Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 87, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

PERSISTENCE OF TRAVELING WAVE SOLUTIONS IN A BIO-REACTOR MODEL WITH STRONG GENERIC DELAY KERNELS AND NONLOCAL EFFECT

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ABSTRACT. In this article, we consider the persistence of nontrivial traveling wave solutions of a bio-reactor system with strong generic delay kernels and nonlocal effect, which models the microbial growth in a flow reactor. By using the geometric singular perturbation theory and the center manifold theorem, we show that traveling wave solutions exist provided that the delays are sufficiently small with the strong generic delay kernels.

1. INTRODUCTION

Recently, traveling wave solutions have been intensively studied in various biological models due to its significant nature in biology, see Aronson and Weinberger [1], Britton [2], Murray [11] and Volpert et al [16].

As a classical bio-reactor model, an autonomous parabolic system describing the evolutionary of microbial growth in a flow reactor has been investigated by several researchers [14, 8], such a model takes the form of

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= d\Delta u - \alpha u_x - f(u)v, \\
\frac{\partial v}{\partial t} &= \Delta v - \alpha v_x + (f(u) - k)v,
\end{aligned}
\tag{1.1}
$$

where $x \in \Omega = [0, L]$ $(L > 0)$, $t > 0$, Δ is the Laplacian operator on $\mathbb{R}$, $u(x,t)$ and $v(x,t)$ denote the concentrations of nutrient and microbial population at position x and time t, respectively, $d > 0$ is the ratio of the diffusivity of the nutrient, $\alpha > 0$ formulates the flow velocity, and $k > 0$ represents the death rate. The nonlinear function f describes the nutrient uptake rate and the growth rate of the microbial at nutrient concentration satisfying

$$
f(u) = 0,\ f'(u) > 0, \quad \text{for } u \geq 0,\ \lim_{u\to\infty} f(u) > k. \tag{1.2}
$$

A typical example of this function is

$$
f(u) = \frac{au}{b+u}, \quad a, b > 0.
$$

2000 *Mathematics Subject Classification.* 35K57, 34C37, 92D25.
Key words and phrases. Traveling wave solutions; R-D equations; singular perturbation; strong generic delay kernels; nonlocal effect.

Submitted February 24, 2013. Published April 5, 2013.

It is well known that time delays are often incorporated into population models for a variety of reasons. Moreover, the individuals have not necessarily been at the same point in space at previous time, because they are moving around. In order to overcome this difficulty, the incorporation of delay necessarily also introduces a nonlocal spatial effect, which can be described by some kinds of spatio-temporal convolutions. In fact, by introducing a kind of spatio-temporal delay, Wang and Yin [17] considered the following bio-reactor model

$$\begin{aligned}\frac{\partial u}{\partial t} &= d\Delta u - \alpha u_x - f(u)v,\\ \frac{\partial v}{\partial t} &= \Delta v - \alpha v_x + ((g * f(u))(x,t) - k)v,\end{aligned} \tag{1.3}$$

where

$$(g * f(u))(x,t) = \int_0^{+\infty}\int_{-\infty}^{+\infty} f(u(x-y,t-s))G(y-\alpha s,s)\kappa(s)\,dy\,ds, \tag{1.4}$$

$$G(y-\alpha s,s) = \frac{1}{\sqrt{4\pi sd}}e^{-\frac{(y-\alpha s)^2}{4sd}}, \quad y\in\mathbb{R}, s>0, \quad \kappa(s) = \frac{1}{\tau}e^{-s/\tau}, \quad \tau>0,$$

the function $\kappa(s)$ is the so-called weak kernel and τ is the average delay. By employing linear chain technique, geometric singular perturbation, and the center manifold theorem, they in [17] proved that the steady traveling wave does not only persist, but also looks qualitatively the same as that without delay if the delay τ is small.

In addition, due to the period of consuming the captured nutrient in which individuals may have different locations, Yang et al [20] considered the spatially random migration of v and obtained the model as

$$\begin{aligned}\frac{\partial u}{\partial t} &= d\Delta u - \alpha u_x - f(u)((g_1 * v)(x,t)),\\ \frac{\partial v}{\partial t} &= \Delta v - \alpha v_x + ((g_2 * f(u))(x,t) - k)v,\end{aligned} \tag{1.5}$$

where

$$(g_1 * v)(x,t) = \int_0^{+\infty}\int_{-\infty}^{+\infty} v(x-y,t-s)G_1(y-\alpha s,s)\kappa_1(s)\,dy\,ds,$$

$$G_1(y-\alpha s,s) = \frac{1}{\sqrt{4\pi s}}e^{-\frac{(y-\alpha s)^2}{4s}}, \quad y\in\mathbb{R}, \quad s>0, \quad \kappa_1(s) = \frac{1}{\tau_1}e^{-s/\tau_1}, \quad \tau>0,$$

and τ_1 is the average delay, and the convolution $(g_2 * f(u))(x,t)$ is defined by (1.4) with $\kappa_2(s) = \frac{1}{\tau_2}e^{-\frac{s}{\tau_2}}$. Further, the authors in [20] proved the persistence of traveling wave solutions. For other results about spatio-temporal delay, we refers to [5, 6, 7, 15, 21].

