Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 73, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SINGULAR BOUNDARY-VALUE PROBLEMS WITH VARIABLE COEFFICIENTS ON THE POSITIVE HALF-LINE

SMAÏL DJEBALI, OUIZA SAIFI, SAMIRA ZAHAR

ABSTRACT. This work concerns the existence and the multiplicity of solutions for singular boundary-value problems with a variable coefficient, posed on the positive half-line. When the nonlinearity is positive but may have a space singularity at the origin, the existence of single and twin positive solutions is obtained by means of the fixed point index theory. The singularity is treated by approximating the nonlinearity, which is assumed to satisfy general growth conditions. When the nonlinearity is not necessarily positive, the Schauder fixed point theorem is combined with the method of upper and lower solutions on unbounded domains to prove existence of solutions. Our results extend those in [18] and are illustrated with examples.

1. INTRODUCTION

This article is devoted to the existence and the multiplicity of positive solutions to the following boundary-value problem posed on the positive half-line:

$$\begin{gathered} x''(t) - k^2(t)x(t) + m(t)f(t, x(t)) = 0, \quad t > 0, \\ x(0) = 0, \quad \lim_{t\to+\infty} x(t) = 0, \end{gathered} \tag{1.1}$$

where the coefficient $k : I \to I$ is a continuous bounded function, $m : I \to I$ is continuous, and $f \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$; here $I = (0, +\infty)$ and $\mathbb{R}^+ = [0, +\infty)$. Boundary-value problems on infinite intervals appear in many problems from mechanics, chemistry, biology, plasma physics, nonlinear mechanics, and non-Newtonian fluid flows (see e.g., [1, 2] and the references therein). For instance, the case $k^2(t) = 1 + 2\omega + t^2 (\omega > 0)$ corresponds to the well-known Holt's equation [15]. The case where the function k is constant is considered in several recent works. In particular, when the nonlinearity f has no space singularity, the existence of solutions to problem (1.1) is obtained in [20] by the Tychonoff fixed point theorem while the Krasnosels'kii fixed point theorem of cone expansion and compression of norm type is employed in [26] to prove existence of multiple solutions (see also [10]). When m is singular at $t = 0$, the authors in [6, 7, 8] have showed the existence of single and multiple positive solutions to (1.1) using the Krasnosels'kii and the Leggett-Williams fixed point theorems. In [24], B. Yan *et al* have obtained some existence

2000 *Mathematics Subject Classification.* 34B15, 34B18, 34B40.
Key words and phrases. Positive solution; variable coefficient; lower and upper solutions; singular problem; half-line; multiplicity; uniqueness; fixed point index.

Submitted July 22, 2012. Published March 17, 2013.

results when f may have a space singularity at $x = 0$, f is allowed to change sign, and k is constant; they have used the upper and lower solution method. The index fixed point theory in a cone with a spacial Banach norm is also used in [23] to study the existence of positive solution to the second-order differential equation

$$(px')' + \lambda(f(t, x) - k^2 x) = 0, \quad t > 0$$

subject to Sturm-Liouville boundary conditions at the origin and at positive infinity. Here $\lambda > 0$, k is constant, and the nonlinearity $f = f(t, x)$ only has time singularity at $t = 0$. Also, the upper and lower solution method is considered in [9] and [17] to investigate some boundary value problems on infinite intervals of the real line. If the constant k is time depending, then the problem is more difficult. In [18], Ma and Zhu have considered the case where k is a bounded continuous function and the nonlinearity $f \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ is semi-positone, has no singularity but satisfies a sublinear polynomial growth condition. They showed that if the parameter λ is small enough, then the problem

$$\begin{gathered} x''(t) - k^2(t)x(t) + \lambda m(t) f(t, x(t)) = 0, \quad t > 0, \\ x(0) = 0, \quad \lim_{t \to +\infty} x(t) = 0, \end{gathered}$$

has a positive solution; the authors have employed the index fixed point theory. Motivated by the papers mentioned above, our aim in this work is two-fold: we not only consider the case where $k = k(t)$ is time-dependant but we also investigate a large class of singular nonlinearities, including the superlinear and sublinear cases. For this, we will employ separately the fixed point index theory and the upper and lower solution techniques.

In Section 2, we first recall some preliminaries needed in this paper. In particular, some properties of the Green's function taken from [18] are recalled. This enables us to reformulate in Section 3 problem (1.1) as a fixed point problem for an integral operator. We study the compactness of a sequence of approximating operators under a quite general growth condition. The fixed point index of an operator defined on a cone of a weighted Banach space is used in Section 4 together with a regularization technique to overcome the singularity. Then the existence of one solution is obtained by using a method of approximation combined with the computation of a fixed point index on an appropriate cone. The nonlinearity satisfies a general growth condition which includes the polynomial case. The existence of twin positive solutions is proved in Section 5 when f is superlinear. Section 6 is devoted to the case when f is not necessarily positive. The existence of bounded solutions is proved by a combination of a regularization technique, the Schauder fixed point theorem, and the method of upper and lower solution (see [4] for a description of this method on bounded domains). In this case, the Nagumo condition is assumed in the nonlinearity. A uniqueness result is also given under a monotonicity condition. The paper ends in Section 7 with three examples of applications illustrating the obtained results while some concluding remarks are presented in Section 8.

2. Preliminaries

In this section, we collect some definitions and lemmas used in this work. Let E be a Banach space. A mapping $A : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets. A nonempty subset $\mathcal{P}$ of a Banach space E is called a cone if $\mathcal{P}$ is convex, closed, and satisfies $\mathcal{P} \cap -\mathcal{P} = \{0\}$ and the condition:

$$\alpha x \in \mathcal{P}, \text{ for all } x \in \mathcal{P} \text{ and all } \alpha \geq 0.$$

Let $\mathcal{P}, \Omega$ be a cone and an open subset of E respectively. The index fixed point of a completely continuous map $A : \overline{\Omega} \cap \mathcal{P} \to \Omega$, $i(A, \Omega \cap \mathcal{P}, \mathcal{P})$, is defined as the Leray-Schauder topological degree of the restriction of $I - A$ on $\Omega \cap \mathcal{P}$; here I refers to the identity operator. The properties of the degree naturally translate to the index. Among them, the existence property states that if $i(A, \Omega \cap \mathcal{P}, \mathcal{P}) \neq 0$, then A has a fixed point. In the following lemma, we recall some important properties we need in this paper. For further details and properties of the fixed point index on cones of Banach spaces, we refer the reader to [5, 11, 14, 25].

Lemma 2.1. *Let Ω be a bounded open set in a real Banach space E, $\mathcal{P}$ a cone of E and $A : \overline{\Omega} \cap \mathcal{P} \to \Omega$ a completely continuous map.*

(i) *If $\lambda Ax \neq x, \forall x \in \partial\Omega \cap \mathcal{P}, \forall \lambda \in (0, 1]$, then $i(A, \Omega \cap \mathcal{P}, \mathcal{P}) = 1$.*
(ii) *If $Ax \nleq x, \forall x \in \partial\Omega \cap \mathcal{P}$, then $i(A, \Omega \cap \mathcal{P}, \mathcal{P}) = 0$.*

To study the boundary-value problem (1.1), we need some restrictions on the bounded function k. Let

$$H := \sup_{t \in I} k(t) \text{ and } h := \inf_{t \in I} k(t) > 0$$

and assume

(H1) the function $k : I \to I$ is continuous, bounded and there exist $d \in [h, H]$, such that for all $\rho > 0$,
$$\lim_{t \to \infty} e^{-\rho t} \int_0^t e^{\rho s} [k^2(s) - d^2] ds \text{ exists;}$$
(H2) the function $k : I \to I$ is continuous and periodic (hence bounded).

The construction of the Green's function is given in [18] by Ma and Zhu where the following properties are discussed. For the asymptotic behavior of solutions of the equation $x''(t) - k^2(t)x(t) = 0$, we also refer to [2, Theorem 7].

