant \tilde{C} independent of β and ϵ . Finally, since $C_\epsilon \rightarrow I$ as $\epsilon \rightarrow 0$ in the strong operator topology, and since all constants on the right are independent of ϵ , we may let $\epsilon \rightarrow 0$ to obtain $\|u(t) - v_\beta(t)\| \leq \tilde{C} \beta^{1-h(t)} M^{h(t)}$ for $0 \leq s \leq t < T$ as desired. \square

5. REGULARIZATION FOR PROBLEM (1.2)

We use the inequality of Theorem 4.3 to prove the main result of the paper, that is the existence of a family of regularizing operators for the ill-posed problem (1.2) where $-A$ generates a holomorphic semigroup (not necessarily bounded) of angle θ on X . Following Definition 1.1, we have the following result.

Theorem 5.1. *Let $-A$ be the infinitesimal generator of a holomorphic semigroup of angle $\theta \in (0, \pi/2]$ on a Banach space X . Then there exists $\lambda \in \mathbb{R}$ such that*

$$\{e^{(\int_s^t a(\tau) d\tau)\lambda} \tilde{V}_\beta(t, s) : \beta > 0, t \in [s, T]\}$$

is a family of regularizing operators for the problem (1.2) where $\tilde{V}_\beta(t, s), 0 \leq s \leq t \leq T$ is the evolution system of Corollary 2.8 corresponding to the operators $f_\beta(t, A - \lambda), 0 \leq t \leq T$ defined by

$$f_\beta(t, A - \lambda) = \begin{cases} a(t)(A - \lambda) - \beta(A - \lambda)^\sigma & \text{if } \theta \in (0, \pi/4] \\ a(t)(A - \lambda)(I + \beta(A - \lambda))^{-1} & \text{if } \theta \in (\pi/4, \pi/2] \end{cases} \quad (5.1)$$

where $\sigma > 1$ when $\theta \in (0, \pi/4]$. The regularization parameter β is chosen as follows: for a given perturbed initial data χ_δ where $\|\chi - \chi_\delta\| \leq \delta$,

$$\beta = \begin{cases} (-2K/\ln \delta)^{\sigma-1} & \text{if } \theta \in (0, \pi/4] \\ -2CT/\ln \delta & \text{if } \theta \in (\pi/4, \pi/2] \end{cases}$$

where K and C are constants independent of δ .

Proof. First, in accordance with Theorem 4.3, assume that $-A$ generates a bounded holomorphic semigroup and that $0 \in \rho(A)$. Let $u(t)$ be a classical solution of (1.2) with initial data χ and assume $u(t)$ satisfies the stabilizing condition of Theorem 4.3, that is $\|A^{2+\kappa}u(t)\| \leq M'$ for all $t \in [s, T]$. Also, let $\|\chi - \chi_\delta\| \leq \delta$.

Let $v_\beta(t)$ be a solution of (1.3) and let $V_\beta(t, s), 0 \leq s \leq t \leq T$ be the evolution system given in Corollary 2.8. Then for $0 \leq s \leq t \leq T$, we have $v_\beta(t) = V_\beta(t, s)\chi$ and

$$\begin{aligned} \|u(t) - V_\beta(t, s)\chi_\delta\| &\leq \|u(t) - v_\beta(t)\| + \|V_\beta(t, s)\chi - V_\beta(t, s)\chi_\delta\| \\ &\leq \|u(t) - v_\beta(t)\| + \delta\|V_\beta(t, s)\|. \end{aligned} \quad (5.2)$$