Based on the biological background, there are other important kernel functions, such as

$$K_1(s) = \frac{s}{\tau_1^2}e^{-s/\tau_1}, \quad K_2(s) = \frac{s}{\tau_2^2}e^{-\frac{s}{\tau_2}}, \quad s>0, \tag{1.6}$$

which are the so-called strong generic delay kernels and are often used in the literatures. For strong generic delay kernels (1.6), by combining the geometric singular perturbation theory with the center manifold theorem, Zhang and Peng [22] considered traveling wave solutions for the diffusive Nicholson' s blowflies equation which is a single scalar equation. See also [9] for some results about Nicholson' s blowflies equation. Motivated by [22], we are interested in the existence of traveling wave solutions of (1.5) with respect to strong generic delay kernels (1.6). That is to say, when the average delays τ_1 and τ_2 are small enough, whether (1.5) has a nontrivial traveling wave solution connecting two equilibrium points $(u^0, 0)$ and $(u_0, 0)$ with $u^0 > u_k > u_0 \geq 0$, where u_k is the positive solution of the equation $f(u) = k$. In this paper, we focus on traveling wave solution moving to the left, against the flow, of the form $u(z)$ and $v(z)$ with $z = x + ct$ satisfying the asymptotic boundary conditions

$$u(-\infty) = u^0, \ v(+\infty) = u_0, \ v(\pm\infty) = 0.$$

We note here that the system (1.5) reduces to (1.1) when the average delays $\tau_1, \tau_2 \to 0$. That is to say the nonlocal interaction vanishes as time delays disappear. In order to seek the traveling wave solutions of (1.5) with the strong generic delay kernels (1.6), we again employ the geometric singular perturbation theory developed by Fenichel [4] to complete our proof. We show that traveling wave solutions exist provided that the delays are sufficiently small by using the geometric singular perturbation theory. We point out that, for strong generic delay kernels, the traveling wave solutions of (1.5) will be recast as an ODE system of order-12 by the linear chain technique, and the process of verifying the condition of the center manifold theorem is more complex than that in [20].

Finally, we would like to mention some results on the traveling wave solutions with delay. In the pioneering work [13], Schaaf first considered the traveling wave solutions for a scalar delayed reaction diffusion equation. Wu [19] treated delay equations with diffusion arising in biological problems. See also [11, 12] and references therein. From the dynamical points of view, traveling wave solutions are some special solutions and can be usually characterized as solutions invariant with respect to transition in space, and these solutions have been widely investigated for nonlinear reaction diffusion equations. In the last few decades, a great amount of research has been devoted to traveling wave solutions. The literature on these solutions is vast. See [10, 18] for the lasted results about traveling wave solutions with delay.

2. Existence of traveling wave solutions

In this section, we prove the existence of traveling wave solutions of (1.5) with (1.6) for sufficiently small $\tau_1 > 0$ and $\tau_2 > 0$. We first recast (1.5) into an ODE system of order-12 by the linear chain technique. Let $p(x,t) = (g_1 * v)(x,t)$ and $w(x,t) = (g_2 * f(u))(x,t)$. Then direct calculations give

$$\begin{aligned} p(x,t) &= \int_0^{+\infty}\int_{-\infty}^{+\infty} v(x-y,t-s)G_1(y-\alpha s,s)K_1(s)\,dy\,ds \\ &= \frac{1}{\tau_1}\int_{-\infty}^{t}\int_{-\infty}^{+\infty} v(y,s)G_1(x-y-\alpha t+\alpha s,t-s)\frac{t-s}{\tau_1}e^{-\frac{t-s}{\tau_1}}\,dy\,ds, \end{aligned} \tag{2.1}$$

$$\begin{aligned} w(x,t) &= \int_0^{+\infty}\int_{-\infty}^{+\infty} f(u(x-y,t-s))G_2(y-\alpha s,s)K_2(s)\,dy\,ds \\ &= \frac{1}{\tau_2}\int_{-\infty}^{t}\int_{-\infty}^{+\infty} f(u(y,s))G_2(x-y-\alpha t+\alpha s,t-s)\frac{t-s}{\tau_2}e^{-\frac{t-s}{\tau_2}}\,dy\,ds, \end{aligned} \tag{2.2}$$

$$\frac{\partial p}{\partial t} = \Delta p - \alpha p_x + \frac{1}{\tau_1}(R_1 - p), \tag{2.3}$$

$$\frac{\partial w}{\partial t} = d\Delta w - \alpha w_x + \frac{1}{\tau_2}(R_2 - w), \tag{2.4}$$

where

$$G_1(y-\alpha s, s) = \frac{1}{\sqrt{4\pi s}} e^{-\frac{(y-\alpha s)^2}{4s}}, \quad y \in \mathbb{R}, s > 0,$$

$$G_2(y-\alpha s, s) = \frac{1}{\sqrt{4\pi s d}} e^{-\frac{(y-\alpha s)^2}{4sd}}, \quad y \in \mathbb{R}, s > 0,$$

$$R_1(x,t) = \frac{1}{\tau_1}\int_{-\infty}^{t}\int_{-\infty}^{+\infty} v(y,s) G_1(x-y-\alpha t+\alpha s, t-s) e^{-\frac{t-s}{\tau_1}}\, dy\, ds,$$

$$R_2(x,t) = \frac{1}{\tau_2}\int_{-\infty}^{t}\int_{-\infty}^{+\infty} f(u(y,s)) G_2(x-y-\alpha t+\alpha s, t-s) e^{-\frac{t-s}{\tau_2}}\, dy\, ds.$$