Lemma 2.2 ([18])**.** *Assume that k is bounded and continuous. Then*

(a) *the Cauchy problem*
$$\begin{gathered} x''(t) - k^2(t)x(t) = 0, \quad t > 0, \\ x(0) = 0, \quad x'(0) = 1 \end{gathered} \tag{2.1}$$
has a unique solution ϕ_1 defined on $\mathbb{R}^+$. Moreover $\phi_1' > 0$ and ϕ_1 is unbounded.
(b) *The limit problem*
$$\begin{gathered} x''(t) - k^2(t)x(t) = 0, \quad t > 0, \\ x(0) = 1, \quad \lim_{t \to +\infty} x(t) = 0 \end{gathered} \tag{2.2}$$
has a unique solution ϕ_2 defined on $\mathbb{R}^+$ with
$$0 < \phi_2 \leq 1, \, \phi_2' < 0.$$

If further (H1) *holds, then*

$$\lim_{t\to\infty}\frac{\phi_2'(t)}{\phi_2(t)}=-d.$$

(c) *If either* (H1) *or* (H2) *holds, then there exists $M>0$ such that*

$$\sup_{t\in\mathbb{R}^+}\phi_1(t)\phi_2(t)<M.$$

Then $\{\phi_1,\phi_2\}$ forms a fundamental system of solutions and thus, regarding the non-homogeneous linear problem, we have the following lemma [18].

Lemma 2.3. *Assume that either* (H1) *or* (H2) *holds. Then for every function $y\in L^1(\mathbb{R}^+)$, the problem*

$$\begin{gathered} x''(t)-k^2(t)x(t)+y(t)=0,\quad t>0,\\ x(0)=0,\quad x(+\infty)=0 \end{gathered}$$

is equivalent to the integral equation

$$x(t)=\int_0^\infty G(t,s)y(s)ds,\quad t>0,$$

where

$$G(t,s)=\begin{cases}\phi_1(t)\phi_2(s), & 0\le t\le s<+\infty,\\ \phi_1(s)\phi_2(t), & 0\le s\le t<+\infty.\end{cases}$$

The Green's function $G(t,s)$ satisfies the following properties:

Lemma 2.4 ([18])**.** (a) *For all $t,s\in\mathbb{R}^+$, $G(t,s)<\frac{1}{2h}$.*

(b) *For every $\theta\in(1,+\infty)$ and all $t,s\in\mathbb{R}^+$,*

$$\phi_2(s)G(s,s)\ge\frac{h}{H}G(t,s)\phi_2^\theta(t)\,.$$

(c) *For all $t,s\in\mathbb{R}^+$,*

$$G(t,s)\ge\gamma(t)G(s,s)\phi_2(s),$$

where $\gamma(t):=\min\{2h\phi_1(t),\phi_2(t)\}$, $t\in\mathbb{R}^+$.

In Sections 3–5, we shall assume that the function $f\in C(I\times\mathbb{R}^+,\mathbb{R}^+)$ is positive and satisfies $\lim_{x\to0^+}f(t,x)=+\infty$, uniformly on compact subintervals of I; i.e., $f(t,x)$ may be singular at $x=0$.

3. Compactness of a sequence of integral operators

Let $\theta>1$, $\widetilde{\gamma}(t)=\gamma(t)\phi_2^\theta(t)$ and $F(t,x)=f\big(t,\frac{x}{\phi_2^\theta(t)}\big)$. Since $\phi_2\le1$,

$$\widetilde{\gamma}(t)\le1,\ \forall t\ge0. \tag{3.1}$$

Consider the growth condition:

(H3) there exist functions $r\in C(\mathbb{R}^+,\mathbb{R}^+)$ and $p\in C(I,I)$ such that

$$0\le F(t,x)\le r(t)p(x),\quad \forall t\in\mathbb{R}^+,\ \forall x\in I \tag{3.2}$$

and there exists a decreasing function $q\in C(I,I)$ such that $\frac{p(x)}{q(x)}$ is increasing and

$$\int_0^{+\infty}G(s,s)\phi_2(s)r(s)m(s)q(c\widetilde{\gamma}(s))ds<+\infty,\quad \text{for each } c>0. \tag{3.3}$$

Let $C_\ell(\mathbb{R}^+,\mathbb{R}) = \{x \in C(\mathbb{R}^+,\mathbb{R}) : \lim_{t\to+\infty} x(t) \text{ exists}\}$. To study problem (1.1), consider the weighted space

$$E = \{x \in C(\mathbb{R}^+,\mathbb{R}) : \lim_{t\to+\infty} x(t)\phi_2^\theta(t) \text{ exists}\}.$$

Clearly E is a Banach space with norm $\|x\| = \sup_{t\in\mathbb{R}^+} |x(t)|\phi_2^\theta(t)$.

Definition 3.1. Let $N \subseteq C_\ell(\mathbb{R}^+,\mathbb{R})$.

(a) N is said to be almost equicontinuous on $\mathbb{R}^+$ if it equicontinuous on every compact interval of $\mathbb{R}^+$.
(b) N is called equiconvergent at $+\infty$ if, given $\varepsilon > 0$, there corresponds $\Lambda(\varepsilon) > 0$ such that $|x(t) - x(+\infty))| < \varepsilon$, for all $t \geq \Lambda(\varepsilon)$, $x \in N$.

Next, we recall a classical compactness criterion due to Corduneanu [3, p. 62]

Theorem 3.2. *Let $N \subseteq C_\ell(\mathbb{R}^+,\mathbb{R})$. Then N is relatively compact in $C_\ell(\mathbb{R}^+,\mathbb{R})$ if the following conditions hold:*

(a) *N is uniformly bounded in $C_l(\mathbb{R}^+,\mathbb{R})$;*
(b) *N is almost equicontinuous;*
(c) *N is equiconvergent at $+\infty$.*

We can easily deduce the following result.

Theorem 3.3. *Let $D \subseteq E$ and*

$$D\phi_2^\theta = \{u : u(t) = x(t)\phi_2^\theta(t),\, x \in D\}.$$

Then D is relatively compact in E if the following conditions hold:

(a) *D is uniformly bounded in E,*
(b) *$D\phi_2^\theta$ is almost equicontinuous on $[0,+\infty)$,*
(c) *$D\phi_2^\theta$ is equiconvergent at $+\infty$.*

Given $f \in C(\mathbb{R}^+ \times I, \mathbb{R}^+)$, define a sequence of functions $\{f_n\}_{n\geq1}$ by

$$f_n(t,x) = f\big(t, \max\{\frac{1}{n\phi_2^\theta(t)}, x\}\big), \quad n \in \{1,2,\dots\},$$

consider the positive cone

$$\mathcal{P} = \{x \in E : x(t) \geq \frac{h}{H}\gamma(t)\|x\|,\, \forall t \geq 0\}$$

and for $x \in \mathcal{P}$, define a sequence of operators

$$A_n x(t) = \int_0^{+\infty} G(t,s)m(s)f_n(s,x(s))\,ds, \quad n \in \{1,2,\dots\}.$$

Theorem 3.4. *Assume that either* (H1), (H3)*, or* (H2)-(H3) *hold. Then, for each $n \geq 1$, the operator A_n sends $\mathcal{P}$ into $\mathcal{P}$ and is completely continuous.*

Proof. **Step 1.** We show that $A_n(\mathcal{P}) \subset \mathcal{P}$. For $x \in \mathcal{P}$, let

$$R_n(s) = \max\{\frac{1}{n}, x(s)\phi_2^\theta(s)\} \quad \text{and} \quad L_n(x) = \frac{p}{q}\big(\max\{\frac{1}{n}, \|x\|\}\big).$$

