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AN APPROACH FOR CONSTRUCTING COEFFICIENTS OF DEGENERATE ELLIPTIC COMPLEX EQUATIONS

GUO CHUN WEN

Abstract. This article deals with the inverse problem for degenerate elliptic systems of first order equations with Riemann-Hilbert type map in simply connected domains. Firstly the formulation and the complex form of the problem for the first-order elliptic systems with the degenerate rank 0 are given, and then the coefficients of the systems are constructed by a new complex analytic method. Here we verify and apply the Hölder continuity of a singular integral operator.

1. Formulation of the inverse problem for degenerate elliptic complex equations of first order

In [1, 2, 3, 5, 6, 7, 15, 16], the authors discussed the inverse problem of second-order elliptic equations without degeneracy. In this article, by using the methods of integral equations and complex analysis, the existence of solutions of the inverse problem for degenerate elliptic complex equations of first order with Riemann-Hilbert type map is discussed.

Let $D(\supset \{0\})$ be a simply connected bounded domain in the complex plane $\mathbb{C}$ with the boundary $\partial D = \Gamma \in C^1_\mu(0 < \mu < 1)$. There is no harm in assuming that the domain D is $\{|z| < 1\}$ with boundary $\Gamma = \{|z| = 1\}$. Consider the linear elliptic systems of first-order equations with degenerate rank 0,

$$\begin{aligned} H_1(y)u_x - H_2(y)v_y &= au + bv \quad \text{in } D \\ H_1(y)v_x + H_2(y)u_y &= cu + dv \quad \text{in } D, \end{aligned} \tag{1.1}$$

in which $H_j(y) = |y|^{m_j/2}h_j(y), h_j(y)\ (j = 1, 2)$ are positive continuous functions in $\overline{D}$, $m_j\,(j = 1, 2, m_2 < \min(1, m_1))$ are positive constants, and $a, b, c, d\,(j = 1, 2)$ are functions of $x + iy\,(\in D)$ satisfying the conditions $a, b, c, d \in L_\infty(D)$, which is called Condition C. In this article, the notation is the same as in references [8, 9, 10, 11, 12, 13, 14, 15, 16]. The following degenerate elliptic system is a special case of system (1.1) with $H_j(y) = |y|^{m_j/2}\ (j = 1, 2)$:

$$\begin{aligned} |y|^{m_1/2}u_x - |y|^{m_2/2}v_y &= au + bv \quad \text{in } D, \\ |y|^{m_1/2}v_x + |y|^{m_2/2}u_y &= cu + dv \quad \text{in } D, \end{aligned} \tag{1.2}$$

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For convenience, we mainly discuss equation (1.2), and equation (1.1) can be similarly discussed. From the elliptic condition in (1.2) (see [13, (1.3), Chpater II]), namely

$$J = 4K_1K_4 - (K_2 + K_3)^2 = 4H^2(y) = 4[H_1(y)/H_2(y)]^2 > 0 \quad \text{in } \overline{D}\backslash\gamma$$

and $J = 0$ on $\gamma = \{-1 < x < 1, y = 0\}$, hence system (1.1) or (1.2) is elliptic system of first-order equations in D with the parabolic degenerate line $\gamma = (-1, 1)$ on the x-axis in $x + iy$-plane. Setting $Y = G(y) = \int_0^y H(t)dt$, $Z = x + iY$ in $\overline{D}$, if $H(y) = |y|^{m/2}h_1(y)/h_2(y)$, $m = m_1 - m_2$, $Y = \int_0^y H(t)dt \le |s_0y|^{(m+2)/2}$, where s_0 is a positive constant, thus we have $s_0|y| \ge |Y|^{2/(m+2)}$. Denote

$$\begin{aligned} &W(z) = u + iv, \\ W_{\overline{z}} &= \frac{1}{2}[H_1(y)W_x + iH_2(y)W_y] = \frac{H_1(y)}{2}[W_x + iW_Y] \\ &= H_1(y)W_{x-iY} = H_1(y)W_{\overline{Z}}, \end{aligned} \tag{1.3}$$

where $dY = H(y)dy = H_1(y)dy/H_2(y)$, $H_2u_y = H_1u_Y$, then the system (1.1) can be written in the complex form

$$\begin{gathered} W_{\overline{z}} = H_1(y)W_{\overline{Z}} = A(z)W + B(z)\overline{W} \quad \text{in } D, \\ A = \frac{1}{4}[a + ic - ib + d], \quad B = \frac{1}{4}[a + ic + ib - d], \end{gathered} \tag{1.4}$$

in which D_Z is the image domain of D with respect to the mapping $Z = Z(z) = x + iY = x + iG(y)$ in D, and denoted by D again for simply, and $z = z(Z)$ is the inverse function of $Z = Z(z)$. For convenience we only discuss the complex equation (1.4) about the number Z replaced by z in Sections 1 and 2 later on.