Differentiating the functions $R_1(x,t)$ and $R_2(x,t)$ with respect to x and t, we have

$$\frac{\partial R_1}{\partial t} = \Delta R_1 - \alpha R_{1x} + \frac{1}{\tau_1}(v - R_1), \tag{2.5}$$

$$\frac{\partial R_2}{\partial t} = d\Delta R_2 - \alpha R_{2x} + \frac{1}{\tau_2}(f(u) - R_2). \tag{2.6}$$

Then, combing (2.3) and (2.5), and combing (2.4) and (2.6), we obtain

$$\begin{aligned} \frac{\partial^2 p}{\partial t^2} &= 2p_{txx} - 2\alpha p_{tx} - p_{xxxx} + 2\alpha p_{xxx} - \alpha^2 p_{xx} \\ &\quad + \frac{2}{\tau_1}(-p_t + p_{xx} - \alpha p_x) + \frac{1}{\tau_1^2}(v - p), \end{aligned} \tag{2.7}$$

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} &= 2dw_{txx} - 2\alpha w_{tx} - d^2 w_{xxxx} + 2d\alpha w_{xxx} - \alpha^2 w_{xx} \\ &\quad + \frac{2}{\tau_2}(-w_t + dw_{xx} - \alpha w_x) + \frac{1}{\tau_2^2}(f(u) - w). \end{aligned} \tag{2.8}$$

Thus we reformulate the the original system (1.5) as

$$\begin{aligned} \frac{\partial u}{\partial t} &= d\Delta u - \alpha u_x - f(u)p, \\ \frac{\partial^2 p}{\partial t^2} &= 2p_{txx} - 2\alpha p_{tx} - p_{xxxx} + 2\alpha p_{xxx} - \alpha^2 p_{xx} \\ &\quad + \frac{2}{\tau_1}(-p_t + p_{xx} - \alpha p_x) + \frac{1}{\tau_1^2}(v - p), \\ \frac{\partial^2 w}{\partial t^2} &= 2dw_{txx} - 2\alpha w_{tx} - d^2 w_{xxxx} + 2d\alpha w_{xxx} - \alpha^2 w_{xx} \\ &\quad + \frac{2}{\tau_2}(-w_t + dw_{xx} - \alpha w_x) + \frac{1}{\tau_2^2}(f(u) - w), \\ \frac{\partial v}{\partial t} &= \Delta v - \alpha v_x + (w - k)v. \end{aligned} \tag{2.9}$$

Obviously, this system is not a delay differential system. The delays in the original problem (1.5) play their roles through the parameters τ_1 and τ_2. Thus, we can deal with the question of traveling wave solutions of (1.5) by seeking the existence of the traveling wave solutions of (2.9). By setting $u(x,t) = u(z)$, $w(x,t) = w(z)$,

$v(x,t) = v(z)$ and $p(x,t) = p(z)$ with $z = x + ct$, we obtain the traveling wave system of (2.9) as follows

$$
\begin{gathered}
Cu' = du'' - f(u)p, \\
C^2 p'' = 2Cp''' - p'''' + \frac{2}{\tau_1}(-Cp' + p'') + \frac{1}{\tau_1^2}(v - p), \\
C^2 w'' = 2dCw''' - d^2 w'''' + \frac{2}{\tau_2}(-Cw' + dw'') + \frac{1}{\tau_2^2}(f(u) - w), \\
Cv' = v'' + (w - k)v,
\end{gathered}
\tag{2.10}
$$

where $C = c + \alpha$. We note that $u(z)$ and $v(z)$ are also the traveling wave solutions of (1.5) with strong generic delay kernels (1.6).

Let $u_1 = du'$, $p_1 = p'$, $p_2 = p_1'$, $p_3 = p_2'$, $w_1 = dw'$, $w_2 = dw_1'$, $w_3 = w_2'$ and $v_1 = v'$, then (2.9) can be recast as a system containing twelve equations of first order

$$
\begin{gathered}
u' = \frac{1}{d}u_1, \quad u_1' = \frac{C}{d}u_1 + f(u)p, \\
p' = p_1, \quad p_1' = p_2, \quad p_2' = p_3, \\
p_3' = -C^2 p_2 + 2Cp_3 + \frac{2}{\tau_1}(-Cp_1 + p_2) + \frac{1}{\tau_1^2}(v - p), \\
w' = \frac{1}{d}w_1, \quad w_1' = \frac{1}{d}w_2, \quad w_2' = w_3, \\
w_3' = -\frac{C^2}{d^2}w_2 + \frac{2C}{d}w_3 + \frac{2}{\tau_2}(-\frac{C}{d}w_1 + \frac{1}{d}w_2) + \frac{1}{\tau_2^2}(f(u) - w), \\
v' = v_1, \quad v_1' = Cv_1 - (w - k)v.
\end{gathered}
\tag{2.11}
$$