Since, for all positive s,

$$\frac{1}{n} \leq \max\{\frac{1}{n}, x(s)\phi_2^\theta(s)\} \leq \max\{\frac{1}{n}, \|x\|\},$$

it follows that

$$\frac{p}{q}\big(R_n(s)\big) \leq L_n(x).$$

By (3.1), $0 < \widetilde{\gamma}(t) \leq 1$. Using $(\mathcal{H}_3)$ and Lemma 2.4, parts (a), (b), for all $t \in \mathbb{R}^+$, we obtain the estimates

$$\begin{aligned}
A_n x(t)\phi_2^\theta(t) &= \int_0^{+\infty} G(t,s)\phi_2^\theta(t)m(s)f_n(s,x(s))\,ds \\
&\leq \frac{H}{h}\int_0^{+\infty} G(s,s)\phi_2(s)m(s)F(s,R_n(s))ds \\
&\leq \frac{H}{h}\int_0^{+\infty} G(s,s)\phi_2(s)m(s)r(s)q(R_n(s))\frac{p}{q}(R_n(s))ds \qquad (3.4)\\
&\leq \frac{H}{h}L_n(x)\int_0^{+\infty} G(s,s)\phi_2(s)m(s)r(s)q(\frac{1}{n})ds \\
&\leq \frac{H}{h}L_n(x)\int_0^{+\infty} G(s,s)\phi_2(s)m(s)r(s)q(\frac{1}{n}\widetilde{\gamma}(s))ds.
\end{aligned}$$

Hence $\sup_{t\geq 0}|A_n x(t)|\phi_2^\theta(t) < \infty$. Similarly, for $x \in \mathcal{P}$, by Lemma 2.4, parts (b), (c), for all positive t, we have

$$\begin{aligned}
A_n x(t) &= \int_0^{+\infty} G(t,s)m(s)f_n(s,x(s))\,ds \\
&\geq \int_0^{+\infty} \gamma(t)\phi_2(s)G(s,s)m(s)f_n(s,x(s))\,ds \\
&\geq \frac{h\gamma(t)}{H}\int_0^{+\infty} G(\xi,s)\phi_2^\theta(\xi)m(s)f_n(s,x(s))\,ds, \quad \forall \xi \geq 0 \\
&\geq \frac{h\gamma(t)}{H}A_n x(\xi)\phi_2^\theta(\xi), \quad \forall \xi \geq 0.
\end{aligned}$$

Passing to the supremum over $\xi \geq 0$, we obtain

$$A_n x(t) \geq \frac{h}{H}\gamma(t)\|A_n x\|, \quad \forall t \geq 0.$$

Therefore, $A_n\mathcal{P} \subseteq \mathcal{P}$.

Step 2. $A_n : \mathcal{P} \to \mathcal{P}$ is continuous. Let a sequence $\{x_j\}_{j\geq 1} \subseteq \mathcal{P}$ be such that $\lim_{j\to+\infty} x_j = x_0 \in \mathcal{P}$. Then there exists $M > 0$, which can be chosen without loss of generality greater than 1, such that $\|x_j\| < M$ for all $j \in \mathbb{N}$. By the continuity of f_n, we have

$$|f_n(s,x_j(s)) - f_n(s,x_0(s))| \to 0, \quad \text{as } j \to +\infty.$$

Moreover,

$$\begin{aligned}
\|A_n x_j - A_n x_0\| &= \sup_{t\geq 0}|A_n x_j(t) - A_n x_0(t)|\phi_2^\theta(t) \\
&\leq \sup_{t\geq 0}\int_0^{+\infty} G(t,s)\phi_2^\theta(t)m(s)|f_n(s,x_j(s)) - f_n(s,x_0(s))|\,ds \\
&\leq \frac{H}{h}\int_0^{+\infty} G(s,s)\phi_2(s)m(s)|f_n(s,x_j(s)) - f_n(s,x_0(s))|\,ds.
\end{aligned}$$

Since

$$G(s,s)\phi_2(s)m(s)|f_n(s,x_j(s))-f_n(s,x_0(s))| \le \frac{p(M)}{hq(M)}\phi_2(s)m(s)r(s)q\big(\frac{1}{n}\widetilde{\gamma}(s)\big),$$

the Lebesgue dominated convergence theorem and the continuity of f_n guarantee that the right-hand term tends to zero, as $j \to +\infty$. Hence A_n is continuous, for each $n \in \{1,2,\dots\}$.

Step 3. Let $D \subseteq \mathcal{P}$ be a bounded subset. Then there exists $M>1$ such that

$$\|x\| \le M, \quad \forall x \in D.$$

(a) $A_n(D)$ is a bounded subset of E. Indeed, using (3.4), we have

$$\begin{aligned}\|A_n x\| &= \sup_{t\ge 0}|A_n x(t)|\phi_2^\theta(t)\\ &\le \frac{Hp(M)}{hq(M)}\int_0^{+\infty} G(s,s)\phi_2(s)m(s)r(s)q(\frac{1}{n}\widetilde{\gamma}(s))ds<\infty.\end{aligned}$$

(b) Now, we show that the functions $\{A_n x(.)\phi_2^\theta(.),\, x\in D\}$ are almost equicontinuous on $[0,+\infty)$. For a given $\Lambda>0$, $x\in D$, and $t,t'\in[0,\Lambda]$ $(t>t')$, proceeding as in Step 1, we obtain the estimates:

$$\begin{aligned}&|A_n x(t)\phi_2^\theta(t)-A_n x(t')\phi_2^\theta(t')|\\ &\le \int_0^{+\infty}|G(t,s)\phi_2^\theta(t)-G(t',s)\phi_2^\theta(t')|m(s)f_n(s,x(s))\,ds\\ &\le \frac{p}{q}(M)\int_0^{\Lambda}|G(t,s)\phi_2^\theta(t)-G(t',s)\phi_2^\theta(t')|m(s)r(s)q(\frac{1}{n}\widetilde{\gamma}(s))ds\\ &\quad+\frac{p}{q}(M)\int_\Lambda^{+\infty}|G(t,s)\phi_2^\theta(t)-G(t',s)\phi_2^\theta(t')|m(s)r(s)q(\frac{1}{n}\widetilde{\gamma}(s))ds\\ &\le \frac{p}{q}(M)\int_0^{\Lambda}|G(t,s)\phi_2^\theta(t)-G(t',s)\phi_2^\theta(t')|m(s)r(s)q(\frac{1}{n}\widetilde{\gamma}(s))ds\\ &\quad+\frac{p}{q}(M)|\phi_1(t)\phi_2^\theta(t)-\phi_1(t')\phi_2^\theta(t')|\int_\Lambda^{+\infty}\phi_2(s)m(s)r(s)q(\frac{1}{n}\widetilde{\gamma}(s))ds.\end{aligned}$$

Then, for every $\varepsilon>0$ and $\Lambda>0$, there exists $\delta>0$ such that for all $x\in D$,

$$|A_n x(t)\phi_2^\theta(t)-A_n x(t')\phi_2^\theta(t')|<\varepsilon,$$

for all $t,t'\in[0,\Lambda]$ with $|t-t'|<\delta$.

(c) The functions $\{A_n x(.)\phi_2^\theta(.),\, x\in D\}$ are almost equiconvergent. Let $\sigma := \frac{\theta-1}{2} > 0$. Since $\lim_{t\to+\infty}\phi_2(t)=0$, then for every $\varepsilon>0$, there exists $\Lambda>0$ such that for all $t>\Lambda$

$$\phi_2(t) \le \Big(\frac{\varepsilon h}{H\int_0^{+\infty}G(s,s)\phi_2(s)m(s)r(s)q(\frac{1}{n}\widetilde{\gamma}(s))ds}\Big)^{1/\sigma}.$$

Using Lemma 2.4 (b), we deduce that for the above $\varepsilon>0$, there exists $\Lambda>0$, such that for $x\in D$ and $t>\Lambda$, we have

$$\begin{aligned}0\le A_n x(t)\phi_2^\theta(t) &= \int_0^{+\infty}G(t,s)\phi_2^\theta(t)m(s)f_n(s,x(s))\,ds\\ &\le \phi_2^\sigma(t)\int_0^{+\infty}G(t,s)\phi_2^{\sigma+1}(t)m(s)f_n(s,x(s))\,ds\end{aligned}$$

$$\leq \phi_2^\sigma(t)\frac{H}{h}K\int_0^{+\infty} G(s,s)\phi_2(s)m(s)r(s)q(\frac{1}{n}\widetilde{\gamma}(s))ds \leq \varepsilon.$$

Hence the functions $\{A_n x(.)\phi_2^\theta(.),\, x \in D\}$ are almost equiconvergent. Consequently, for each n, the operator A_n is completely continuous. □

4. Existence of at least one positive solution

We sue the hypotheses:

(H4) there exists $R > 0$ such that

$$hRq(R) > Hp(R)\int_0^{+\infty} G(s,s)\phi_2(s)m(s)r(s)q\big(\frac{h}{H}\widetilde{\gamma}(s)R\big)ds.$$

(H5) There exists $\psi \in C(\mathbb{R}^+, I)$ such that

$$F(t,x) \geq \psi(t),\ \forall t \in \mathbb{R}^+,\ \forall x \in (0,R]$$

with

$$\int_0^{+\infty} G(s,s)\phi_2(s)m(s)\psi(s)ds < +\infty.$$

Note that (H4) is equivalent to

$$\sup_{c>0} \frac{2h^2cq(c)}{Hp(c)\int_0^{+\infty}\phi_2(s)m(s)r(s)q(\frac{h}{H}\widetilde{\gamma}(s)c)ds} > 1.$$

Now we prove our first existence result.