Introduce the Riemann-Hilbert boundary conditions for the equation (1.4) as follows:

$$\begin{gathered} \operatorname{Re}[\overline{\lambda(z)}W(z)] = r(z) + f(z) = f_1(z), \quad z \in \Gamma, \\ \operatorname{Im}[\overline{\lambda(a_j)}W(a_j)] = b_j, \quad j = 1, \dots, 2K + 1, \ K \ge 0, \end{gathered} \tag{1.5}$$

where

$$f(z) = \begin{cases} 0, & K \ge 0, \\ g_0 + \operatorname{Re}\sum_{m=1}^{-K-1}(g_m^+ + ig_m^-)z^m, & K < 0, \end{cases}$$

in which $\lambda(z)$ $(\neq 0)$, $r(z) \in C_\alpha(L)$, $\alpha(0 < \alpha < 1)$ is a positive constant, g_0, $g_m^\pm$ $(m = 1, \dots, -K - 1, K < 0)$ are unknown real constants to be determined appropriately, $a_j(\in \Gamma = \{|z| = 1\}, j = 1, \dots, 2K + 1, K \ge 0)$ are distinct points, and $b_j(j = 1, \dots, 2K + 1)$ are all real constants, in which $K = \frac{1}{2\pi}\Delta_\Gamma \arg \lambda(z)$ is called the index of $\lambda(z)$ on Γ. The above Riemann-Hilbert boundary value problem is called Problem RH for equation (1.4). Under Condition C, the solution $W(z)$ of Problem RH in D can be found. From [8, (5.114) and (5.115), Chapter VI], we see that Problem RH of equation (1.4) possesses the important application to the shell and elasticity.

It is clear that the above solution $W(z)$ satisfies the following Riemann-Hilbert type boundary condition for the equation (1.4):

$$\operatorname{Im}[\overline{\lambda(z)}W(z)] = f_2(z) \quad \text{on } \Gamma, \tag{1.6}$$

and then the boundary condition of Riemann-Hilbert to Riemann-Hilbert type map can be written as follows

$$\begin{gathered}\overline{\lambda(z)}W(z) = f_1(z) + if_2(z) \quad \text{on } \Gamma, \text{ i.e.} \\ W(z) = h(z) = [f_1(z) + if_2(z)]/\overline{\lambda(z)} \quad \text{on } \Gamma,\end{gathered} \tag{1.7}$$

which will be called Problem RR for the complex equation (1.4) (or (1.1)), where $h(z) \in C_\alpha(\Gamma)$ is a complex function. Thus we can define the Riemann-Hilbert to Riemann-Hilbert type map $\Lambda : C_\alpha(\Gamma) \to C_\alpha(\Gamma)$, i.e. $f_1(z) \to f_2(z)$ by $\Lambda f_1 = f_2$

Our inverse problem is to determine the coefficient a, b, c, d of equation (1.1) (or $A(z), B(z)$ in (1.4)) from the map Λ. Obviously the function $f_1(z) + if_2(z)$ corresponds to the function $h(z)$ one by one. Denote by R_h the set of $\{h(z)\}$. It is clear that for any function $f_1(z)$ of the set $C_\alpha(\Gamma)$ in the Riemann-Hilbert boundary condition (1.5), there is a set $\{f_2(z)\}$ of the functions of Riemann-Hilbert type boundary condition (1.6), where $R_h = \{h(z)\}$ is corresponding to the complex equation (1.4). Inversely from the set $R_h = \{h(z)\}$, one complex equation in (1.4) can be determined, which will be verified later on.

In Section 3, we prove Theorems 3.1 and 3.2, which are important results in the present paper. In fact we first assume that the coefficients $A = B = 0$, $H = H(y)$ of the complex equation (1.4) in the ε-neighborhood $D_\varepsilon = D \cap \{|\operatorname{Im} z| < \varepsilon\}$ of $D \cap \{\operatorname{Im} z = 0\}$, note that the above coefficients $A(z)$, $B(z)$ weakly converge to $A(z), B(z)$ in D as $\varepsilon \to 0$, and on the basis of Theorem 3.1 below, we see the Hölder continuity of solution $W(Z)$ and $TW_{\overline{Z}} = T[AW + B\overline{W}]/H_1$ of the complex equation (1.4) with above coefficients and $TW_{\overline{Z}} = T[AW + B\overline{W}]/H_1$ (see [8, 11, 13], hence from $\{W(z)\}$ and $TW_{\overline{Z}}$, we can choose the subsequences, which uniformly converges the Hölder continuous functions in $\overline{D}$ respectively. From this, we can also obtain the corresponding Pompeiu and Plemelj-Sokhotzki formulas about $W(z)$ in $\overline{D}$.

2. Existence of solutions of the inverse problem for degenerate elliptic complex equations of first order

We introduce a singular integral operator

$$\tilde{T}f(z) = T\Big(\frac{f}{H_1}\Big) = -\frac{1}{\pi}\int\int_D \frac{f(\zeta)/H_1(y)}{\zeta - Z}d\sigma_\zeta,$$

where $|y|^\tau f(z) \in L_\infty(D)$ with $\tau = \max(1 - m_1/2, 0)$, m_1 is a positive constant, $H_1(y)$ is as stated in (1.1). Suppose that $f(z) = 0$ in $\mathbb{C}\backslash\overline{D}$. Then $|y|^\tau f(z) \in L_\infty(\mathbb{C})$, from Theorem 3.1 below, it follows $(\tilde{T}f)_{\bar{z}} = f(z)/H_1$ in $\mathbb{C}$. We consider the first-order complex equation with singular coefficients

$$\begin{gathered}H_1 W_{\overline{z}} - A(z)W - B(z)\overline{W} = 0, \quad \text{i.e.,} \\ H_1(y)[g(z)]_{\overline{z}} - A(z)g(z) - B(z)\overline{g(z)} = 0 \quad \text{in } \mathbb{C},\end{gathered} \tag{2.1}$$

where $G_1(y) = \int_0^y H_1(y)dy$, $g(z) = W(z)$. Applying the Pompeiu formula (see [8, Chapters I and III]), the corresponding integral equation of the complex equation (2.1) is as follows

$$g(z) - T[(Ag + B\overline{g})/H_1] = \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta \quad \text{in } D. \tag{2.2}$$

For simplicity we can consider only the integral equation

$$g(z) - T[(Ag + B\overline{g})/H_1] = 1$$

or i in D later on. On the basis of Theorem 3.1 below, we know that the integral in (2.2) is a completely continuous operator, hence by using the similar method as in [8, Sec. 5, Chapter III] and the proof of [15, Lemma 2.2], we can verify that the above integral equation has a unique solution.