Note that this system has the equilibrium of the form

$$
(u, u_1, p, p_1, p_2, p_3, w, w_1, w_2, w_3, v, v_1) = (u^0, 0, 0, 0, 0, 0, f(u^0), 0, 0, 0, 0, 0).
$$

Furthermore, we introduce two small parameters $\tau_1 = \varepsilon^2\tilde{\tau}_1$ and $\tau_2 = \varepsilon^2\tilde{\tau}_2$, and define $u = \tilde{u}$, $u_1 = \tilde{u}_1$, $p = \tilde{p}$, $\varepsilon p_1 = \tilde{p}_1$, $\varepsilon^2 p_2 = \tilde{p}_2$, $\varepsilon^3 p_3 = \tilde{p}_3$, $w = \tilde{w}$, $\varepsilon w_1 = \tilde{w}_1$, $\varepsilon^2 w_2 = \tilde{w}_2$, $\varepsilon^3 w_3 = \tilde{w}_3$, $v = \tilde{v}, v_1 = \tilde{v}_1$, and drop the tildes. Then (2.11) can be recast into

$$
\begin{gathered}
u' = \frac{1}{d}u_1, \quad u_1' = \frac{C}{d}u_1 + f(u)p, \\
\varepsilon p' = p_1, \quad \varepsilon p_1' = p_2, \quad \varepsilon p_2' = p_3, \\
\varepsilon p_3' = -C^2\varepsilon^2 p_2 + 2C\varepsilon p_3 + \frac{2\varepsilon^2}{\tau_1}(-\frac{C}{\varepsilon}p_1 + \frac{1}{\varepsilon^2}p_2) + \frac{1}{\tau_1^2}(v - p), \\
\varepsilon w' = \frac{1}{d}w_1, \quad \varepsilon w_1' = \frac{1}{d}w_2, \quad \varepsilon w_2' = w_3, \\
\varepsilon w_3' = -\frac{C^2\varepsilon^2}{d^2}w_2 + \frac{2C\varepsilon}{d}w_3 + \frac{2\varepsilon^2}{\tau_2}(-\frac{C}{d\varepsilon}w_1 + \frac{1}{d\varepsilon^2}w_2) + \frac{1}{\tau_2^2}(f(u) - w), \\
v' = v_1, \quad v_1' = Cv_1 - (w - k)v,
\end{gathered}
\tag{2.12}
$$

which is a standard form of singular perturbation problem. When $\varepsilon = 0$, (2.12) reduces to

$$\begin{aligned} u' &= \frac{1}{d}u_1, \\ u_1' &= \frac{C}{d}u_1 + f(u)v, \\ v' &= v_1, \\ v_1' &= Cv_1 - (f(u)-k)v, \end{aligned} \tag{2.13}$$

in which $u(z)$ and $v(z)$ are the traveling wave solutions of (1.1). System (2.12) is referred as a slow system if $\varepsilon > 0$ is sufficiently small. Note that when $\varepsilon = 0$ it does not define a dynamic system in $\mathbb{R}^{12}$. This problem can be overcome through the transformation $z = \varepsilon\eta$, under which (2.12) becomes

$$\begin{aligned} u_\eta' &= \frac{\varepsilon}{d}u_1, \quad u_{1\eta}' = \frac{C\varepsilon}{d}u_1 + \varepsilon f(u)p, \\ p_\eta' &= p_1, \quad p_{1\eta}' = p_2, \quad p_{2\eta}' = p_3, \\ p_{3\eta}' &= -C^2\varepsilon^2 p_2 + 2C\varepsilon p_3 + \frac{2\varepsilon^2}{\tau_1}(-\frac{C}{\varepsilon}p_1 + \frac{1}{\varepsilon^2}p_2) + \frac{1}{\tau_1^2}(v-p), \\ w_\eta' &= \frac{1}{d}w_1, \quad w_{1\eta}' = \frac{1}{d}w_2, \\ w_{2\eta}' &= w_3, \\ w_{3\eta}' &= -\frac{C^2\varepsilon^2}{d^2}w_2 + \frac{2C\varepsilon}{d}w_3 + \frac{2\varepsilon^2}{\tau_2}(-\frac{C}{d\varepsilon}w_1 + \frac{1}{d\varepsilon^2}w_2) + \frac{1}{\tau_2^2}(f(u)-w), \\ v_\eta' &= \varepsilon v_1, \quad v_{1\eta}' = C\varepsilon v_1 - \varepsilon(w-k)v. \end{aligned} \tag{2.14}$$

This is called the fast system. The slow system and the fast system are equivalent when $\varepsilon > 0$.