First consider $0 \leq s \leq t < T$. If $\theta \in (0, \pi/4]$ so that $f_\beta(t, A)$ is defined as $f_\beta(t, A) = a(t)A - \beta A^\sigma$, then from Corollary 2.6, we have $\|V_\beta(t, s)\| \leq K'e^{K\beta^{-1/(\sigma-1)t}}$ for small β where K and K' are constants independent of β . Choose $\beta = (-2K/\ln \delta)^{\sigma-1}$. Then $\beta \rightarrow 0$ as $\delta \rightarrow 0$, and by (5.2) and Theorem 4.3, we have

$$\begin{aligned} \|u(t) - V_\beta(t, s)\chi_\delta\| &\leq \tilde{C}\beta^{1-h(t)}M^{h(t)} + \delta K'e^{K\beta^{-1/(\sigma-1)t}} \\ &= \tilde{C}\beta^{1-h(t)}M^{h(t)} + \sqrt{\delta}K' \\ &\rightarrow 0 \quad \text{as } \delta \rightarrow 0. \end{aligned} \quad (5.3)$$

If on the other hand $\theta \in (\pi/4, \pi/2]$, in which case $f_\beta(t, A)$ is defined as $f_\beta(t, A) = a(t)A(I + \beta A)^{-1}$, then from Proposition 2.7 we have $\|V_\beta(t, s)\| \leq e^{CT/\beta}$ where C is a constant independent of β . In this case, choose $\beta = -2CT/\ln \delta$. Then similarly $\beta \rightarrow 0$ as $\delta \rightarrow 0$, and

$$\begin{aligned} \|u(t) - V_\beta(t, s)\chi_\delta\| &\leq \tilde{C}\beta^{1-h(t)}M^{h(t)} + \delta e^{CT/\beta} \\ &= \tilde{C}\beta^{1-h(t)}M^{h(t)} + \sqrt{\delta} \\ &\rightarrow 0 \quad \text{as } \delta \rightarrow 0. \end{aligned} \quad (5.4)$$

Finally, for the case that $t = T$, from inequalities (4.7) and (4.8), it is easily shown (following the remainder of the proof of Theorem 4.3 with $t = T$) that $\|u(T) - v_\beta(T)\| \leq \beta N$ for some constant N independent of β . Then by (5.2), in the case of either approximation, we have that $\beta \rightarrow 0$ as $\delta \rightarrow 0$ and

$$\begin{aligned} \|u(T) - V_\beta(T, s)\chi_\delta\| &\leq \|u(T) - v_\beta(T)\| + \delta\|V_\beta(T, s)\| \\ &\leq \beta N + \sqrt{\delta}(K' + 1) \\ &\rightarrow 0 \quad \text{as } \delta \rightarrow 0. \end{aligned} \quad (5.5)$$

Combining (5.3), (5.4), and (5.5) proves that $\{V_\beta(t, s) : \beta > 0, t \in [s, T]\}$ is a family of regularizing operators for problem (1.2).

Now, for the general case, assume that $-A$ generates a holomorphic semigroup of angle θ on X . It is known that for $\theta' \in (0, \theta)$, then there exists $\lambda \in \mathbb{R}$ such that $-A + \lambda$ is the infinitesimal generator of a bounded holomorphic semigroup of angle θ' on X and $0 \in \rho(A - \lambda)$ (cf. [18, Section X.8, p. 253]). Let $u(t)$ be a classical solution of (1.2) with initial data $\chi \in X$. It is easily shown that $w(t) = e^{-(\int_s^t a(\tau)d\tau)\lambda}u(t)$ is then a classical solution of the evolution problem

$$\begin{aligned} \frac{dw}{dt} &= a(t)(A - \lambda)w(t) \quad 0 \leq s \leq t < T \\ w(s) &= \chi. \end{aligned} \quad (5.6)$$

Then since $-(A - \lambda)$ generates a bounded holomorphic semigroup of angle θ' and $0 \in \rho(A - \lambda)$, we have by the bounded case argument above that $\{\tilde{V}_\beta(t, s) : \beta > 0, t \in [s, T]\}$ is a family of regularizing operators for the problem (5.6) where $\tilde{V}_\beta(t, s), 0 \leq s \leq t \leq T$ is the evolution system of Corollary 2.8 corresponding to the family of operators $f_\beta(t, A - \lambda), 0 \leq t \leq T$ defined by (5.1). Hence, given $\delta > 0$ and $\|\chi - \chi_\delta\| \leq \delta$, there exists $\beta > 0$, such that $\beta \rightarrow 0$ as $\delta \rightarrow 0$ and

