Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 70, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

POINT RUPTURE SOLUTIONS OF A SINGULAR ELLIPTIC EQUATION

HUIQIANG JIANG, ATTOU MILOUA

Abstract. We consider the elliptic equation

$$\Delta u = f(u)$$

in a region $\Omega \subset \mathbb{R}^2$, where f is a positive continuous function satisfying

$$\lim_{u\to 0^+} f(u) = \infty.$$

Motivated by the thin film equations, a solution u is said to be a point rupture solution if for some $p \in \Omega$, $u(p) = 0$ and $u(p) > 0$ in $\Omega\backslash\{p\}$. Our main result is a sufficient condition on f for the existence of radial point rupture solutions.

1. Introduction

Let Ω be a region in $\mathbb{R}^2$ and f be a continuous function defined on $(0, \infty)$ satisfying

$$\lim_{v\to 0^+} f(v) = \infty.$$

We are interested in the elliptic equation

$$\Delta u = f(u) \quad \text{in } \Omega \tag{1.1}$$

with Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. A solution to (1.1) is said to be a point rupture solution if for some $p \in \Omega$, $u(p) = 0$ and $u(x) > 0$ for any $x \in \Omega\backslash\{p\}$.

In the lubrication model of thin films, u will be the thickness of the thin film over a planar region Ω and the dynamic of the thin film can be modeled by the fourth order partial differential equation

$$u_t = -\nabla\cdot(u^m\nabla u) - \nabla\cdot(u^n\nabla\Delta u). \tag{1.2}$$

Here the fourth-order term in the equation reflects surface tension effects, and the second-order term can reflect gravity, van der Waals interactions, thermocapillary effects or the geometry of the solid substrate. This class of model equation occurs in connection with many physical systems involving fluid interfaces. When $n = 1$ and $m = 1$, it describes a thin jet in a Hele-Shaw cell [1, 5, 7, 8, 15]; when $n = m = 3$ it describes fluid droplets hanging from a ceiling [9]; when $n = 0$ and $m = 1$, it is a

2000 *Mathematics Subject Classification.* 49Q20, 35J60, 35Q35.
Key words and phrases. Thin film; point rupture; radial solution; singular equation.

Submitted Submitted May 2, 2012. Published March 13, 2013.
Supported by grant DMS-1200599 from National Science Foundation.

modified Kuramoto-Sivashinsky equation which describes solidification of a hyper-cooled melt[3, 4], and when $n = 3$, $m = -1$, it models van der Waals force driven thin film [6, 11, 16, 17, 18].

Many mathematically rigorous works have been done when the space dimension is one. Laugesen and Pugh [14] considered positive periodic steady states and touchdown steady states in a more general setting. The dynamics of a special type of thin film equation has been investigated by Bernis and Friedman [2]. They established the existence of weak solutions and showed that the support of the thin film will expand with time. On the other hand, when the space dimension is two, the physically realistic dimension, the dynamics of (1.2) is not well understood. Naturally, we start with its steady state. When $n - m \neq 1$, let

$$p = -\frac{1}{m-n+1} u^{m-n+1} - \Delta u,$$

which can be viewed as the pressure of the fluid. We can rewrite (1.2) as

$$u_t = \nabla(u^n \nabla p).$$

Now let $\Omega \subset R^2$ be the bottom of a cylindrical container occupied by the thin film fluid, we assume that there is no flux across the boundary, which yields the boundary condition

$$\frac{\partial p}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

We also ignore the wetting or non-wetting effect, and assume that the fluid surface is perpendicular to the boundary of the container, i.e.,

$$\frac{\partial h}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

Whenever $m - n \neq -1$ or -2, we can associate (1.2)) with the energy

$$E(u) = \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 - \frac{1}{(m-n+1)(m-n+2)} u^{m-n+2}\Big),$$

and formally, using (1.3) , (1.4)), we have

$$\frac{d}{dt} E(u) = -\int_\Omega u^n |\nabla p|^2.$$

Hence, for a thin film fluid at rest, p has to be a constant, and u satisfies

$$-\Delta u - \frac{1}{m-n+1} u^{m-n+1} = p \quad \text{in } \Omega,$$

which is an elliptic equation.