Theorem 4.1. *Assume that either* (H1), (H3)–(H5) *or* (H2)–(H5) *hold. Then, problem* (1.1) *has at least one positive solution.*

Proof. **Step 1.** With R being given by (H4), we define $\Omega_1 = \{x \in E : \|x\| < R\}$, and then claim that $x \neq \lambda A_n x$ for all $x \in \partial\Omega_1 \cap \mathcal{P}, \lambda \in (0,1]$ and $n \geq n_0 > 1/R$. On the contrary, suppose that there exist $n \geq n_0$, $x_0 \in \partial\Omega_1 \cap \mathcal{P}$ and $\lambda_0 \in (0,1]$ such that $x_0 = \lambda_0 A_n x_0$. Since $x_0 \in \partial\Omega_1 \cap \mathcal{P}$, we have

$$x_0(t) \geq \frac{h}{H}\gamma(t)\|x_0\| = \frac{h}{H}\gamma(t)R, \quad \forall t \in \mathbb{R}^+.$$

Then

$$x_0(t)\phi_2^\theta(t) \geq \frac{h}{H}\widetilde{\gamma}(t)\|x_0\| = \frac{h}{H}\widetilde{\gamma}(t)R$$

and so

$$\begin{aligned}
R &= \|x_0\| = \lambda_0\|A_n x_0\| \\
&\leq \sup_{t\geq 0}\int_0^{+\infty} G(t,s)\phi_2^\theta(t)m(s)f_n(s,x_0(s))\,ds \\
&\leq \frac{H}{h}\int_0^{+\infty} G(s,s)\phi_2(s)m(s)F\Big(s,\max\{\frac{1}{n},x_0(s)\phi_2^\theta(s)\}\Big)\,ds \\
&\leq \frac{H}{h}\int_0^{+\infty} G(s,s)\phi_2(s)m(s)r(s)q\Big(\max\{\frac{1}{n},x_0(s)\phi_2^\theta(s)\}\Big) \\
&\quad\times \frac{p}{q}\Big(\max\{\frac{1}{n},x_0(s)\phi_2^\theta(s)\}\Big)\,ds \\
&\leq \frac{H}{h}\frac{p(R)}{q(R)}\int_0^{+\infty} G(s,s)\phi_2(s)m(s)r(s)q(\frac{h\widetilde{\gamma}(s)}{H}R)\,ds.
\end{aligned}$$

As a consequence

$$2h^2Rq(R) \le Hp(R)\int_0^{+\infty} \phi_2(s)m(s)r(s)q\Big(\frac{h\widetilde{\gamma}(s)}{H}R\Big)\,ds,$$

which is contradictory. By Lemma 2.1, we infer that

$$i(A_n, \Omega_1 \cap \mathcal{P}, \mathcal{P}) = 1, \quad \text{for all } n \in \{n_0, n_0+1, \dots\}. \tag{4.1}$$

By the existence property of the fixed point index, there exists an $x_n \in \Omega_1 \cap \mathcal{P}$ such that $A_n x_n = x_n$, $\forall n \ge n_0$. Writing

$$\begin{aligned} f_n(t, x_n(t)) &= f\Big(t, \max\{\frac{1}{n\phi_2^\theta(t)}, x_n(t)\}\Big) \\ &= f\Big(t, \frac{1}{\phi_2^\theta(t)}\max\{\frac{1}{n}, \phi_2^\theta(t)x_n(t)\}\Big) \\ &= F\Big(t, \max\{\frac{1}{n}, \phi_2^\theta(t)x_n(t)\}\Big), \end{aligned}$$

noting that $\|x_n\| < R$, and using (H5), we obtain

$$f_n(t, x_n(t)) \ge \psi(t), \quad t \ge 0,\ n \ge n_0.$$

Let

$$c^* := \int_0^{+\infty} G(s,s)\phi_2(s)m(s)\psi(s)ds < +\infty.$$

Then

$$\begin{aligned} x_n(t) &= A_n x_n(t) \\ &= \int_0^{+\infty} G(t,s)m(s)f_n(s, x_n(s))\,ds \\ &\ge \int_0^{+\infty} G(t,s)m(s)\psi(s)ds \\ &\ge \gamma(t)\int_0^{+\infty} G(s,s)\phi_2(s)m(s)\psi(s)ds \\ &= c^*\gamma(t). \end{aligned}$$

Hence $x_n(t)\phi_2^\theta(t) \ge c^*\widetilde{\gamma}(t)$ for all $t \ge 0$. From (H3), we finally deduce that

$$f_n(s, x_n(s)) = F\Big(s, \max\{\frac{1}{n}, x_n(s)\phi_2^\theta(s)\}\Big) \le r(s)q(c^*\widetilde{\gamma}(s))\frac{p}{q}(R).$$

Step 2. The sequence $\{x_n\}_{n\ge n_0}$ is relatively compact.
(a) $\{x_n\}_{n\ge n_0}$ is uniformly bounded for

$$\begin{aligned} \|x_n\| &= \sup_{t\ge 0}|x_n(t)|\phi_2^\theta(t) \\ &\le \sup_{t\ge 0}\int_0^{+\infty} G(t,s)\phi_2^\theta(t)m(s)f_n(s, x_n(s))\,ds \\ &\le \frac{H}{h}\frac{p(R)}{q(R)}\int_0^{+\infty} \phi_2(s)m(s)r(s)q(c^*\widetilde{\gamma}(s))\,ds < \infty. \end{aligned}$$

(b) Almost equicontinuity. For all $\Lambda > 0$ and $t, t' \in [0, \Lambda]$, we have

$$|x_n(t)\phi_2^\theta(t) - x_n(t')\phi_2^\theta(t')|$$

$$
\begin{aligned}
&= \int_0^{+\infty} |G(t,s)\phi_2^\theta(t) - G(t',s)\phi_2^\theta(t')| m(s) f_n(s, x_n(s))\, ds \\
&\le \int_0^{\Lambda} |G(t,s)\phi_2^\theta(t) - G(t',s)\phi_2^\theta(t')| m(s) r(s) q(\widetilde{\gamma}(s)c^*) \frac{p(R)}{q(R)}\, ds \\
&\quad + |\phi_1(t)\phi_2^\theta(t) - \phi_1(t')\phi_2^\theta(t')| \int_\Lambda^{+\infty} \phi_2(s) m(s) r(s) q(\widetilde{\gamma}(s)c^*) \frac{p(R)}{q(R)}\, ds.
\end{aligned}
$$

Then by (3.3), for every $\varepsilon > 0$ and $\Lambda > 0$, there exists $\delta > 0$ such that $|x_n(t)\phi_2^\theta(t) - x_n(t')\phi_2^\theta(t')| < \varepsilon$ for all $t, t' \in [0, \Lambda]$ with $|t - t'| < \delta$. Hence $\{x_n(.)\phi_2^\theta(.)\}_{n \ge n_0}$ is almost equicontinuous.