We first prove the following lemma (see [7]).

Lemma 2.1. *The function $g(z) = h_j(z)$ $(h_j(z), j = 1,2)$ are a solutions of the integral equations*

$$\begin{gathered} g(z) - T(A/H_1)g - T(B/H_1)\overline{g} = \begin{cases} 1 \\ i \end{cases} \quad \text{in } \overline{D}, \\ g(z) = \begin{cases} h_1(z) \\ h_2(z) \end{cases} \quad \text{on } \Gamma, \end{gathered} \tag{2.3}$$

if and only if it is a solution of the integral equation

$$\begin{gathered} \frac{1}{2}g(z) + \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta = \begin{cases} 1, \\ i, \end{cases} \quad g(\zeta) = \begin{cases} h_1(\zeta), \\ h_2(\zeta), \end{cases} \quad \text{i.e.,} \\ \frac{h_1(z)}{2} + \frac{1}{2\pi i}\int_\Gamma \frac{h_1(\zeta)}{\zeta - z}d\zeta = 1, \quad \frac{h_2(z)}{2} + \frac{1}{2\pi i}\int_\Gamma \frac{h_2(\zeta)}{\zeta - z}d\zeta = i \quad \text{on } \Gamma \end{gathered} \tag{2.4}$$

respectively.

Proof. It is clear that we need to discuss only the case of h_1. If $g(z)$ is a solution of the first integral equation in (2.3), then $g_{\overline{z}} = Ag/H_1 + B\overline{g}/H_1$. On the basis of the Pompeiu formula

$$g(z) = \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta + T[g(\zeta)]_{\overline{\zeta}} = \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta + T[Ag/H_1 + B\overline{g}/H_1] \tag{2.5}$$

in D (see [8, Chapters I and III]), we have

$$g(z,k) - TAg/H_1 - TB\overline{g}/H_1 = 1 = \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta \quad \text{in } D, \tag{2.6}$$

where $g(\zeta) = h_1(\zeta)$ on Γ. Moreover by using the Plemelj-Sokhotzki formula for Cauchy type integral (see [4, 9])

$$1 = \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta + \frac{1}{2}g(z), \quad g(\zeta) = h_1(\zeta) \quad \text{on } Ga,$$

this is the first formula in (2.4).

Conversely if the first integral equation in (2.4) is true, then by the conditions in Section 1, there exists a solution of equation $g_{\overline{z}} = Ag/H_1 + B\overline{g}/H_1$ in $\overline{D}$ with the boundary values $g(\zeta) = h_1(\zeta)$ on Γ, thus we have (2.5), where the integral $\frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta$ in D is analytic, whose boundary value on Γ is

$$\lim_{z'(\in D)\to z(\in\Gamma)} \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z'}d\zeta = \frac{1}{2}g(z) + \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta = 1,$$

hence

$$\frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta = 1 \quad \text{in } D,$$

and the first formula in (2.3) is true. □

Theorem 2.2. *Under the above conditions, the functions $h_1(z)$, $h_2(z)$ as stated in Section* 1 *are the solutions of the system of integral equations*

$$\begin{gathered}\frac{h_1}{2}+Sh_1=1,\quad \frac{h_2}{2}+Sh_2=i,\\ Sh_1=\frac{1}{2\pi i}\int_\Gamma\frac{h_1(\zeta)}{\zeta-t}d\zeta,\quad Sh_2=\frac{1}{2\pi i}\int_\Gamma\frac{h_2(\zeta)}{\zeta-t}d\zeta.\end{gathered}\tag{2.7}$$

Proof. From the theory of integral equations (see [4, 6, 15]), we can derive the solutions h_1 and h_2 of (2.7). In fact, on the basis of Lemma 2.1, we can find the solutions of the integral equations

$$\begin{gathered}W_1(z)=1+T[(AW_1+B\overline{W}_1)/H_1]\quad\text{in } D,\\ W_2(z)=i+T[(AW_2+B\overline{W_2})/H_1]\quad\text{in } D.\end{gathered}$$

By using the Pompeiu formula, the above equations can be rewritten as

$$\begin{gathered}W_1(z)=\frac{1}{2\pi i}\int_L\frac{W_1(t)}{t-z}dt-\frac{1}{\pi}\int\int_D\frac{AW_1(\zeta)+B\overline{W_1(\zeta)}}{(\zeta-z)H_1}d\sigma_\zeta\quad\text{in } D,\\ W_2(z)=\frac{1}{2\pi i}\int_L\frac{W_2(t)}{t-z}dt-\frac{1}{\pi}\int\int_D\frac{AW_2(\zeta)+B\overline{W_2(\zeta)}}{(\zeta-z)H_1}d\sigma_\zeta\quad\text{in } D\end{gathered}$$

and $W_1(z)=h_1(z)$ and $W_2(z)=h_2(z)$ on Γ. Because the functions $\frac{1}{2\pi i}\int_L\frac{h_j(\zeta)}{\zeta-z}dt$ $(j=1,2)$ are analytic in $D'=\mathbb{C}\backslash\overline{D}$ (see [6]), we can analytically extend $h_j(z)\,(j=1,2)$ to the domain D'; i.e., define