If we further assume $m - n + 1 < -1$, which includes the van der Waals force case. We can write the equation as

$$\Delta u = \frac{1}{\alpha} u^{-\alpha} - p \quad \text{in } \Omega, \tag{1.5}$$

where p is an unknown constant and

$$\alpha = -(m - n + 1) > 1.$$

For van der Waals force driven thin film, $\alpha = 3$.

Hence, (1.1) can be viewed as a generalization of the stationary thin film equation with van der Waals force.

The rupture set $\Sigma = \{x \in \Omega : u(x) = 0\}$ corresponds to "dry spots" in the thin films, which is of great significance in the coatings industry where nonuniformities are very undesirable. In a joint work of Lin and the first author, an estimate on the Hausdorff dimension of the rupture set to (1.5) was obtained using geometric measure theory, under the assumption that the total energy is finite [12] and such estimate seems the first estimate for such problems.

We conjecture that the ruptures are discrete for finite energy solutions, and we expect that the radial point rupture solutions will serve as the blow up profile of the solution near any point rupture. The main purpose of this paper is on the existence of radial point rupture solution. And we are only interested in the local solutions in a neighborhood of the point rupture. Since the equation has no singularity away from the rupture, the possible extension of point rupture solution to a global solution could be carried out using similar arguments in [13] where the case $f(u) = u^{-\alpha} - 1$, $\alpha > 1$, is completely studied.

Now we state our main result.

Theorem 1.1. *Let $\sigma^* > 0$ and f be a continuous, monotone decreasing positive function on $(0, \sigma^*]$ such that*

$$\lim_{v\to 0^+} f(v) = \infty.$$

Let

$$G(v) = \int_0^v \frac{1}{f(s)} ds. \tag{1.6}$$

Assume in addition that

$$\frac{v}{f(v)G(v)} \in L^1[0, \sigma^*]. \tag{1.7}$$

Then there exists $r^* > 0$ and a radial point rupture solution u_0 to (1.1) in $B_{r^*}(0)$ such that $u_0 = u_0(r)$ is continuous on $[0, r^*]$,

$$u_0(0) = 0, \quad u_0(r) > 0 \text{ for any } r \in (0, r^*],$$

and u is a weak solution to (1.1) in $B_{r^*}(0)$. Moreover, u_0 is monotone increasing and

$$G^{-1}\big(\frac{1}{4}r^2\big) \le u_0(r) \le \int_0^{G^{-1}(r^2/4)} \frac{v}{f(v)G(v)} dv \quad \text{for any } r \in [0, r^*].$$

Remark 1.2. Here the technical assumption (1.7) is not very strong, for example, if $f(v) = v^{-\alpha}$, for some $\alpha > 0$, we would have

$$\frac{v}{f(v)G(v)} = \frac{v}{v^{-\alpha}(\frac{1}{1+\alpha}v^{1+\alpha})} = 1 + \alpha \in L^1[0, \sigma^*].$$

Such assumption also holds for some singularity of exponential growth, for example, if

$$f(v) = v^{p+1}e^{1/v^p}, \quad 0 < p < 1,$$

we have

$$\frac{v}{f(v)G(v)} = \frac{p}{v^p} \in L^1[0, \sigma^*].$$

Remark 1.3. The assumption that f is monotone decreasing can be replaced by the assumption that f is a product of a monotone deceasing function and a bounded positive function.

Such result is a generalization of the existence result obtained by the first author and Ni in [13] for $f(v) = \frac{1}{\alpha}v^{-\alpha} - 1$ with $\alpha > 1$ where uniqueness of the radial rupture solution is also established. In space dimension $N \ge 3$, the existence result has also been obtained by Guo, Ye and Zhou[10], where the technical assumption (1.7)) is not needed.