In the slow system (2.12), the flow is confined to the set

$$\begin{aligned} M_0 = \big\{&(u,u_1,p,p_1,p_2,p_3,w,w_1,w_2,w_3,v,v_1) \in \mathbb{R}^{12} : p_1 = p_2 = p_3 = 0, \\ &p = v,\ w_1 = w_2 = w_3 = 0,\ w = f(u)\big\}, \end{aligned}$$

which is a four-dimensional invariant manifold for system (2.12) with $\varepsilon = 0$. Note that M_0 consists of the equilibria of the fast system when $\varepsilon = 0$. If this invariant manifold is normally hyperbolic, then we can obtain an invariant manifold M_ε of system (2.12) for $\varepsilon > 0$, which is close to M_0. The restriction of (2.12) to this invariant manifold M_ε yields a four-dimensional system.

From Fenichel [4], to verify normal hyperbolicity of M_0, we must check that the linearization of the fast system (2.14), restricted to M_0, has precisely dim M_0 eigenvalues on the imaginary axis, with the remainder of the spectrum being hyperbolic. Direct calculations show that the matrix of the linearization of (2.14) restricted to M_0 has 12 eigenvalues: 0, 0, 0, 0, $\frac{1}{\sqrt{\tau_1}}$, $\frac{1}{\sqrt{\tau_1}}$, $-\frac{1}{\sqrt{\tau_1}}$, $-\frac{1}{\sqrt{\tau_1}}$, $\frac{1}{\sqrt{d\tau_2}}$, $\frac{1}{\sqrt{d\tau_2}}$, $-\frac{1}{\sqrt{d\tau_2}}$, $-\frac{1}{\sqrt{d\tau_2}}$. Obviously, we have the correct number of eigenvalues on the imaginary axis and the other eigenvalues are hyperbolic. Thus the invariant manifold M_0 is normally hyperbolic in the sense of Fenichel [4]. By the geometric singular perturbation theory, we know that the slow system (2.8) has an invariant manifold M_ε, which can be written as

$$M_\varepsilon = \Big\{(u,u_1,p,p_1,p_2,p_3,w,w_1,w_2,w_3,v,v_1) \in \mathbb{R}^{12} : p = v + q(u,u_1,v,v_1,\varepsilon),$$

$$\begin{aligned}&p_1 = l(u,u_1,v,v_1,\varepsilon),\ p_2 = m(u,u_1,v,v_1,\varepsilon),\ p_3 = n(u,u_1,v,v_1,\varepsilon),\\ &w = f(u) +\ r(u,u_1,v,v_1,\varepsilon),\ w_1 = h(u,u_1,v,v_1,\varepsilon),\ w_2 = i(u,u_1,v,v_1,\varepsilon),\\ &w_3 = j(u,u_1,v,v_1,\varepsilon)\Big\},\end{aligned}$$

where the functions q, l, m, n, r, h, i and j depend on ε and satisfy

$$\begin{aligned}&q(u,u_1,v,v_1,0) = l(u,u_1,v,v_1,0) = m(u,u_1,v,v_1,0) = n(u,u_1,v,v_1,0)\\ &= r(u,u_1,v,v_1,0) = h(u,u_1,v,v_1,0) = i(u,u_1,v,v_1,0) = j(u,u_1,v,v_1,0) = 0.\end{aligned}$$

Thus we can expand q, l, m, n, r, h, i and j into the form of Taylor series about ε,

$$\begin{aligned}q(u,u_1,v,v_1,\varepsilon) &= q_1(u,u_1,v,v_1)\varepsilon + q_2(u,u_1,v,v_1)\varepsilon^2 + \dots\\ l(u,u_1,v,v_1,\varepsilon) &= l_1(u,u_1,v,v_1)\varepsilon + l_2(u,u_1,v,v_1)\varepsilon^2 + \dots\\ m(u,u_1,v,v_1,\varepsilon) &= m_1(u,u_1,v,v_1)\varepsilon + m_2(u,u_1,v,v_1)\varepsilon^2 + \dots\\ n(u,u_1,v,v_1,\varepsilon) &= n_1(u,u_1,v,v_1)\varepsilon + n_2(u,u_1,v,v_1)\varepsilon^2 + \dots\\ r(u,u_1,v,v_1,\varepsilon) &= r_1(u,u_1,v,v_1)\varepsilon + r_2(u,u_1,v,v_1)\varepsilon^2 + \dots\\ h(u,u_1,v,v_1,\varepsilon) &= h_1(u,u_1,v,v_1)\varepsilon + h_2(u,u_1,v,v_1)\varepsilon^2 + \dots\\ i(u,u_1,v,v_1,\varepsilon) &= i_1(u,u_1,v,v_1)\varepsilon + i_2(u,u_1,v,v_1)\varepsilon^2 + \dots\\ j(u,u_1,v,v_1,\varepsilon) &= j_1(u,u_1,v,v_1)\varepsilon + j_2(u,u_1,v,v_1)\varepsilon^2 + \dots\end{aligned} \tag{2.15}$$

