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GLOBAL BRANCHING FOR DISCONTINUOUS PROBLEMS INVOLVING THE p-LAPLACIAN

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Abstract. In this article, we study elliptic problems with discontinuous nonlinearities involving the p-Laplacian both in bounded and unbounded domains. We prove that there exists a global branch of positive solutions under some suitable assumptions of the nonlinearities. Our results extend the corresponding ones of the Laplacian due to Ambrosetti, et al.

1. Introduction

Imaging the loss of transparency and vegetation in some shallow lake subject to human-induced eutrophication [17, 18]. Denote by u the nutrient loading level and $f(u)$ the turbidity. There is a critical thresholds $u = a$ at which the water shifts abruptly from clear to turbid. That is to say f has a jumping at $u = a$. Beside the above, there are several problems in Plasma Physics give rise to equations with discontinuous nonlinearities, see for example [5]. These discontinuous examples in the real world inspires us to study the problems with discontinuous nonlinearities.

The main purpose of this article is to establish the existence of global branch of positive solutions for some elliptic problems with discontinuous nonlinearities involving the p-Laplacian. We refer to [8] and the references therein for the existence of solutions in the case of problems involving the p-Laplacian. Many nonlinear problems in physics and mechanics are formulated in equations that contain the p-Laplacian, we refer to [7] and the references therein for the specific setting about the p-laplacian.

In Section 2 we shall consider the problem

$$\begin{gathered} -\Delta_p u = f(u-a) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $a \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain, $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian $(1 < p < N)$ and f satisfies the following conditions:

(F1) $f(s) = 0$ for all $s \leq 0$;
(F2) $f \in C^{0,\alpha}(\mathbb{R}^+, \mathbb{R}^+)$ and is nondecreasing;

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(F3) there exists $0 < c_1 \leq \lambda_1(\Omega)$, $c_2 > 0$ and $0 < d \leq c_2$, such that $d < f(s) < c_1 s^{p-1} + c_2$ for all $s > 0$, where $\lambda_1(\Omega)$ denote the first eigenvalue of $-\Delta_p$ on Ω with zero Dirichlet boundary condition.

Taking a as the bifurcation parameter, we shall prove that there is a global branch S of pairs (a, u) with $u > 0$ solution of (1.1), bifurcating from (0,0) and having a turning point at some value $a_* > 0$, that is to say (1.1) has at least two positive solutions for all $a \in (0, a_*)$.

In Section 3, we study the problem

$$\begin{gathered} -\Delta_p u = f(u-a) \quad \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty \end{gathered} \tag{1.2}$$

with $N \geq 3$.

Problem (1.2) is approximated by problem (1.1) with $\Omega = B(R) = \{x \in \mathbb{R}^N : |x| < R\}$. By a limiting procedure based upon an a priori estimate of the free boundary $\{u = a\}$, we show that (1.2) possesses an unbounded branch S in $\mathbb{R}^+ \times D^{1,p}(\mathbb{R}^N)$ of positive solution pairs bifurcating from $(0, 0)$.

In recent years, many authors have studied the existence of solutions for problem (1.1) from several points of view and with different approaches; see for example [1, 2, 4].

Our work is motivated by the articles [1, 2, 4, 5, 6]. In [1], using the Clarke dual principle and Nonsmooth Mountain pass theorem, Ambrosetti and Badiale show that there exists a positive constant a_0 such that problem

$$\begin{gathered} -\Delta u = f(u-a) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.3}$$

has two positive solutions for $a \in (0, a_0)$. Using nonsmooth critical point theory, Arcoya and Calahorrano [6] extend the result to problem (1.1). In [2], using global bifurcation theory, Ambrosetti, Calahorrano and Dobarro prove that there is a global branch S of (1.3) when Ω is a ball or Steiner symmetric set, bifurcating from $(0, 0)$ and having a turning point at some value $a^* > 0$. They also show that the problem

$$\begin{gathered} -\Delta u = f(u-a) \quad \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty \end{gathered} \tag{1.4}$$

possesses an unbounded branch S of positive solutions bifurcating from $(0, 0)$.

Of course, the natural question is whether or not these results also hold for problems (1.1) and (1.2). In this article we shall give an affirmative answer to this question. We also would like to point out that in [2] the authors show that (1.2) possesses an unbounded branch S of positive solutions bifurcating from $(0, 0)$, and that under the conditions (F1)–(F2), (F4) (f4) there exist $c \geq d \geq 0$ such that $d \leq f(s) \leq c$ for all $s > 0$.

Obviously, the assumption (F4) is too restircitve. In the present paper, we shall remove this assumption and prove that (1.2) possesses an unbounded branch S of positive solutions bifurcating from $(0, 0)$. Furthermore, we also prove that S is unbounded in the a-direction. So even in the case of $p \equiv 2$, our results are also extending the related results of [2]. We also point out that most of the results in this paper also hold when $p \geq N$. However, if $p \geq N$ problem (1.2) may have no radial positive solution since for any $a > 0$, any positive, radial solution of (1.2) must satisfies (3.7) which may not hold if $p \geq N$.