$$\begin{gathered}w_1(z)=1-\frac{1}{2\pi i}\int_\Gamma\frac{h_1(\zeta)}{\zeta-z}d\zeta\quad z\in\mathbb{C}\backslash\overline{D},\\ w_2(z)=i-\frac{1}{2\pi i}\int_\Gamma\frac{h_2(\zeta)}{\zeta-z}d\zeta,\quad z\in\mathbb{C}\backslash\overline{D},\end{gathered}\tag{2.8}$$

which are analytic in D' with the boundary values $h_1(z)$, $h_2(z)$ on Γ respectively. According to the Plemelj-Sokhotzki formula for Cauchy type integrals, we immediately obtain the formulas

$$\begin{gathered}h_1(t)=1-\lim_{z(\in D')\to t(\in\Gamma)}\frac{1}{2\pi i}\int_\Gamma\frac{h_1(\zeta)}{\zeta-z}d\zeta=1+\frac{1}{2}h_1(t)-Sh_1\quad z\in\mathbb{C}\backslash\overline{D},\\ h_2(t)=i-\lim_{z(\in D')\to t(\in\Gamma)}\frac{1}{2\pi i}\int_\Gamma\frac{h_2(\zeta)}{\zeta-z}d\zeta=i+\frac{1}{2}h_1(t)-Sh_2\quad z\in\mathbb{C}\backslash\overline{D}.\end{gathered}$$

This is just the formula (2.7) with $h_j(t)$, $j=1,2$. □

Theorem 2.3. *For the inverse problem of Problem RR for equation* (1.1) *with Condition C, we can reconstruct the coefficients $a(z),b(z),c(z)$ and $d(z)$.*

Proof. We shall find two solutions $\phi_1(z)=W_1(z)$ and $i\phi_2(z)=W_2(z)$ of complex equation

$$[\phi]_{\bar z}-A/H_1\phi-B\overline{\phi}/H_1=0\quad\text{in }\mathbb{C}\tag{2.9}$$

with the conditions $\phi_1(z)\to 1$ and $i\phi_2(z)\to i$ as $z\to\infty$. In fact the above solutions $F(z)=\phi_1(z)$, $G(z)=i\phi_2(z)$ are also the solutions of integral equations

$$\begin{gathered}F(z)-T[(AF+B\overline{F})/H_1\}=1\quad\text{in }\mathbb{C},\\ G(z)-T[(AG+B\overline{G})/H_1\}=i\quad\text{in }\mathbb{C}.\end{gathered}\tag{2.10}$$

As stated in Lemma 2.1 and Theorem 2.2, we can require that the above solutions satisfy the boundary conditions

$$F(z) = h_1(z),\ G(z) = h_2(z) \quad \text{on } \Gamma,$$

where $h_1(z),\ h_2(z) \in R_h$.

Noting that $F(z)$, $G(z)$ satisfy the complex equations

$$\begin{gathered} F_{\bar z} - \{(AF + B\overline{F})/H_1\} = 0 \quad \text{in } \mathbb{C}, \\ G_{\bar z} - \{(AG + B\overline{G})/H_1\} = 0 \quad \text{in } \mathbb{C}. \end{gathered} \tag{2.11}$$

Moreover, on the basis of Lemma 2.4 below, we have

$$\operatorname{Im}[F(z)\overline{G(z)}] = [F(z)\overline{G(z)} - \overline{F(z)}G(z)]/2i \neq 0 \quad \text{in } D. \tag{2.12}$$

Thus from (2.11), the coefficients A/H_1 and B/H_1 can be determined as follows

$$\begin{gathered} A/H_1 = \frac{F_{\bar z}\overline{G} - G_{\bar z}\overline{F}}{F\overline{G} - \overline{F}G}, \quad B/H_1 = -\frac{F_{\bar z}G - G_{\bar z}F}{F\overline{G} - \overline{F}G} \quad \text{in } D; \text{ i.e.,} \\ A = H_1\frac{F_{\bar z}\overline{G} - G_{\bar z}\overline{F}}{F\overline{G} - \overline{F}G}, \quad B = -H_1\frac{F_{\bar z}G - G_{\bar z}F}{F\overline{G} - \overline{F}G} \quad \text{in } D. \end{gathered}$$

From the above formulas, the coefficients $a(z)$ and $b(z)$ of the system (1.1) are obtained; i.e.,

$$a(z) + ic(z) = 2[A(z) + B(z)],\ d(z) - ib(z) = 2[A(z) - B(z)] \quad \text{in } D.$$

□

Lemma 2.4. *For the solution* $[F(z), G(z)]$ *of the system* (2.11)*, we can get the inequality* (2.12)*.*

Proof. Suppose that (2.12) is not true, then there exists a point $z_0 \in D$ such that $\operatorname{Im}[\overline{F(z_0)}G(z_0)] = 0$; i.e.,

$$\begin{vmatrix} \operatorname{Re} F(z_0) & \operatorname{Im} F(z_0) \\ \operatorname{Re} G(z_0) & \operatorname{Im} G(z_0) \end{vmatrix} = 0.$$