Note that M_ε is the invariant manifold for the flow of (2.12). Thus differentiating $p = v + q(u,u_1,v,v_1,\varepsilon)$ with respect to z, we have

$$p' = \frac{\partial q}{\partial u}u' + \frac{\partial q}{\partial u_1}u_1' + \big(1 + \frac{\partial q}{\partial v}\big)v' + \frac{\partial q}{\partial v_1}v_1'. \tag{2.16}$$

Substituting (2.16) into (2.12) and restricting to M_ε, we have

$$\varepsilon\big\{\frac{1}{d}\frac{\partial q}{\partial u}u_1 + \frac{\partial q}{\partial u_1}\big[\frac{C}{d}u_1 + f(u)(v+q)\big] + [1+\frac{\partial q}{\partial v}]v_1 + \frac{\partial q}{\partial v_1}[Cv_1 + kv - (f(u)+r)v]\big\} = l. \tag{2.17}$$

Similarly, from $p_1 = l(u,u_1,v,v_1,\varepsilon)$ and (2.12), we have

$$\varepsilon\big\{\frac{1}{d}\frac{\partial l}{\partial u}u_1 + \frac{\partial l}{\partial u_1}[\frac{C}{d}u_1 + f(u)(v+q)] + \frac{\partial l}{\partial v}v_1 + \frac{\partial l}{\partial v_1}[Cv_1 + kv - (f(u)+r)v]\big\} = m. \tag{2.18}$$

From $p_2 = m(u,u_1,v,v_1,\varepsilon)$ and (2.12), we have

$$\varepsilon\big\{\frac{1}{d}\frac{\partial m}{\partial u}u_1 + \frac{\partial m}{\partial u_1}[\frac{C}{d}u_1 + f(u)(v+q)] + \frac{\partial m}{\partial v}v_1 + \frac{\partial m}{\partial v_1}[Cv_1 + kv - (f(u)+r)v]\big\} = n. \tag{2.19}$$

From $p_3 = n(u,u_1,v,v_1,\varepsilon)$ and (2.12), we have

$$\begin{aligned}&\varepsilon\big\{\frac{1}{d}\frac{\partial n}{\partial u}u_1 + \frac{\partial n}{\partial u_1}[\frac{C}{d}u_1 + f(u)(v+q)] + \frac{\partial n}{\partial v}v_1 + \frac{\partial n}{\partial v_1}[Cv_1 + kv - (f(u)+r)v]\big\}\\ &= \big(-C^2\varepsilon^2 + \frac{2}{\tau_1}\big)m + 2C\varepsilon\big(n - \frac{l}{\tau_1}\big) - \frac{q}{\tau_1^2}.\end{aligned} \tag{2.20}$$

From $w = f(u) + r(u, u_1, v, v_1, \varepsilon)$ and (2.12), we have

$$\begin{aligned} &\varepsilon\Big\{\frac{1}{d}[f'(u) + \frac{\partial r}{\partial u}]u_1 + \frac{\partial r}{\partial u_1}[\frac{C}{d}u_1 + f(u)(v+q)] + \frac{\partial r}{\partial v}v_1 \\ &+ \frac{\partial r}{\partial v_1}[Cv_1 + kv - (f(u)+r)v]\Big\} \\ &= \frac{1}{d}h. \end{aligned} \tag{2.21}$$

From $w_1 = h(u, u_1, v, v_1, \varepsilon)$ and (2.12), we have

$$\varepsilon\big\{\frac{1}{d}\frac{\partial h}{\partial u}u_1 + \frac{\partial h}{\partial u_1}[\frac{C}{d}u_1 + f(u)(v+q)] + \frac{\partial h}{\partial v}v_1 + \frac{\partial h}{\partial v_1}[Cv_1 + kv - (f(u)+r)v]\big\} = \frac{1}{d}i. \tag{2.22}$$

From $w_2 = i(u, u_1, v, v_1, \varepsilon)$ and (2.12), we have

$$\varepsilon\big\{\frac{1}{d}\frac{\partial i}{\partial u}u_1 + \frac{\partial i}{\partial u_1}[\frac{C}{d}u_1 + f(u)(v+q)] + \frac{\partial i}{\partial v}v_1 + \frac{\partial i}{\partial v_1}[Cv_1 + kv - (f(u)+r)v]\big\} = j. \tag{2.23}$$

From $w_3 = j(u, u_1, v, v_1, \varepsilon)$ and (2.12), we have

$$\begin{aligned} &\varepsilon\Big\{\frac{1}{d}\frac{\partial j}{\partial u}u_1 + \frac{\partial j}{\partial u_1}[\frac{C}{d}u_1 + f(u)(v+q)] + \frac{\partial j}{\partial v}v_1 + \frac{\partial j}{\partial v_1}[Cv_1 + kv - (f(u)+r)v]\Big\} \\ &= \big(-\frac{C^2\varepsilon^2}{d^2} + \frac{2}{d\tau_2}\big)i + \frac{2C\varepsilon}{d}\big(j - \frac{h}{\tau_2}\big) - \frac{r}{\tau_2^2}. \end{aligned} \tag{2.24}$$