(c) The sequence $\{x_n(.)\phi_2^\theta(.)\}_{n \ge n_0}$ is equiconvergent at $+\infty$. Let $\sigma := \frac{\theta - 1}{2} > 0$. Since $\lim_{t \to +\infty} \phi_2(t) = 0$, then for some given $\varepsilon > 0$, there exists $\Lambda > 0$ such that for all $t > \Lambda$

$$
\phi_2(t) \le \Big(\frac{\varepsilon h q(R)}{H p(R) \int_0^{+\infty} G(s,s)\phi_2(s) m(s) r(s) q(c^* \widetilde{\gamma}(s)) ds} \Big)^{1/\sigma}.
$$

From Lemma 2.4 (b), we deduce that for the above $\varepsilon > 0$, there exists $\Lambda > 0$, such that for $n \ge n_0$ and $t > \Lambda$,

$$
\begin{aligned}
0 \le x_n(t)\phi_2^\theta(t) &= \int_0^{+\infty} G(t,s)\phi_2^\theta(t) m(s) f_n(s, x(s))\, ds \\
&\le \phi_2^\sigma(t) \int_0^{+\infty} G(t,s)\phi_2^{\sigma+1}(t) m(s) f_n(s, x(s))\, ds \\
&\le \phi_2^\sigma(t) \frac{H}{h} \frac{p(R)}{q(R)} \int_0^{+\infty} G(s,s)\phi_2(s) m(s) r(s) q(c^* \widetilde{\gamma}(s)) ds \le \varepsilon.
\end{aligned}
$$

Then the sequence $\{x_n(.)\phi_2^\theta(.)\}_{n \ge n_0}$ is equiconvergent at $+\infty$. By Theorem 3.3, there exists a subsequence $\{x_{n_k}\}_{k \ge 1}$ with $\lim_{k \to +\infty} x_{n_k} = x_0$. Since $x_{n_k}(t) \ge c^* \widetilde{\gamma}(t), \forall k \ge 1$ and $\forall t \ge 0$, we have $x_0(t) \ge c^* \widetilde{\gamma}(t), \forall t \ge 0$. Hence

$$
\int_0^{+\infty} G(t,s) m(s) f(s, x_0(s))\, ds < +\infty.
$$

The continuity of f guarantees that, for all $s \in \mathbb{R}^+$:

$$
\begin{aligned}
\lim_{k \to +\infty} f_{n_k}(s, x_{n_k}(s)) &= \lim_{k \to +\infty} f(s, \max\{1/\phi_2^\theta(s) n_k, x_{n_k}(s)\}) \\
&= f(s, \max\{0, x_0(s)\}) \\
&= f(s, x_0(s)).
\end{aligned}
$$

With the Lebesgue dominated convergence theorem, we conclude that

$$
\begin{aligned}
x_0(t) &= \lim_{k \to +\infty} x_{n_k}(t) \\
&= \lim_{k \to +\infty} \int_0^{+\infty} G(t,s) m(s) f_{n_k}(s, x_{n_k}(s))\, ds \\
&= \int_0^{+\infty} G(t,s) m(s) f(s, x_0(s))\, ds,
\end{aligned}
$$

proving that x_0 is a positive solution of (1.1) with $\|x_0\| \le R$. Now, using (H4) the same reasoning as in Step 1 guarantees that $\|x_0\| < R$. $\square$

5. Existence of at least two positive solutions

With R given by (H4), we define the assumptions:

(H6) there exists $[\alpha,\beta]\subset I$ and $R'>R$ such that
$$f(t,x)>N^*x,\quad \forall t\in[\alpha,\beta],\ \forall x\geq R',$$
where
$$N^*=1+\frac{1}{r\min_{t\in[\alpha,\beta]}\widetilde{\gamma}(t)\int_\alpha^\beta G(s,s)\phi_2(s)m(s)\,ds}$$
and $r=\frac{h}{H}\min_{t\in[\alpha,\beta]}\gamma(t)$.

(H5') There exists $\psi\in C(\mathbb{R}^+,I)$ such that
$$F(t,x)\geq\psi(t),\quad \forall t\in\mathbb{R}^+,\ \forall x\in(0,R'/r]$$
with
$$\int_0^{+\infty}G(s,s)\phi_2(s)m(s)\psi(s)ds<+\infty.$$

Note that (H6) is satisfied for instance in the super-linear case:
$$\lim_{x\to+\infty}\frac{f(t,x)}{x}=+\infty,\qquad \text{uniformly for } t\in[\alpha,\beta].$$

Theorem 5.1. *Assume that either* (H1), (H3), (H4), (H5'), (H6) *or* (H2), (H3), (H4), (H5'), (H6) *hold. Then problem* (1.1) *has at least two positive solutions.*

Proof. By Theorem 4.1, there exists a positive solution x_0 such that $\|x_0\|<R$, where R is as introduced in (H4). Let
$$\Omega_2=\{x\in E:\ \|x\|<R'/r\},$$
where R' is as introduced in (H6). We show that $A_nx\not\leq x$ for all $x\in\partial\Omega_2\cap\mathcal{P}$ and $n\in\{1,2,\dots\}$. Suppose on the contrary that there exists $n\in\{1,2,\dots\}$ and $x_0\in\partial\Omega_2\cap\mathcal{P}$ such that $A_nx_0\leq x_0$. Since $x_0\in\mathcal{P}$, we have
$$x_0(t)\geq\frac{h}{H}\gamma(t)\|x_0\|\geq\frac{h}{H}\min_{s\in[\alpha,\beta]}\gamma(s)\frac{R'}{r}\geq R',\quad \forall t\in[\alpha,\beta].$$
Then for every $t\in[\alpha,\beta]$, we have
$$\begin{aligned}
x_0(t)\phi_2^\theta(t)&\geq A_nx_0(t)\phi_2^\theta(t)\\
&=\int_0^{+\infty}G(t,s)\phi_2^\theta(t)m(s)f_n(s,x_0(s))\,ds\\
&\geq\gamma(t)\phi_2^\theta(t)\int_\alpha^\beta G(s,s)\phi_2(s)m(s)f_n(s,x_0(s))\,ds\\
&\geq\widetilde{\gamma}(t)\int_\alpha^\beta G(s,s)\phi_2(s)m(s)N^*\max\{\frac{1}{\phi_2^\theta(s)n},x_0(s)\}\,ds\\
&\geq N^*R'\min_{t\in[\alpha,\beta]}\widetilde{\gamma}(t)\int_\alpha^\beta G(s,s)\phi_2(s)m(s)\,ds>R'/r,
\end{aligned}$$
contradicting $\|x_0\|=R'/r$. Finally, Lemma 2.1 guarantees
$$i(A_n,\Omega_2\cap\mathcal{P},\mathcal{P})=0,\quad \forall n\in\{1,2,\dots\}\tag{5.1}$$

while (4.1) and (5.1) imply that

$$i(A_n, (\Omega_2 \setminus \overline{\Omega}_1) \cap \mathcal{P}, \mathcal{P}) = -1, \quad \forall n \geq n_0. \tag{5.2}$$

The existence property of the fixed point index guarantees that A_n has a second fixed point $y_n \in (\Omega_2 \setminus \overline{\Omega}_1) \cap P$, for all $n \geq n_0$. The sequence $\{y_n\}_{n \geq n_0}$ satisfies $y_n(t) \geq \frac{h}{H}\gamma(t)R$ for all $t \geq 0$ and $\|y_n\| < R'/r$ for all $n \geq n_0$. Arguing as above, we can show that $\{y_n\}_{n \geq n_0}$ has a subsequence $\{y_{n_j}\}_{j \geq 1}$ converging to some limit y_0 solution of (1.1). Moreover

$$\|x_0\| < R \leq \|y_0\| < R'/r\cdot$$

Hence x_0 and y_0 are two distinct positive solutions of problem (1.1). □

6. Upper and lower solutions

We first define upper and lower solutions on the half-line.