$$\begin{aligned} \|u(t) - e^{(\int_s^t a(\tau)d\tau)\lambda}\tilde{V}_\beta(t, s)\chi_\delta\| &= e^{(\int_s^t a(\tau)d\tau)\lambda}\|w(t) - \tilde{V}_\beta(t, s)\chi_\delta\| \\ &\rightarrow 0 \quad \text{as } \delta \rightarrow 0 \end{aligned}$$

for $0 \leq s \leq t \leq T$, proving that $\{e^{(\int_s^t a(\tau)d\tau)\lambda}\tilde{V}_\beta(t, s) : \beta > 0, t \in [s, T]\}$ is a family of regularizing operators for the problem (1.2). \square

6. EXAMPLES IN L^p SPACES

In this final section, we apply the theory of regularization in Section 5 to ill-posed partial differential equations in L^p spaces where A is a strongly elliptic differential operator. We will use the following notation (cf. [17, Chapter 7.1]). For an n -tuple of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ (called a multi-index), we define $|\alpha| = \sum_{i=1}^n \alpha_i$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Also, denote $D_k = \partial/\partial x_k$ and $D = (D_1, D_2, \dots, D_n)$. Then D^α is defined by

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \dots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

Finally, for a fixed domain Ω in \mathbb{R}^n , $W^{m,p}(\Omega)$ will denote the Sobolev space consisting of functions $u \in L^p(\Omega)$ whose derivatives $D^\alpha u$, in the sense of distributions, of order $k \leq m$ are in $L^p(\Omega)$. Also, $W_0^{m,p}(\Omega)$ denotes the subspace of functions in $W^{m,p}(\Omega)$ with compact support in Ω .

Let Ω be a bounded domain in \mathbb{R}^n with smooth boundary $\partial\Omega$. Consider the differential operator of order $2m$,

$$P(x, D) = \sum_{|\alpha| \leq 2m} h_\alpha(x) D^\alpha \tag{6.1}$$

where the coefficients $h_\alpha(x)$ are sufficiently smooth complex-valued functions of x in $\bar{\Omega}$, the closure of Ω .

Definition 6.1 ([17, Definition 7.2.1]). The operator $P(x, D)$ is called *strongly elliptic* if there exists a constant $c > 0$ such that

$$\text{Re}\{(-1)^m P_{2m}(x, \xi)\} \geq c|\xi|^{2m}$$

for all $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^n$, where $P_{2m}(x, \xi) = \sum_{|\alpha|=2m} h_\alpha(x)\xi^\alpha$.

Example 6.2. Following [7, Example 5.2], consider the nonautonomous problem

$$\begin{aligned} \frac{\partial}{\partial t} u(t, x) &= a(t)P(D)u(t, x), \quad (t, x) \in [s, T] \times \mathbb{R}^n \\ u(s, x) &= \psi(x), \quad x \in \mathbb{R}^n \end{aligned} \tag{6.2}$$

where $a \in C([0, T] : \mathbb{R}^+)$ and $P : \mathbb{R}^n \rightarrow \mathbb{C}$ is a polynomial of order $2m$ such that $A = P(D)$ is strongly elliptic with domain $W^{2m,p}(\mathbb{R}^n)$. Set

$$\mu_1 = \sup_{|\xi|=1} |\text{Re}P_{2m}(\xi)|, \quad \mu_2 = \sup_{|\xi|=1} |\text{Im}P_{2m}(\xi)|.$$

Then, as seen in [25], $-A = -P(D)$ is the generator of a holomorphic semigroup of angle θ on the Banach space $X = L^p(\mathbb{R}^n)$, $1 < p < \infty$ where

$$\theta = \begin{cases} \arctan(\mu_1/\mu_2) & \text{if } \mu_2 \neq 0 \\ \pi/2 & \text{if } \mu_2 = 0. \end{cases}$$