Thus we have two real constants c_1, c_2, which are not both equal to 0, such that $c_1F(z_0) + c_2G(z_0) = 0$. Next, we prove that the equality of $c_1F(z_0) + c_2G(z_0) = 0$ can not be true. If $W(z_0) = c_1F(z_0) + c_2G(z_0) = 0$, then $W(z) = \Phi(z)e^{\phi(z)} = (z - z_0)\Phi_0(z)e^{\phi(z)}$, where $\Phi(z), \Phi_0(z)$ are analytic functions in D, and

$$\begin{aligned} &(z - z_0)\Phi_0(z)e^{\phi(z)} + \frac{1}{\pi}\int\int_D \frac{(\zeta - z_0)\Phi_0(\zeta)e^{\phi(\zeta)}[A/H_1 + B\overline{W(\zeta)}/H_1W(\zeta)]}{\zeta - z}d\sigma_\zeta \\ &= c_1 + c_2i. \end{aligned}$$

Letting $z \to z_0$, we have

$$\frac{1}{\pi}\int\int_D \Phi_0(\zeta)e^{\phi(\zeta)}[A/H_1 + B\overline{W(\zeta)}/H_1W(\zeta)]d\sigma_\zeta = c_1 + c_2i,$$

and then

$$\begin{aligned} &c_1 + c_2i \\ &= (z - z_0)\Phi_0(z)e^{\phi(z)} \\ &\quad + \frac{1}{\pi}\int\int_D \frac{(\zeta - z + z - z_0)\Phi_0(\zeta)e^{\phi(\zeta)}[A/H_1 + B\overline{W(\zeta)}/H_1W(\zeta)]}{\zeta - z}d\sigma_\zeta \end{aligned}$$

$$
\begin{aligned}
&= (z-z_0)\{\Phi_0(z)e^{\phi(z)} + \frac{1}{\pi}\int\int_D \frac{\Phi_0(\zeta)e^{\phi(\zeta)}[A/H_1 + B\overline{W(\zeta)}/H_1W(\zeta)]}{\zeta - z}d\sigma_\zeta\} \\
&\quad + \frac{1}{\pi}\int\int_D \Phi_0(\zeta)e^{\phi(\zeta)}[A/H_1 + B\overline{W(\zeta)}/H_1W(\zeta)]d\sigma_\zeta.
\end{aligned}
$$

The above equality implies

$$
\Phi_0(z)e^{\phi(z)} + \frac{1}{\pi}\int\int_D \frac{\Phi_0(\zeta)e^{\phi(\zeta)}[A/H_1 + B\overline{W(\zeta)}/H_1W(\zeta)]}{\zeta - z}d\sigma_\zeta = 0 \quad \text{in } D,
$$

and the above homogeneous integral equation only have the trivial solution, namely $\Phi_0(z) = 0 \quad \text{in } D$, thus $W(z) = \Phi(z)e^{\phi(z)} = (z-z_0)\Phi_0(z)e^{\phi(z)} \equiv 0$ in D. This is impossible.

In addition, by using another way, we can prove that the equality $c_1F(z_0) + c_2G(z_0) = 0$ can not be true. According to the method in [8, Section 5, Chapter III], we know that the integral equations

$$
W(z) - T[AW/H_1 + B\overline{W}/H_1] = \begin{cases} c_1 + c_2 i & \text{in } \overline{D}, \\ c_1 + c_2 i & \text{in } \mathbb{C}, \end{cases}
$$

have the unique solutions $W(z) = c_1F(z) + c_2G(z)$ in $\overline{D}$ and $\mathbb{C}$ respectively, where $A, B \in L_p(\overline{D})$ and $A = B = 0$ in $\mathbb{C}\backslash\overline{D}$, this shows that the function $W(z)$ in $\overline{D}$ can be continuously extended in $\mathbb{C}$. Moreover according to the method in [8, 13], the solution $W(z)$ can be expressed as $W(z) = \Phi(z)e^{T[A/H_1 + B\overline{W}/H_1W]}$ in $\mathbb{C}$. Note that $T[A/H_1 + B\overline{W}/H_1W] \to 0$ as $z \to \infty$, and the entire function $\Phi(z)$ in $\mathbb{C}$ satisfies the condition $\Phi(z) \to c_1 + c_2 i$ as $z \to \infty$, hence $\Phi(z) = c_1 + c_2 i$ in $\mathbb{C}$, thus $W(z) = (c_1 + c_2 i)e^{T[A/H_1 + B\overline{W}/H_1W]}$ in $\overline{D}$ and $W(z_0) = c_1F(z_0) + c_2G(z_0) \neq 0$. This contradiction verifies that (2.12) is true.

For the above discussion, we see that four real coefficients $a(z), b(z), c(z), d(z)$ of system (1.1) or two complex coefficients $A(z)$, $B(z)$ of the complex equation (1.4) can be determined by two boundary functions $h_1(z), h_2(z)$ in the set R_h. □

3. Hölder continuity of a singular integral operator

It is clear that the complex equation

$$
W_{\overline{Z}} = 0 \quad \text{in } \overline{D_Z} \tag{3.1}
$$

is a special case of equation (1.4), where D_Z is a bounded simply connected domain with boundary $\partial D \in C^1_\mu\,(0 < \mu < 1)$. On the basis of [11, Theorem 1.3, Chapter I], we can find a unique solution of Problem RH for equation (3.1) in $\overline{D_Z}$.