Definition 6.1. (a) We say that α is a lower solution of problem (1.1) if $\alpha \in \mathcal{C}^0(\mathbb{R}^+) \cap \mathcal{C}^2(I)$ and

$$\begin{gathered} \alpha''(t) - k^2(t)\alpha(t) + m(t)f(t, \alpha(t)) \geq 0, \quad t > 0 \\ \alpha(0) \leq 0, \quad \alpha(+\infty) \leq 0. \end{gathered}$$

(b) A function β is an upper solution of problem (1.1) if $\beta \in \mathcal{C}^0(\mathbb{R}^+) \cap \mathcal{C}^2(I)$ and

$$\begin{gathered} \beta''(t) - k^2(t)\beta(t) + m(t)f(t, \beta(t)) \leq 0, \quad t > 0 \\ \beta(0) \geq 0, \quad \beta(+\infty) \geq 0. \end{gathered}$$

In this section, we assume that the nonlinearity $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous, but not necessarily positive. We enunciate some growth assumptions:

(H7) There exist $\alpha \leq \beta$ lower and upper solutions of problem (1.1) respectively.
(H8) There exists a continuous function $\psi : I \to \mathbb{R}^+$ such that

$$\int_0^\infty m(s)\psi(s)ds < \infty, \tag{6.1}$$

$$|f(t, x)| \leq \psi(t), \ \forall (t, x) \in D_\alpha^\beta, \tag{6.2}$$

where $D_\alpha^\beta := \{(t, x) \in I \times \mathbb{R} : \alpha(t) \leq x \leq \beta(t)\}$.

Consider the Banach space

$$X = \{x \in \mathcal{C}^0(\mathbb{R}^+) \mid \lim_{t \to +\infty} x(t) = 0\}$$

with the sup-norm $\|x\| = \sup_{t \in [0, \infty)} |x(t)|$. Define the truncation function $\widetilde{f}$ by

$$\widetilde{f}(t, x) = \begin{cases} f(t, \beta(t)), & \beta(t) \leq x, \\ f(t, x), & \alpha(t) \leq x \leq \beta(t), \\ f(t, \alpha(t)), & x \leq \alpha(t) \end{cases}$$

and consider the modified problem

$$\begin{gathered} x''(t) - k^2(t)x(t) + m(t)\widetilde{f}(t, x(t)) = 0, \quad t > 0 \\ x(0) = 0, \quad x(+\infty) = 0. \end{gathered} \tag{6.3}$$

Lemma 6.2. *Under Assumption* (H7)*, all possible solutions of problem* (6.3) *satisfy*

$$\alpha(t) \leq x(t) \leq \beta(t), \quad \forall t \geq 0.$$

Proof. Suppose, on the contrary that $\sup_{t\in[0,\infty)}(x-\beta)(t)>0$. Since $(x-\beta)(+\infty)=-\beta(+\infty)\le 0$ and $(x-\beta)(0)=-\beta(0)\le 0$, then there is $t_0\in(0,\infty)$ such that $x(t_0)-\beta(t_0)=\sup_{t>0}(x-\beta)(t)>0$ and $(x''-\beta'')(t_0)\le 0$. In addition, by definition of an upper solution, we have

$$\begin{aligned}
(x''-\beta'')(t_0) &= k^2(t_0)x(t_0)-m(t_0)\widetilde{f}(t_0,x(t_0))-\beta''(t_0)\\
&\ge k^2(t_0)x(t_0)-m(t_0)\widetilde{f}(t_0,x(t_0))-k^2(t_0)\beta(t_0)+m(t_0)f(t_0,\beta(t_0))\\
&= k^2(t_0)(x-\beta)(t_0)-m(t_0)[\widetilde{f}(t_0,x(t_0))-f(t_0,\beta(t_0))]\\
&= k^2(t_0)(x-\beta)(t_0)>0,
\end{aligned}$$

leading to a contradiction. Similarly, we can prove that $x(t)\ge\alpha(t)$, for all $t\ge 0$. $\square$

6.1. **Existence of bounded solutions.** Our main existence result in this section is as follows.

Theorem 6.3. *Assume that either Assumptions* (H1), (H7), (H8) *or* (H2), (H7), (H8) *hold. Then problem* (1.1) *has at least one solution $x\in X$ with the representation*

$$x(t)=\int_0^\infty G(t,s)m(t)f(s,x(s))ds$$

and such that

$$\alpha(t)\le x(t)\le\beta(t),\quad \forall t\in\mathbb{R}^+.$$

Proof. **Step 1.** We show that problem (6.3) has at least one solution in X. Let us consider the operator $T:X\to X$ defined by

$$(Tx)(t)=\int_0^\infty G(t,s)m(s)\widetilde{f}(s,x(s))ds. \tag{6.4}$$

Then solving problem (6.3) amount to proving the existence of a fixed point for T.
(a) $T:X\to X$ is well defined. Given $x\in X$, from (6.1), (6.2), and the monotonicity of ϕ_1,ϕ_2, we have

$$\begin{aligned}
(Tx)(t) &\le \int_0^\infty G(t,s)m(s)\widetilde{f}(s,x(s))ds\\
&\le \int_0^\infty G(t,s)m(s)\psi(s)ds\\
&\le \int_0^t \phi_1(s)\phi_2(t)m(s)\psi(s)ds+\int_t^\infty \phi_1(t)\phi_2(s)m(s)\psi(s)ds\\
&\le \phi_1(t)\int_0^t \phi_2(t)m(s)\psi(s)ds+\phi_1(t)\int_t^\infty \phi_2(t)m(s)\psi(s)ds\\
&= \phi_1(t)\phi_2(t)\int_0^\infty m(s)\psi(s)ds\\
&\le M\int_0^\infty m(s)\psi(s)ds.
\end{aligned}$$

Hence the integral in (6.4) is well defined. Moreover Tx is continuous and by (6.1) and (6.2), the map $s\mapsto m(s)\widetilde{f}(s,x(s))$ is L^1; hence Lemma 2.3 implies that $\lim_{t\to+\infty}Tx(t)=0$ and so $Tx\in X$.

(b) $T: X \to X$ is continuous. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence converging to some limit x in X. We have

$$\int_0^\infty m(s)\left|\widetilde{f}(s,x_n(s)) - \widetilde{f}(s,x(s))\right| ds \le 2\int_0^\infty m(s)\psi(s)ds < \infty$$

and

$$\begin{aligned}\|Tx_n - Tx\| &= \sup_{t\in[0,\infty)} \Big|\int_0^\infty G(t,s)m(s)[\widetilde{f}(s,x_n(s)) - \widetilde{f}(s,x(s))]ds\Big| \\ &\le \frac{1}{2h}\int_0^\infty m(s)|\widetilde{f}(s,x_n(s)) - \widetilde{f}(s,x(s))|ds.\end{aligned}$$

By the Lebesgue dominated convergence theorem, the right-hand side term tends to 0, as $n \to +\infty$, proving our claim.

(c) $T: X \to X$ is compact. For every $x \in X$ as above, we have

$$\|Tx\| \le \frac{1}{2h}\int_0^\infty m(s)\psi(s)ds < \infty,$$

hence T is bounded. Now, for a given $\Lambda > 0$ and $t_1, t_2 \in [0,\Lambda]$, we have the estimates:

$$\begin{aligned}|(Tx)(t_2) - (Tx)(t_1)| &\le \int_0^\infty |G(t_2,s) - G(t_1,s)|m(s)\psi(s)ds \\ &\le \int_0^\Lambda |G(t_2,s) - G(t_1,s)|m(s)\psi(s)ds \\ &\quad + \int_\Lambda^\infty |\phi_1(t_2)\phi_2(s) - \phi_1(t_1)\phi_2(s)|m(s)\psi(s)ds \\ &\le \int_0^\Lambda |G(t_2,s) - G(t_1,s)|m(s)\psi(s)ds \\ &\quad + |\phi_1(t_2) - \phi_1(t_1)|\int_0^\infty m(s)\psi(s)ds.\end{aligned}$$

By (6.1) and the continuity of the Green's function, the Lebesgue dominated convergence theorem guarantees that

$$\lim_{|t_1-t_2|\to 0}\int_0^\Lambda |G(t_2,s) - G(t_1,s)|m(s)\psi(s)ds = 0.$$

In addition, (6.1) and the continuity of ϕ_1 imply that

$$\lim_{|t_1-t_2|\to 0}|\phi_1(t_2) - \phi_1(t_1)|\int_0^\infty m(s)\psi(s)ds = 0.$$