Substituting (2.15) into (2.17)-(2.24), and comparing coefficients of ε and ε^2, we obtain

$$\begin{gathered} q_1(u, u_1, v, v_1) = 0, \quad q_2(u, u_1, v, v_1) = 2\tau_1 v(k - f(u)), \\ l_1(u, u_1, v, v_1) = v_1, \quad l_2(u, u_1, v, v_1) = 0, \\ m_1(u, u_1, v, v_1) = 0, \quad m_2(u, u_1, v, v_1) = Cv_1 + kv - f(u)v, \\ n_1(u, u_1, v, v_1) = n_2(u, u_1, v, v_1) = 0, \\ r_1(u, u_1, v, v_1) = 0, \quad r_2(u, u_1, v, v_1) = \frac{2\tau_2}{d}(f''(u)u_1^2 + df'(u)f(u)v), \\ h_1(u, u_1, v, v_1) = f'(u)u_1, \quad h_2(u, u_1, v, v_1) = 0, \\ i_1(u, u_1, v, v_1) = 0, \quad i_2(u, u_1, v, v_1) = f''(u)u_1^2 + Cf'(u)u_1 + df'(u)f(u)v, \\ j_1(u, u_1, v, v_1) = j_2(u, u_1, v, v_1) = 0. \end{gathered} \tag{2.25}$$

Thus, on M_ε, slow system (2.12) reduces to

$$\begin{gathered} u' = \frac{1}{d}u_1, \\ u_1' = \frac{C}{d}u_1 + f(u)v + f(u)q, \\ v' = v_1, \\ v_1' = Cv_1 + kv - f(u)v - rv, \end{gathered} \tag{2.26}$$

where $q = q_2(u, u_1, v, v_1)\varepsilon^2 + o(\varepsilon^2)$ and $r = r_2(u, u_1, v, v_1)\varepsilon^2 + o(\varepsilon^2)$.

From [8, Theorem 1.1], under the condition (1.2), we know that there exists $0 < u_0 < u_k$, such that (2.13) has a traveling wave solution $(u(x+ct), v(x+ct))$ connecting $(u_0, 0)$ to $(u^0, 0)$ for each $u^0 > u_k$, and $C = c+\alpha > C^* = \sqrt{4(f(u^0) - k)}$.

Thus, there exists a $u_0 \in (0, u_k)$ such that a positive branch of stable manifold $W_0^s(u_0)$ of $(u_0, 0, 0, 0)$ of (2.13) connects to $(u^0, 0, 0, 0)$.

In what follows, we start to prove that positive branch of stable manifold $W_\varepsilon^s(u_0)$ of $(u_0, 0, 0, 0)$ of (2.26) connecting to some $(\hat{u}^0, 0, 0, 0)$, which closes to $(u^0, 0, 0, 0)$ for sufficiently small $\varepsilon > 0$. Denoting the forward orbit of (2.26) through a point $x_\varepsilon = x_\varepsilon(0)$ by $\{x_\varepsilon(z) : z \geq 0\}$, which depends continuously on ε and can be used to describe the local stable manifold, we obtain that the forward orbit of $\{x_\varepsilon(z) : z \geq 0\}$ means $\{x_\varepsilon(z) : z \geq -Z(Z \gg 1)\}$ with endpoint $x_\varepsilon(-z)$ by a compact piece of the global stable manifold. As in [17], we define the backward orbit, and expect that such a compact piece of $W_\varepsilon^s(u_0)$ has an endpoint near $(u^0, 0, 0, 0)$ if $\varepsilon > 0$ is sufficiently small.

By a similar argument in [17], applying the center manifold theory to the time-reversed system (2.26) with the equation $\varepsilon' = 0$, and translating $(u^0, 0, 0, 0)$ to origin by letting

$$S = u - u^0 - \frac{1}{C\zeta}[ru_1 + f(u^0)v_1 - Cf(u^0)v], \quad \zeta = f(u^0) - k > 0, \tag{2.27}$$

we have

$$\begin{gathered} S' = \frac{k}{C\zeta}(f(u^0) - f(u)) + \frac{f(u)}{C}q - \frac{f(u^0)}{C\zeta}rv, \\ u_1' = -\frac{C}{d}u_1 + (f(u^0) - f(u))v - f(u^0)v - f(u)q, \\ v' = -v_1, \\ v_1' = -Cv_1 + \zeta v - (f(u^0) - f(u))v + rv, \\ \varepsilon' = 0, \end{gathered} \tag{2.28}$$

where $q = q_2(u, u_1, v, v_1)\varepsilon^2 + o(\varepsilon^2)$, $r = r_2(u, u_1, v, v_1)\varepsilon^2 + o(\varepsilon^2)$, and u is determined by (2.27). Let $X = (S, \varepsilon)$ and $Y = (u_1, v, v_1)$. Then (2.28) has the form

$$\begin{gathered} X' = AX + P(X, Y), \\ Y' = BY + Q(X, Y), \end{gathered}$$

where A is the zero matrix,

$$B = \begin{pmatrix} -\frac{C}{d} & -f(u^0) & 0 \\ 0 & 0 & -1 \\ 0 & \zeta & -C \end{pmatrix}$$

is a stable matrix, P and Q are higher order terms and satisfy $P(0, 0) = P'(0, 0) = 0$ and $Q(0, 0) = Q'(0, 0) = 0$. Then by the results in [3], we can obtain the following lemma, in which we can see what happens to the backward orbit through this endpoint.