Now we consider the function $g(Z) \in L_\infty(D_Z)$, and first extend the function $g(Z)$ to the exterior of $\overline{D_Z}$ in $\mathbb{C}$, i.e. set $g(Z) = 0$ in $\mathbb{C}\backslash\overline{D_Z}$, hence we can only discuss the domain $D_0 = \{|x| < R_0\} \cap \{\operatorname{Im} Y \neq 0\} \supset \overline{D_Z}$, here $Z = x + iY$ and R_0 is an appropriately large positive number. In the following we shall verify that the integral

$$
\begin{gathered}
\Psi(Z) = T\Big(\frac{g}{H_1}\Big) = -\frac{1}{\pi}\int\int_{D_0} \frac{g(t)/H_1(\operatorname{Im} t)}{t - Z}d\sigma_t \quad \text{in } D_0, \\
L_\infty[g(Z), D_0] \le k_3,
\end{gathered} \tag{3.2}
$$

satisfies the estimate (3.3) below, where $H_j(y) = y^{m_j/2}h_j(y)$ $(m_j > 0, j = 1, 2, m_2 < \min(1, m_1))$ are as stated in Section 1, and $H_1(y) = H_1[\operatorname{Im} z(Z)]$, $z(Z)$ is as stated in (1.3). It is clear that the function $g(Z)/H_1(y) = g(Z)/H_1[\operatorname{Im} z(Z)]$ belongs to the space $L_1(D_0)$ and in general is not belonging to the space $L_p(D_0)\,(p>2)$, and the integral $\Psi(Z_0)$ is definite when $\operatorname{Im} Z_0 \neq 0$. If $Z_0 \in D_0$ and $\operatorname{Im} Z_0 = 0$, we can define the integral $\Psi(Z_0)$ as the limit of the corresponding integral over $D_0 \cap \{|\operatorname{Re} t - \operatorname{Re} Z_0| \geq \varepsilon\} \cap \{|\operatorname{Im} t - \operatorname{Im} Z_0| \geq \varepsilon\}$ as $\varepsilon \to 0$, where ε is a sufficiently small positive number. The Hölder continuity of the singular integral will be proved by the following method.

Theorem 3.1. *If the function $g(Z)$ in D_Z satisfies the condition in* (3.2)*, and $H_1(y) = y^{m_1/2}h_1(y)$, where m_1 is a positive number, $h_1(y)$ is a continuous positive function, then the integral in* (3.2) *satisfies the estimate*

$$C_\beta[\Psi(Z), \overline{D_Z}] \leq M_1, \tag{3.3}$$

in which $\beta = (2-m_2)/(m+2) - \delta$, $m = m_1 - m_2$, δ is a sufficiently small positive constant, and $M_1 = M_1(\beta, k_3, H_1, D_Z)$ is a positive constant.

Proof. We first give the estimates for $\Psi(Z)$ of (3.2) in $D \cap \{\operatorname{Im} Y \geq 0\}$, and verify the boundedness of the function in (3.2). As stated Section 1, if $H_1(y) = y^{m_1/2}h_1(y)$, then $H_1(y) \geq sY^{m_1/(m+2)}$, where s is a positive constant. For any two points $Z_0 = x_0 \in \gamma = (-1,1)$ on x-axis and $Z_1 = x_1 + iY_1 (Y_1 > 0) \in D_0$ satisfying the condition $2\operatorname{Im} Z_1/\sqrt{3} \leq |Z_1 - Z_0| \leq 2\operatorname{Im} Z_1$, this means that the inner angle at Z_0 of the triangle $Z_0Z_1Z_2$ $(Z_2 = x_0 + iY_1 \in D_0)$ is not less than $\pi/6$ and not greater than $\pi/3$, choose a sufficiently large positive number q, from the Hölder inequality, we have $L_1[\Psi(Z), D_0] \leq L_q[g(Z), D_0]L_p[1/H_1(\operatorname{Im} t)(t-Z), D_0]$, where $p = q/(q-1)$ (>1) is close to 1. In fact we can derive it as follows

$$\begin{aligned} |\Psi(Z_0)| &\leq \Big|\frac{1}{\pi}\int\int_{D_0} \frac{g(t)/H_1(\operatorname{Im} t)}{t - Z_0} d\sigma_t\Big| \\ &\leq \frac{1}{s\pi} L_q[g(Z), D_0]\Big[\int\int_{D_0}\Big|\frac{1}{t^{m_1/(m+2)}(t-Z_0)}\Big|^p d\sigma_t\Big]^{1/p} \\ &= \frac{1}{s\pi} L_q[g(Z), D_0]\, J_1^{1/p}, \end{aligned} \tag{3.4}$$

in which

$$\begin{aligned} J_1 &= \int\int_{D_0}\Big|\frac{1}{t^{m_1/(m+2)}(t-Z_0)}\Big|^p d\sigma_t \\ &\leq \int\int_{D_0} \frac{1}{|t|^{pm_1/(m+2)}|\operatorname{Im}(t-Z_0)|^{p\beta_0}|\operatorname{Re}(t-Z_0)|^{p(1-\beta_0)}} d\sigma_t \\ &\leq \Big|\int_0^{d_0} \frac{1}{Y^{pm_1/(m+2)}|Y-Y_0|^{p\beta_0}} dY \int_{d_1}^{d_2} \frac{1}{|x-x_0|^{p(1-\beta_0)}} dx\Big| \leq k_4, \end{aligned}$$

where $d_0 = \max_{Z\in\overline{D_0}} \operatorname{Im} Z$, $d_1 = \min_{Z\in\overline{D_0}} \operatorname{Re} Z$, $d_2 = \max_{Z\in\overline{D_0}} \operatorname{Re} Z$, $\beta_0 = (2 - m_2)/(m+2) - \varepsilon$, $\varepsilon\,(< 1/p - m_1/(m+2))$ is a sufficiently small positive constant, we can choose $\varepsilon = 2(p-1)/p\,(\leq (2-m_2)/(m+2))$, such that $p(1-\beta_0) < 1$ and $p[m_1/(m+2) + \beta_0] < 1$, and $k_4 = k_4(\beta, k_3, H_1, D_0)$ is a non-negative constant.