Hence the right-hand side term goes to 0, as $|t_1 - t_2| \to 0$, proving that the family $\{Tx\}$ is almost equicontinuous. Finally, to prove equiconvergence at $+\infty$, we first note that from Lemma 2.3, we have $\lim_{t\to\infty} Tx(t) = 0$. Thus using the fact that

$$\lim_{t\to\infty}\phi_2(t) = 0 \quad \text{and} \quad \int_0^\infty m(s)\psi(s)ds < \infty,$$

we have that for every $\varepsilon > 0$, there exists $\Lambda > 0$ such that

$$0 \le \sup_{x\in X}|Tx(t) - 0| = \sup_{x\in X}\int_0^\infty G(t,s)m(s)\widetilde{f}(s,x(s))ds$$

$$
\begin{aligned}
&\le \int_0^t \phi_1(s)\phi_2(t)m(s)\psi(s)ds + \int_t^\infty \phi_1(t)\phi_2(s)m(s)\psi(s)ds \\
&= \int_0^\Lambda \phi_1(s)\phi_2(t)m(s)\psi(s)ds + \int_\Lambda^t \phi_1(s)\phi_2(t)m(s)\psi(s)ds \\
&\quad + \int_t^\infty \phi_1(t)\phi_2(s)m(s)\psi(s)ds \\
&\le \int_0^\Lambda \phi_1(s)\phi_2(t)m(s)\psi(s)ds + \int_\Lambda^\infty \phi_1(t)\phi_2(t)m(s)\psi(s)ds \\
&\quad + \int_\Lambda^\infty \phi_1(t)\phi_2(t)m(s)\psi(s)ds \\
&\le \phi_1(\Lambda)\phi_2(t)\int_0^\Lambda m(s)\psi(s)ds + M\int_\Lambda^\infty m(s)\psi(s)ds \\
&\quad + M\int_\Lambda^\infty m(s)\psi(s)ds \\
&\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

The compactness of T then follows from Theorem 3.2. By the Schauder fixed theorem point, we conclude that T has at least a fixed point $x \in X$ and then problem (6.3) has a continuous, bounded solution.

Step 2. Problem (1.1) has at least one solution in X. By Lemma 6.2, every solution x of problem (6.3) satisfies the estimates

$$\alpha(t) \le x(t) \le \beta(t), \quad \forall t \ge 0.$$

Then $\widetilde{f}(t, x(t)) = f(t, x(t))$ and x is a solution of problem (1.1), which completes the proof of the theorem. A similar result is obtained when Assumptions (H2), (H7), (H8) hold; the details of the proof are omitted. □

6.2. **Uniqueness result.** The following result complements Theorem 6.3.

Proposition 6.4. *In addition to the hypotheses in Theorem 6.3, assume that*

(H9) $x_1 \ge x_2 \to f(t, x_1) \le f(t, x_2)$ *for all* $t > 0$.

Then problem (1.1) *has a unique solution x such that, for every $t \in \mathbb{R}$,*

$$\alpha(t) \le x(t) \le \beta(t), \quad \forall t \ge 0.$$

Proof. Suppose that there exist two distinct solutions x_1, x_2 to problem (1.1) and let $z = x_1 - x_2$. Assume that $z(t_1) > 0$ for some t_1. Since $z(+\infty) = z(0) = 0$, z has a positive maximum at some $t_0 < \infty$. Hence

$$
\begin{aligned}
0 \ge z''(t_0) &= k^2(t_0)z(t_0) - m(t_0)f(t_0, x_1(t_0)) + m(t_0)f(t_0, x_2(t_0)) \\
&= k^2(t_0)z(t_0) + m(t_0)[f(t_0, x_2(t_0)) - f(t_0, x_1(t_0))] > 0,
\end{aligned}
$$

leading to a contradiction and completing the proof. □

7. Applications

Example 7.1. Consider the boundary-value problem

$$
\begin{gathered}
x''(t) - (\sin t + 3)^2 x(t) + e^{-t}(t + x(t)) = 0, \quad t > 0, \\
x(0) = 0, \quad x(+\infty) = 0.
\end{gathered}
\tag{7.1}
$$

Then $\alpha(t) \equiv 0$ and $\beta(t) = t$ are respectively lower-solution and upper-solution. Moreover

$$m(t)|f(t,x)| \le e^{-t}(1+t) \in L^1, \ \forall (t,x) \in D_\alpha^\beta.$$

Then Assumptions (H2), (H7)-(H9) are satisfied. As a consequence, Theorem 6.3 and Proposition 6.4 imply that problem (7.1) has exactly one nontrivial solution x such that

$$0 \le x(t) \le t, \quad \forall t \ge 0.$$

Example 7.2. Let $m, n, p \in \{1, 2, \dots\}$ and $\delta \ge 0$ an arbitrary real parameter. Consider the boundary-value problem:

$$\begin{gathered} x''(t) - (2 + \delta + |\sin^m t \cos^n t|)^p x(t) + e^{-t}(t+1)(1 + x^2(t)) = 0, \quad t > 0, \\ x(0) = 0, \quad x(+\infty) = 0. \end{gathered} \tag{7.2}$$

Clearly the trivial solution is a lower-solution while $\beta(t) = t+1$ is an upper-solution. Indeed, since $e^{-t}(t^2 + 2t + 2) \le 2$, for $t \ge 0$, we have

$$\begin{aligned} &\beta''(t) - (2 + \delta + |\sin^m t \cos^n t|)^p \beta(t) + e^{-t}(t+1)(1 + \beta^2(t)) \\ &= -(2 + \delta + |\sin^m t \cos^n t|)^p (1+t) + e^{-t}(t+1)(t^2 + 2t + 2) \\ &\le -2^p(t+1) + 2(1+t) = (t+1)(-2^p + 2) \le 0. \end{aligned}$$

In addition,

$$m(t)|f(t,x)| \le e^{-t}(1+t)(t^2 + 2t + 2) \in L^1, \quad \forall (t,x) \in D_\alpha^\beta.$$

Then Assumptions (H2), (H7), (H8) are satisfied. By Theorem 6.3 problem (7.2) has at least one nontrivial solution x such that

$$0 \le x(t) \le (1+t), \ \forall t \ge 0.$$

Example 7.3. Consider the singular boundary-value problem

$$\begin{gathered} x''(t) - (\sin(t) + 3)^2 x(t) + \frac{1}{4} e^{-t} \gamma(t) \frac{\phi_2^4(t) x^2 + 1}{\sqrt{x}} = 0, \\ x(0) = 0, \quad \lim_{t \to +\infty} x(t) = 0. \end{gathered} \tag{7.3}$$

Let $f(t,x) = \gamma(t) \frac{\phi_2^4(t)x^2+1}{\sqrt{x}}$, $m(t) = \frac{1}{4} e^{-t}$, and $k(t) = \sin(t)+3$. With $\theta = 2$, we have $F(t,x) = \phi_2(t)\gamma(t)\frac{x^2+1}{\sqrt{x}} \cdot$. The functions ϕ_2 and γ are as introduced in Lemmas 2.2, 2.4. Then Assumptions (H2), (H3), (H4), (H5'), and (H6) are satisfied. Indeed: For (H2), k is continuous, periodic, hence bounded with $h = 2$ and $H = 4$. For (H3), the function $q(x) = \frac{1}{x} : I \to I$ is continuous, decreasing, and

$$F(t,x) \le r(t) p(x), \quad \forall t \ge 0, \ \forall x > 0,$$

with $p(x) = \frac{x^2+1}{\sqrt{x}}$ and $r(t) = \phi_2(t)\gamma(t)$. The function $\frac{p}{q}(x) = \sqrt{x}(x^2 + 1)$ is increasing on I and for any $c > 0$, we have

$$\int_0^{+\infty} \phi_2(s) m(s) r(s) q(c\widetilde{\gamma}(s)) ds = \frac{1}{4c} < +\infty.$$