Lemma 2.1. *Let $u^0 > u_k$ and $\delta_0 < u^0 - u_k$. Then there are $\delta \in (0, \delta_0)$ and $\varepsilon_0 > 0$ such that the solution $x_\varepsilon(z) = (u_\varepsilon(z), u_{1\varepsilon}(z), v_\varepsilon(z), v_{1\varepsilon}(z))$ of (2.26) satisfies $|x_\varepsilon(z) - (u^0, 0, 0, 0)| < \delta_0$ for all $z < 0$, when $|x_\varepsilon(0) - (u^0, 0, 0, 0)| < \delta$. Furthermore, there are $x_\varepsilon(z) \to (\hat{u}^0, 0, 0, 0)$ when $z \to 0$, and $(\hat{u}^0, 0, 0, 0) \to (u^0, 0, 0, 0)$ when $\varepsilon \to 0$.*

We omit the proof of the above lemma and refer to [14, Lemma 3.1] for a similar proof. Finally, we are in a position to give and prove the main result of this paper.

Theorem 2.2. *Let* $u^0 > u_k$ *and* $C = c + \alpha > C^* = \sqrt{4(f(u^0) - k)}$. *Then there is* $0 < u_0 < u_k$ *such that for any sufficiently small* $\tau_1, \tau_2 > 0$, (1.5) *with* (1.6) *admits a traveling wave solution* $(u(x + ct), v(x + ct))$ *connecting* $(\hat{u}^0, 0)$ *to* $(u_0, 0)$, *and* $\hat{u}^0 \to u^0$ *as* $\tau_i \to 0$ $(i = 1, 2)$. *Moreover, the solution* $(u(x + ct), v(x + ct))$ *satisfies* $u'(z) > 0$ *for* $z \in \mathbb{R}$, *and* $v(z) > 0$ $(z \in \mathbb{R})$ *is unimodal.*

Proof. It follows from the invariant manifold theorem [4] that "compact pieces" of the positive branch of the stable manifold $W^s_\varepsilon(u_0)$ of $(u_0, 0, 0, 0)$ to system (2.26), lie within a small neighborhood of $W^s_\varepsilon(u_0)$ and are diffeomorphic to $W^s_\varepsilon(u_0)$ for any $0 < u_0 < u_k$. At the same time, according to [8, Theorem 1.1], there exists a positive branch of stable manifold $W^s_0(u_0)$ connecting $(u_0, 0, 0, 0)$ to $(u^0, 0, 0, 0)$. By the stable manifold theorem and continuous dependence of the solutions on parameters over any finite time interval, there exists $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$, the compact piece of $W^s_\varepsilon(u_0)$ has endpoint within distance δ of $(u^0, 0, 0, 0)$. Assume that $\varepsilon_1 < \varepsilon_0$ with ε_0 defined in Lemma 2.1, then by Lemma 2.1, the backward continuation of the compact piece of $W^s_\varepsilon(u_0)$ is asymptotic to some point $(\hat{u}^0, 0, 0, 0)$, which implies that there exists a heteroclinic orbit of (2.11) connecting $(\hat{u}^0, 0, 0, 0, 0, f(\hat{u}_0), 0, 0, 0, 0, 0)$ to $(u_0, 0, 0, 0, 0, f(u_0), 0, 0, 0, 0, 0)$.

According to Lemma 2.1, we can adopt the argument, as in the proof of [14, Theorem 3.2] to prove $v(z) > 0$ for all $z \in \mathbb{R}$. Here we just remark that the polar coordinate for (v, v_1) in reversed time scale has the form

$$\rho\rho' = -Cv_1^2 + (f(u) - k - 1)vv_1 + rvv_1,$$
$$\rho^2\theta = (f(u) - k - \frac{C^2}{4})v^2 + (v_1 - \frac{C}{2}v)^2,$$

where $r = r_2(u, u_1, v, v_1)\varepsilon^2 + o(\varepsilon^2)$.

The property that $u'(z) < 0$ and $v(z)$ is unimodal can be proved just by taking $K_1(s) = \frac{s}{\tau_1^2}e^{-s/\tau_1}$ and $K_2(s) = \frac{s}{\tau_2^2}e^{-\frac{s}{\tau_2}}$ instead of $\kappa_1(s) = \frac{1}{\tau_1}e^{-s/\tau_1}$ and $\kappa_2(s) = \frac{1}{\tau_2}e^{-\frac{s}{\tau_2}}$ in the proof of [20, Theorem 3.3] and by using the standard theory of ordinary differential equation and the condition (1.2). The details of proof are omitted here. This completes the proof. □

Acknowledgements. The first author was supported by grants 11201402 from NSF of China, and ZR2010AQ006 from the NSF of Shandong Province of China.

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