Next we estimate the Hölder continuity of the integral $\Psi(Z)$ in $\overline{D_0}$; i.e.,

$$
\begin{aligned}
&|\Psi(Z_1)-\Psi(Z_0)|\\
&\le \frac{|Z_1-Z_0|}{\pi}\Big|\int\int_{D_0}\frac{g(t)/H_1(\operatorname{Im}t)}{(t-Z_0)(t-Z_1)}d\sigma_t\Big|\\
&\le \frac{|Z_1-Z_0|}{s\pi}L_q[g(Z),D_0]\Big[\int\int_{D_0}\Big|\frac{1}{t^{m_1/(m+2)}(t-Z_0)(t-Z_1)}\Big|^p d\sigma_t\Big]^{1/p},
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
J_2&=\int\int_{D_0}\Big|\frac{1}{t^{m_1/(m+2)}(t-Z_0)(t-Z_1)}\Big|^p d\sigma_t\\
&\le\int\int_{D_0}\frac{|\operatorname{Re}(t-Z_0)|^{p(\beta_0/2-1)}|\operatorname{Re}(t-Z_1)|^{p(\beta_0/2-1)}}{|t|^{pm_1/(m+2)}|\operatorname{Im}(t-Z_0)|^{p\beta_0/2}|\operatorname{Im}(t-Z_1)|^{p\beta_0/2}}d\sigma_t\\
&\le\int_0^{d_0}\frac{1}{Y^{pm_1/(m+2)}|\operatorname{Im}(Y-Z_0)|^{p\beta_0/2}|\operatorname{Im}(Y-Z_1)|^{p\beta_0/2}}dY\\
&\quad\times\int_{d_1}^{d_2}\frac{1}{|\operatorname{Re}(t-Z_0)|^{p(1-\beta_0/2)}|\operatorname{Re}(t-Z_1)|^{p(1-\beta_0/2)}}d\operatorname{Re}t\\
&\le k_5\int_{d_1}^{d_2}\frac{1}{|x-x_0)|^{p(1-\beta_0/2)}|x-x_1|^{p(1-\beta_0/2)}}dx,
\end{aligned}
$$

where $\beta_0=(2-m_2)/(m+2)-\varepsilon$ is chosen as before and

$$
k_5=\max_{Z_0,Z_1\in D_0}\int_0^{d_0}[Y^{pm_1/(m+2)}|\operatorname{Im}(Y-Z_0)|^{p\beta_0/2}|\operatorname{Im}(Y-Z_1)|^{p\beta_0/2}]^{-1}dY.
$$

Denote $\rho_0=|\operatorname{Re}(Z_1-Z_0)|=|x_1-x_0|$, $L_1=D_0\cap\{|x-x_0|\le 2\rho_0, Y=Y_0\}$ and $L_2=D_0\cap\{2\rho_0<|x-x_0|\le 2\rho_1<\infty, Y=Y_0\}\supset[d_1,d_2]\backslash L_1$, where ρ_1 is a sufficiently large positive number, we can derive

$$
\begin{aligned}
J_2&\le k_5\Big[\int_{L_1}\frac{1}{|x-x_0|^{p(1-\beta_0/2)}|x-x_1|^{p(1-\beta_0/2)}}dx\\
&\quad+\int_{L_2}\frac{1}{|x-x_0|^{p(1-\beta_0/2)}|x-x_1|^{p(1-\beta_0/2)}}dx\Big]\\
&\le k_5\Big[|x_1-x_0|^{1-2p+p\beta_0}\int_{|\xi|\le 2}\frac{1}{|\xi|^{p(1-\beta_0/2)}|\xi\pm 1|^{p(1-\beta_0/2)}}d\xi\\
&\quad+k_6\Big|\int_{2\rho_0}^{2\rho_1}\rho^{p\beta_0-2p}d\rho\Big|\Big]\\
&\le k_7|x_1-x_0|^{1-p(2-\beta_0)}\\
&=k_7|x_1-x_0|^{p((2-m_2)/(m+2)-\varepsilon+1/p-2)},
\end{aligned}
$$

in which we use $|x-x_0|=\xi|x_1-x_0|$, $|x-x_1|=|x-x_0-(x_1-x_0)|=|\xi\pm1||x_1-x_0|$ if $x\in L_1$, $|x-x_0|=\rho\le 2|x-x_1|$ if $x\in L_2$, choose that p (>1) is close to 1 such that $1-p(2-\beta_0)<0$, and $k_j=k_j(\beta,k_3,H,D_0)$ $(j=6,7)$ are non-negative constants. Thus we obtain

$$
|\Psi(Z_1)-\Psi(Z_0)|\le k_7|Z_1-Z_0||x_1-x_0|^{(2-m_2)/(m+2)-\varepsilon+1/p-2}\le k_8|Z_1-Z_0|^\beta,\tag{3.6}
$$

in which we use that the inner angle at Z_0 of the triangle $Z_0Z_1Z_2$ ($Z_2=x_0+iY_1\in D_0$) is not less than $\pi/6$ and not greater than $\pi/3$, and choose $\varepsilon=2(p-1)/p$,

$\beta = (2-m_2)/(m+2)-\delta$, $\delta = 3(p-1)/p$, $k_8 = k_8(\beta, k_3, H_1, D_0)$ is a non-negative constant. The above points $Z_0 = x_0$, $Z_1 = x_1+iY_1$ can be replaced by $Z_0 = x_0+iY_0$, $Z_1 = x_1+iY_1 \in \overline{D_0}$, $0 < Y_0 < Y_1$ and $2(Y_1-Y_0)/\sqrt{3} \le |Z_1-Z_0| \le 2(Y_1-Y_0)$.