If $\mu_1 \leq \mu_2$ so that $\theta \in (0, \pi/4]$, then by Theorem 5.1 and (5.1), for some $\lambda \in \mathbb{R}$, the approximate well-posed problem (1.3) becomes

$$\begin{aligned} \frac{\partial}{\partial t} v(t, x) &= a(t)(P(D) - \lambda)v(t, x) - \beta(P(D) - \lambda)^\sigma v(t, x) \\ &\text{for } (t, x) \in [s, T] \times \mathbb{R}^n, \\ v(s, x) &= \psi(x) \quad \text{for } x \in \mathbb{R}^n, \end{aligned} \quad (6.3)$$

and $\{e^{(\int_s^t a(\tau)d\tau)\lambda}\tilde{V}_\beta(t, s) : \beta > 0, t \in [s, T]\}$ is a family of regularizing operators for the ill-posed problem (6.2) where $\tilde{V}_\beta(t, s), 0 \leq s \leq t \leq T$ is the evolution system of Corollary 2.8 corresponding to the operators

$$f_\beta(t, P(D) - \lambda) = a(t)(P(D) - \lambda) - \beta(P(D) - \lambda)^\sigma.$$

On the other hand, if $\mu_1 > \mu_2$ or if $\mu_2 = 0$ so that $\theta \in (\pi/4, \pi/2]$, then for some $\lambda \in \mathbb{R}$, (1.3) becomes

$$\begin{aligned} (1 - \beta\lambda + \beta P(D)) \frac{\partial}{\partial t} v(t, x) &= a(t)(P(D) - \lambda)v(t, x) \\ &\text{for } (t, x) \in [s, T] \times \mathbb{R}^n, \\ v(s, x) &= \psi(x) \quad \text{for } x \in \mathbb{R}^n. \end{aligned}$$

Again, by Theorem 5.1, $\{e^{(\int_s^t a(\tau)d\tau)\lambda}\tilde{V}_\beta(t, s) : \beta > 0, t \in [s, T]\}$ is a family of regularizing operators for the ill-posed problem (6.2) where $\tilde{V}_\beta(t, s), 0 \leq s \leq t \leq T$ is the evolution system of Corollary 2.8, in this case corresponding to the operators $f_\beta(t, P(D) - \lambda) = a(t)(P(D) - \lambda)(I + \beta(P(D) - \lambda))^{-1}$. Note, as mentioned in the introduction, the model (6.3) may still be used with $\sigma = 2$ if $\theta > \pi/4$.

Example 6.3. Following [17, Chapter 7.6], consider the nonautonomous problem

$$\begin{aligned} \frac{\partial}{\partial t} u(t, x) &= a(t)P(x, D)u(t, x) \quad \text{for } (t, x) \in [s, T] \times \Omega \\ D^\alpha u(t, x) &= 0, \quad |\alpha| < m \quad \text{for } (t, x) \in [s, T] \times \partial\Omega \\ u(s, x) &= \psi(x) \quad \text{for } x \in \Omega, \end{aligned} \quad (6.4)$$

where $a \in C([0, T] : \mathbb{R}^+)$ and $P(x, D)$ as defined in (6.1) is strongly elliptic. For $1 < p < \infty$, define the operator A_p by $\text{Dom}(A_p) = W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$ and

$$A_p u = P(x, D)u \quad \text{for } u \in \text{Dom}(A_p).$$

Then by [17, Theorem 7.3.5], $-A_p$ is the infinitesimal generator of a holomorphic semigroup of angle θ on the Banach space $X = L^p(\Omega)$ for some $\theta \in (0, \pi/2)$. As discussed in [7, Example 5.3], the exact value of θ is difficult to determine in this situation. However, as in the methods from Example 6.2, whether $\theta \in (0, \pi/4]$ or $\theta \in (\pi/4, \pi/2)$, Theorem 5.1 yields that $\{e^{(\int_s^t a(\tau)d\tau)\lambda}\tilde{V}_\beta(t, s) : \beta > 0, t \in [s, T]\}$ is a family of regularizing operators for the ill-posed problem (6.4).

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