2. Proof of main results

For any $\sigma \in (0, \sigma^*)$, we use u_σ to denote the unique solution to the initial value problem

$$\begin{gathered} u_{rr} + \frac{1}{r} u_r = f(u), \\ u(0) = \sigma, \quad u'(0) = 0. \end{gathered} \tag{2.1}$$

Lemma 2.1. *There exists $r_\sigma > 0$ such that u_σ is defined on $[0, r_\sigma]$ with $u_\sigma(r_\sigma) = \sigma^*$. Moreover, $u'_\sigma(r) > 0$ on $(0, r_\sigma]$ and*

$$G^{-1}\big(\frac{1}{4} r^2\big) \le u_\sigma(r) \le \sigma + \int_0^{G^{-1}(r^2/4)} \frac{v}{f(v) G(v)} dv \quad \text{on } [0, r_\sigma]. \tag{2.2}$$

Proof. For simplicity, we suppress the σ subscript in this proof. We write

$$u_{rr} + \frac{1}{r} u_r = f(u)$$

in the form of $(r u_r)_r = r f(u) \ge 0$, so we have

$$r u_r = \int_0^r s f(u(s)) ds \ge 0.$$

In particular, u is monotone increasing and u can be extended whenever $f(u)$ is defined and bounded. Hence, there exists $r_\sigma > 0$ such that u_σ is defined on $[0, r_\sigma]$ with $u_\sigma(r_\sigma) = \sigma^*$. Since u is monotone increasing and f is monotone decreasing, we have

$$r u_r = \int_0^r s f(u(s)) ds \ge f(u(r)) \int_0^r s ds = \frac{1}{2} r^2 f(u(r)),$$

hence,

$$\frac{u_r}{f(u)} \ge \frac{1}{2} r.$$

Integrating again, we have

$$G(u(r)) \ge G(\sigma) + \frac{1}{4} r^2 \ge \frac{1}{4} r^2.$$

Since G is strictly monotone increasing, we have

$$u(r) \ge G^{-1}\big(\frac{1}{4} r^2\big).$$

On the other hand,

$$r u_r = \int_0^r s f(u(s)) ds \le \int_0^r f(G^{-1}(\frac{1}{4} s^2)) s ds.$$

Let $v = G^{-1}(\frac{1}{4} s^2)$, we have $G(v) = \frac{1}{4} s^2$, and

$$\frac{1}{f(v)} dv = \frac{1}{2} s ds.$$

Hence,

$$\int_0^r f(G^{-1}(\frac{1}{4} s^2)) s \, ds = \int_0^{G^{-1}(r^2/4)} 2 dv = 2 G^{-1}(\frac{1}{4} r^2).$$

Hence,

$$u_r \le \frac{2}{r} G^{-1}\big(\frac{1}{4} r^2\big)$$

which yields

$$\begin{aligned} u(r) &\le \sigma + \int_0^r \frac{2}{s} G^{-1}(\frac{1}{4} s^2) ds \\ &= \sigma + \int_0^{G^{-1}\left(\frac{1}{4}r^2\right)} \frac{v}{f(v)G(v)} dv. \end{aligned}$$

∎

The bounds on u_σ imply the following result.

Corollary 2.2. *There exists $r^* > 0$ such that for any $\sigma \in (0, \frac{\sigma^*}{2}]$,*

$$r_\sigma \ge r^*.$$

We can take

$$r^* = 2\sqrt{G(H^{-1}(\frac{\sigma^*}{2}))},$$

where

$$H(u) = \int_0^u \frac{v}{f(v)G(v)} dv.$$

Proof. For any $\sigma \in (0, \sigma^*/2]$,

$$\begin{aligned} \sigma^* = u_\sigma(r_\sigma) &\le \sigma + \int_0^{G^{-1}(\frac{1}{4}r_\sigma^2)} \frac{v}{f(v)G(v)} dv \\ &\le \frac{\sigma^*}{2} + \int_0^{G^{-1}(\frac{1}{4}r_\sigma^2)} \frac{v}{f(v)G(v)} dv. \end{aligned}$$

Hence,

$$\int_0^{G^{-1}(\frac{1}{4}r_\sigma^2)} \frac{v}{f(v)G(v)} dv \ge \frac{\sigma^*}{2}.$$

Since $\frac{v}{f(v)G(v)}$ is integrable, the function

$$H(u) = \int_0^u \frac{v}{f(v)G(v)} dv$$

is strictly monotone increasing, so

$$H\Big(G^{-1}\big(\frac{1}{4} r_\sigma^2\big)\Big) \ge \frac{\sigma^*}{2}$$

implies

$$r_\sigma \ge 2\sqrt{G\Big(H^{-1}\big(\frac{\sigma^*}{2}\big)\Big)}.$$

∎

The point rupture solution can be constructed as the limit of u_σ as $\sigma \to 0$.