Finally we consider any two points $Z_1 = x_1+iY_1$, $Z_2 = x_2+iY_1$ and $x_1 < x_2$, from the above estimates, the following estimate can be derived

$$\begin{aligned} &|\Psi(Z_1)-\Psi(Z_2)| \le |\Psi(Z_1)-\Psi(Z_3)|+|\Psi(Z_3)-\Psi(Z_2)| \\ &\le k_8|Z_1-Z_3|^\beta + k_8|Z_3-Z_2|^\beta \le k_9|Z_1-Z_2|^\beta, \end{aligned} \tag{3.7}$$

where $Z_3 = (x_1+x_2)/2+i[Y_1+(x_2-x_1)/(2\sqrt{3})]$. If $Z_1 = x_1+iY_1$, $Z_2 = x_1+iY_2$, $Y_1 < Y_2$, and we choose $Z_3 = x_1+(Y_2-Y_1)/2\sqrt{3}+i(Y_2+Y_1)/2$, and can also get (3.7). If $Z_1 = x_1+iY_1$, $Z_2 = x_2+iY_2$, $x_1 < x_2, Y_1 < Y_2$, and we choose $Z_3 = x_2+iY_1$, obviously

$$|\Psi(Z_1)-\Psi(Z_2)| \le |\Psi(Z_1)-\Psi(Z_3)|+|\Psi(Z_3)-\Psi(Z_2)|,$$

and $|\Psi(Z_1)-\Psi(Z_3)|$, $|\Psi(Z_3)-\Psi(Z_2)|$ can be estimated by the above way, hence we can obtain the estimate of $|\Psi(Z_1)-\Psi(Z_2)|$. For the function $\Psi(Z)$ of (3.2) in $D\cap\{\operatorname{Im} Y \le 0\}$, the similar estimates can be also derived. Hence we have the estimate (3.3). □

Theorem 3.2. *If the condition $H_1(y) = y^{m_1/2}h_1(y)$ in Theorem 3.1 is replaced by $H_1(y) = y^\eta h_1(y)$, herein η is a positive constant satisfying the inequality $\eta < (m+2)/(2-m_2)$, then by the same method we can prove that the integral $\Psi(Z) = T(g/H_1)$ satisfies the estimate*

$$C_\beta[\Psi(Z), D_Z] \le M_1, \tag{3.8}$$

in which $\beta = 1-\eta(2-m_2)/(m+2)-\delta$, δ is a sufficiently small positive constant, and $M_1 = M_1(\beta, k_3, H_1, D_Z)$ is a positive constant. In particular if $H_1(y) = y$; i.e., $\eta = 1$, then we can choose $\beta = m_1/(m+2)-\delta$, δ is a sufficiently small positive constant.

References

[1] R.-M. Brown, G. Uhlmann; *Uniqueness in the inverse conductivity problem for smooth conductivities in two dimensions,* Comm in Partial Differential Equations, **22** (1997): 1009-1027.

[2] V. Isakov; *Inverse problems for partial differential equations,* Springer-Verlag, New York, (2004).

[3] A. Kirsch; *An introduction to the mathematical theory of inverse problems,* Springer-Verlag, New York, (1996).

[4] N.-I. Mushelishvili; *Singular integral equations,* Noordhoff, Groningen, (1953).

[5] L.-Y. Sung; *An inverse scattering problem for the Davey-Stewartson II equations, I, II, III,* J Math Anal Appl, **183** (1994): 121-154, 389-325, 477-494.

[6] A. Tamasan; *On the scattering method for the $\overline{\partial}$-equation and reconstruction of convection coefficients,* Inverse Problem, **20** (2004): 1807-1817.

[7] Z.-L. Tong, J. Cheng, M. Yamamoto; *A method for constructing the convection coefficients of elliptic equations from Dirichlet to Neumann map,* (in Chinese) Science in China, Ser A, **34** (2004): 752-766.

[8] I.-N. Vekua; *Generalized analytic functions,* Pergamon, Oxford, (1962).

[9] G.-C, Wen; *Conformal mappings and boundary value problems,* Translations of Mathematics Monographs 106, Amer. Math. Soc., Providence, RI, (1992).

[10] G.-C, Wen; *Linear and quasilinear complex equations of hyperbolic and mixed Type,* Taylor & Francis, London, (2002).

[11] G.-C, Wen; *Elliptic, hyperbolic and mixed complex equations with parabolic degeneracy,* World Scientific, Singapore,(2008).

[12] G.-C, Wen; *Recent progress in theory and applications of modern complex analysis,* Science Press, Beijing, (2010).

[13] G.-C, Wen, H. Begehr; *Boundary value problems for elliptic equations and systems,* Longman Scientific and Technical Company, Harlow, (1990).

[14] G.-C, Wen, D.-C, Chen, Z.-L. Xu; *Nonlinear complex analysis and its applications,* Mathematics Monograph Series 12, Science Press, Beijing, (2008).

[15] G.-C. Wen, Z,-L. Xu; *The inverse problem for linear elliptic systems of first order with Riemann-Hilbert type map in multiply connected domains,* Complex Variables and Elliptic Equations, **54** (2009): 811-823.

[16] G.-C. Wen, Z,-L. Xu, F.-M. Yang; *The inverse problem for elliptic equations from Dirichlet to Neumann map in multiply connected domains,* Boundary Value Problems, **305291** (2009): 1-16.

Guo Chun Wen

LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China

E-mail address: Wengc@math.pku.edu.cn