For (H4), since

$$\sup_{c>0} \frac{2h^2 c q(c)}{H p(c) \int_0^{+\infty} \phi_2(s) m(s) r(s) q(\frac{h}{H} \widetilde{\gamma}(s) c) ds} = \sup_{c>0} \frac{16 c \sqrt{c}}{c^2 + 1} > 1,$$

there exists $R > 0$ satisfying (H4). For (H6), in any subinterval $[\alpha, \beta] \subset (0, +\infty)$,

$$\lim_{x \to +\infty} \frac{f(t,x)}{x} = +\infty, \quad \text{uniformly for } t \in [\alpha, \beta];$$

hence (H6) is satisfied with some $R' > R$. For satisfying (H5'), there exists $\psi(t) = \phi_2(t)\gamma(t)\frac{\sqrt{r}}{\sqrt{R'}} \in C(\mathbb{R}^+, I)$ such that

$$F(t,x) \geq \psi(t), \quad \forall t \in \mathbb{R}^+, \ \forall x \in (0, R'/r]$$

with

$$\int_0^{+\infty} G(s,s)\phi_2(s)m(s)\psi_c(s)ds < +\infty.$$

Therefore, all conditions of Theorem 5.1 are met and then problem (7.3) has at least two positive solutions.

Concluding remarks. In this work, we have obtained some existence results and even a uniqueness theorem for problem (1.1). This problem has the particularity that the derivation operator is time depending. As far as we know, this problem was only considered in [2, 18] where the nonlinearity is positone and in [12, 13] where bounded solutions were sought for the following boundary conditions:

$$\begin{gathered} -x''(t) + k^2(t)x(t) = f(t, x(t)), \quad t > 0, \\ x(0) = x_0, \quad x \text{ is bounded}, \end{gathered}$$

k being bounded from below; the method of upper and lower solutions has been employed. The nonlinearity f is allowed to change sign and has a space singularity. In each case, we have developed the upper and lower solution method on infinite intervals of the real line together with the index fixed point theory to prove existence of single or twin solutions in appropriate cones of a weighted Banach space. Quite general growth conditions of the right-hand side nonlinearity, including superlinearities, were assumed. Indeed Assumption (H3) allows singular nonlinearities of the form $p(x) = x^p x^{-q}$ for positive p, q. Finally, notice that Theorems 4.1 provides a solution lying in a ball of a Banach space and thus may be the trivial one. To avoid such a solution, one may add the assumption that $f(t, 0) \not\equiv 0$ in Theorem 4.1 and $\alpha \not\equiv 0$ in Theorem 6.3. As for the second solution obtained in Theorem 5.1, it is of course positive. We believe that this work can make a contribution in the study of a class of Sturm-Liouville boundary values problems on the half-line with time-depending derivation operator.

Acknowledgments. The authors are grateful to the anonymous referee for his/her careful reading of the manuscript and the pertinent remarks that have substantially improved the first draft.

References

[1] R. P. Agarwal, D. O'Regan; *Infinite Interval Problems for Differential, Difference and Integral Equations,* Kluwer Academic Publisher, Dordrecht, 2001.

[2] R. P. Agarwal, O.G. Mustafa, Y.V. Rogovchenko; *Existence and asymptotic behavior of solutions of boundary value problem on an infinite interval,* Math. Comput. Model., **41** (2005) 135–157.

[3] C. Corduneanu; *Integral Equations and Stability of Feedback Systems,* Academic Press, New York, 1973.

[4] C. De Coster, P. Habets; *Two Point Boundary Value Problems: Lower and Upper Solutions,* Elsevier, Amsterdam, 2006.

[5] K. Deimling; *Nonlinear Functional Analysis,* Springer-Verlag, Berlin, Heidelberg, 1985.
[6] S. Djebali, K. Mebarki; *Existence results for a class of BVPs on the positive half-line,* Comm. Appl. Nonlin. Anal., **4(2)** (2007) 13–31.
[7] S. Djebali, K. Mebarki; *Multiple positive solutions for singular BVPs on the positive half-line,* Comput. Math. Appl., **55(12)** (2008) 2940–2952.
[8] S. Djebali, T. Moussaoui; *A class of second order BVPs on infinite intervals,* Elec. J. Qual. Theo. Diff. Eq., **4** (2006) 1–19.
[9] S. Djebali, O. Saifi; *Positive solutions for singular BVPs with sign changing and derivative depending nonlinearity on the positive half-line,* Acta Appl. Math., **110(2)** (2010) 639–665.
[10] S. Djebali, O. Saifi, B. Yan; *Positive solutions for singular bvps on the positive half-line arising from epidemiology and combustion theory*, Acta Math. Scientia, **32B(2)** (2012) 672–694.
[11] J. Dugundji, A. Granas; *Fixed Point Theory,* Springer Monographs in Mathematics, Springer Verlag, New York, 2003.
[12] P. W. Eloe, L. J. Grimm, J. D. Mashburn; *A boundary value problem on an unbounded domain*, Diff. Eq. Dyn. Syst., **8(2)** (2000) 125–140.
[13] P. W. Eloe, E. R. Kaufmann, C. C. Tisdell; *Multiple solutions of a boundary value problem on an unbounded domain*, Dyn. Syst. Appl., **15(1)** (2006) 53–63.
[14] D. J. Guo, V. Lakshmikantham; *Nonlinear Problems in Abstract Cones,* Academic Press, 1988.
[15] J. F. Holt; *Numerical solution of nonlinear two-point boundary value problems by finite difference methods,* Commun. ACM, **7** (1964) 366–373.
[16] H. Lian and W. Ge; *Existence of positive solutions for Sturm-Liouville boundary value problems on the half-line,* J. Math. Anal. Appl., **321** (2006) 781–792.
[17] H. Lian, P. Wang, W. Ge; *Unbounded upper and lower solutions method for Sturm Liouville boundary value problem on infinite intervals,* Nonlin. Anal. T.M.A., **70(7)** (2009) 2627–2633.
[18] R. Ma, B. Zhu; *Existence of positive solutions for a semipositone boundary value problem on the half line,* Comput. Math. Appl., **58(8)** (2009) 1672–1686.
[19] J. D. Murray; *Mathematical Biology*, Biomathematics Texts, **V19**, Springer-Verlag, 1989.
[20] D. O'Regan; *Existence Theory for Ordinary Differential Equations,* Kluwer, Dordrecht, 1997.
[21] D. O'Regan, B. Yan, R. P. Agarwal; *Solutions in weighted spaces of singular boundary value problems on the half-line,* J. Comput. Appl. Math., **205** (2007) 751–763.
[22] B. Przeradzki; *Travelling waves for reaction-diffusion equations with time depending nonlinearities,* J. Math. Anal. Appl., **281** (2003) 164–170.
[23] Y. Wang, L. Liu, Y. Wu; *Positive solutions of singular boundary value problems on the half-line,* Appl. Math. Comput., **197(2)** (2008) 789–796.
[24] B. Yan, D. O'Regan, R. P. Agarwal,; *Positive solutions to singular boundary value problems with sign changing nonlinearities on the half-Line via upper and lower solutions,* Acta Math. Sinica. English Series, **23(8)** (2007) 1447–1456.
[25] E. Zeidler; *Nonlinear Functional Analysis and its Applications. Vol. I: Fixed Point Theorems,* Springer-Verlag, New York, 1986.
[26] M. Zima; *On positive solutions of boundary value problems on the half-line,* J. Math. Anal. Appl., **259** (2001) 127–136.

Smaïl Djebali
Laboratoire "Théorie du Point Fixe et Applications", École Normale Supérieure, Kouba, B.P. 92, 16050 Kouba. Algiers, Algeria
E-mail address: djebali@ens-kouba.dz, djebali@hotmail.com

Ouiza Saifi
Department of Economics, Faculty of Economic and Management Sciences, Algiers University 3, Algeria
E-mail address: saifi_kouba@yahoo.fr

Samira Zahar
Department of Mathematics, A.E. Mira University, 06000. Bejaia, Algeria
E-mail address: zahar_samira@yahoo.fr