Proposition 2.3. *There exists a sequence* $\{\sigma_k\}_{k=1}^{\infty} \subset (0, \frac{\sigma^*}{2}]$ *satisfying* $\lim_{k\to\infty} \sigma_k = 0$*, such that* $u_{\sigma_k} \to u_0$ *uniformly in* $\overline{B_{r^*}(0)}$ *as* $k \to \infty$*, for some function* $u_0 \in C^0(\overline{B_{r^*}(0)}) \cap C^2(\overline{B_{r^*}(0)}\backslash\{0\})$*. Moreover,* u_0 *is a classical solution to* (1.1) *in* $B_{r^*}(0)\backslash\{0\}$ *and*

$$G^{-1}\big(\frac{1}{4}r^2\big) \le u_0(r) \le \int_0^{G^{-1}\left(\frac{1}{4}r^2\right)} \frac{v}{f(v)G(v)}dv \ \text{on } [0, r^*].$$

Proof. For any $\varepsilon > 0$, u_σ, $\sigma \in (0, \sigma^*/2]$ is a family of uniformly bounded classical solutions to

$$\Delta u = f(u) \text{ in } \overline{B_{r^*}(0)}\backslash B_\varepsilon(0),$$

hence by a diagonal argument, there exists a sequence $\{\sigma_k\}_{k=1}^{\infty} \subset (0, \frac{\sigma^*}{2}]$ satisfying $\lim_{k\to\infty} \sigma_k = 0$, such that $u_{\sigma_k} \to u_0$ locally uniformly in $\overline{B_{r^*}(0)}\backslash\{0\}$ as $k \to \infty$. Now (2.2) implies

$$G^{-1}\big(\frac{1}{4}r^2\big) \le u_0(r) \le \int_0^{G^{-1}\left(\frac{1}{4}r^2\right)} \frac{v}{f(v)G(v)}dv \text{ on } [0, r^*].$$

Since

$$\lim_{r\to 0} \int_0^{G^{-1}(r^2/4)} \frac{v}{f(v)G(v)}dv = 0,$$

it is not difficulty to see, from the bounds of u_σ and u_0, that $u_{\sigma_k} \to u_0$ uniformly in $\overline{B_{r^*}(0)}$ as $k \to \infty$. $\square$

Remark 2.4. The above limit should be independent of the choice of $\{\sigma_k\}_{k=1}^{\infty}$. Actually, we expect that $u_\sigma \to u_0$ uniformly on $[0, r^*]$ as $\sigma \to 0$. Unfortunately, we are unable to provide a proof here.

To show that u_0 is a weak solution. We need the following lemma.

Lemma 2.5.

$$\lim_{r\to 0^+} r u_0'(r) = 0. \tag{2.3}$$

Proof. For any $r \in (0, r^*)$, we have

$$(r u_0'(r))' = r f(u_0) > 0.$$

Hence, $r u_0'(r)$ is monotone increasing in $(0, r^*)$. Since $r u_0'(r) \ge 0$ in $(0, r^*)$,

$$\beta = \lim_{r\to 0^+} r u_0'(r) \ge 0$$

is well defined. If $\beta > 0$, we have for r sufficiently small, say $r \in (0, \tilde{r}]$,

$$r u_0'(r) \ge \frac{\beta}{2}$$

hence, for any $r \in (0, \tilde{r}]$,

$$u_0(r) = u_0(\tilde{r}) - \int_r^{\tilde{r}} u_0'(r)dr \le u_0(\tilde{r}) - \int_r^{\tilde{r}} \frac{\beta}{2r}dr.$$

which contradicts to the fact that u_0 is continuous at 0 if we let $r \to 0^+$. Hence $\beta = 0$ and (2.3) holds. $\square$

Proposition 2.6. $f(u_0) \in L^1(B_{r^*}(0))$ *and* u_0 *is a weak solution to* (1.1)*) in* $B_{r^*}(0)$*.*

Proof. For any test function $\varphi \in C_c^\infty(B_{r^*}(0))$, we have

$$\begin{aligned}
&\int_{B_{r^*}(0)} u_0 \Delta\varphi dx \\
&= \lim_{\varepsilon\to 0^+} \int_{B_{r^*}(0)\setminus\overline{B_\varepsilon(0)}} u_0 \Delta\varphi dx \\
&= \lim_{\varepsilon\to 0^+} \Big(\int_{B_{r^*}(0)\setminus\overline{B_\varepsilon(0)}} \Delta u_0 \varphi dx - \int_{\partial B_\varepsilon(0)} \Big(u_0 \frac{\partial\varphi}{\partial n} - \varphi \frac{\partial u_0}{\partial n} \Big) ds_x \Big) \\
&= \lim_{\varepsilon\to 0^+} \Big(\int_{B_{r^*}(0)\setminus\overline{B_\varepsilon(0)}} f(u_0)\varphi dx - \int_{\partial B_\varepsilon(0)} u_0 \frac{\partial\varphi}{\partial n} ds_x + \int_{\partial B_\varepsilon(0)} \varphi \frac{\partial u_0}{\partial n} ds_x \Big).
\end{aligned}$$

Now for any $\varepsilon \in (0, r^*)$, since $u_0(\varepsilon) \le u_0(r^*) \le \delta^*$, we have

$$\begin{aligned}
\Big| \int_{\partial B_\varepsilon(0)} u_0 \frac{\partial\varphi}{\partial n} ds_x \Big| &\le u_0(\varepsilon) \|\nabla\varphi\|_{L^\infty(B_{r^*}(0))} |\partial B_\varepsilon(0)| \\
&\le 2\pi\varepsilon u_0(\varepsilon) \|\nabla\varphi\|_{L^\infty(B_{r^*}(0))} \to 0
\end{aligned}$$

as $\varepsilon \to 0^+$. On the other hand, (2.3)) implies that

$$\Big| \int_{\partial B_\varepsilon(0)} \varphi \frac{\partial u_0}{\partial n} ds_x \Big| \le 2\pi\varepsilon u_0'(\varepsilon) \|\varphi\|_{L^\infty(B_{r^*}(0))} \to 0$$

as $\varepsilon \to 0^+$. Hence, we have for any $\varphi \in C_c^\infty(B_{r^*}(0))$,

$$\int_{B_{r^*}(0)} u_0 \Delta\varphi dx = \lim_{\varepsilon\to 0^+} \int_{B_{r^*}(0)\setminus\overline{B_\varepsilon(0)}} f(u_0)\varphi dx.$$

Choosing φ such that $\varphi \equiv 1$ near the origin, the above limit implies that $f(u_0)$ is integrable near the origin. Since $f(u_0)$ is a positive continuous function in $B_{r^*}(0)\setminus\{0\}$, we conclude $f(u_0) \in L^1(B_{r^*}(0))$. So we have for any test function $\varphi \in C_c^\infty(B_{r^*}(0))$,

$$\int_{B_{r^*}(0)} u_0 \Delta\varphi dx = \lim_{\varepsilon\to 0^+} \int_{B_{r^*}(0)\setminus\overline{B_\varepsilon(0)}} f(u_0)\varphi dx = \int_{B_{r^*}(0)} f(u_0)\varphi dx;$$

i.e., u_0 is a weak solution to (1.1) in $B_{r^*}(0)$. $\square$

The main theorem is a combination of Proposition 2.3 and Proposition 2.6.

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Huiqiang Jiang
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA
E-mail address: `hqjiang@pitt.edu`

Attou Miloua
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA
E-mail address: `atm33@pitt.edu`