2. Bounded domain setting

In this section we study problem (1.1) with a bounded domain Ω in $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$. Let ϕ_1 be the first eigenfunction corresponding to $\lambda_1 = \lambda_1(\Omega)$ such that $\int_\Omega \phi_1^p \, dx = 1$, $\phi_1 > 0$.

By a positive weak solution of (1.1) we mean a $u \in W_0^{1,p}(\Omega)$ such that $u \geq 0$ in the weak sense and satisfies

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int_\Omega f(u-a)\varphi \, dx$$

for any $\varphi \in W_0^{1,p}(\Omega)$.

Note that, by (F1) and the maximum principle, any positive solution u which is $\not\equiv 0$ is strictly positive: in fact, $u(x) > a$ for some $x \in \Omega$. Moreover, the discontinuity is not influent for $a \leq 0$ and, taking into account (F3), (1.1) has nontrivial, positive, weak solutions for such range of parameters. Since $f(s)$ is only discontinuous at $s = 0$, problem reduces to an equality, except possibly on the subset $T(a) = \{x \in \Omega : u(x) = a\}$. Since f is sublinear and continuous in $\Omega \setminus T(a)$, [15, Theorem E.0.20] implies $u \in L^\infty(\Omega \setminus T(a))$. Further, using the $C^{1,\alpha}$ regularity results for quasilinear elliptic equations with p-growth condition, see e.g. [12, 19], we have $u \in C^{1,\alpha}(\overline{\Omega} \setminus T(a))$ for any positive weak solution of (1.1), where $0 < \alpha < 1$. Therefore, u solves pointwise (1.1) on $\Omega \setminus T(a)$ for any positive weak solution of (1.1).

Set $E = C(\overline{\Omega})$ with the usual norm $|u|_\infty$, $V(b) = \{u \in E : |u|_\infty < b\}$, $\Sigma(\Omega) = \{(a,u) \in \mathbb{R}^+ \times E : u \text{ is a positive solution of (1.1)}\}$ and

$$\Sigma_0(\Omega) = \{u : u \text{ is a positive solution of (1.1) for } a = 0\}$$

Theorem 2.1. *Let* (F1)–(F3). *Then there exists a global branch* $S(\Omega) \subset cl(\Sigma(\Omega))$ *such that:*

(i) $(0,0) \in S(\Omega)$ *and if* $(a,0) \in S(\Omega)$ *then* $a = 0$*;*
(ii) $S(\Omega)$ *is bounded in* $\mathbb{R}^+ \times E$ *and* $S(\Omega) \cap (\Sigma_0(\Omega)) \neq \emptyset$*;*
(iii) *if* $(a,u) \in S(\Omega)$*, with* $0 < |u|_\infty$ *small, then* $a > 0$*. As a consequence, there is* $a_* > 0$ *such that for all* $a \in (0,a_*)$ (1.1) *has at least two distinct positive solutions with* $(a,u) \in S(\Omega)$*.*

Proof. The proof is divided into two steps. For simplicity of notations the dependence on Ω is understood and omitted during all the proof, but where a precision is worthwhile.

Step 1. (Smooth approximation of (1.1)). For $\varepsilon > 0$, let $f_\varepsilon \in C^{0,\alpha}(\mathbb{R},\mathbb{R})$ be defined by

$$f_\varepsilon(s) = \begin{cases} 0, & \text{if } s < -\varepsilon, \\ f(0+)\big(1 - \frac{|s|^{p-1}}{\varepsilon^{p-1}}\big) & \text{if } -\varepsilon \leq s \leq 0, \\ f(s), & \text{if } s > 0 \end{cases}$$

and consider the smooth problems

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f_\varepsilon(u-a) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

Let $G = (-\Delta_p)^{-1}$ denote the inverse to the 0-Dirichlet p-Laplacian. Also let $\Phi(\varepsilon,a,u) = u - Gf_\varepsilon(u-a)$ and denote by $\Sigma(\varepsilon)$ the set of $(a,u) \in \mathbb{R}^+ \times E$ such that $\Phi(\varepsilon,a,u) = 0$.

First of all we note that there exists $a_* = a_*(\Omega) > 0$ and $b_* = b_*(\Omega) > 0$ such that $\Sigma(\varepsilon) \subset [0, a_*] \times V(b_*)$ for all $\varepsilon > 0$. In fact, by (F3) there is $a_* > 0$ such that $f_\varepsilon(u - a) < c_1 u^{p-1}$ for all $a > a_*$. If u is any (positive) solution of (2.1) with $a > a_*$, one has

$$\int_\Omega |\nabla u|^p \, dx = \int_\Omega f_\varepsilon(u - a) u \, dx < c_1 \int_\Omega u^p \, dx \le \frac{c_1}{\lambda_1} \int_\Omega |\nabla u|^p \, dx,$$

which is a contradiction. On the other hand, since f_ε is p-sublinear and continuous, [15, Theorem E.0.20] implies $u \in L^\infty(\Omega)$. Again, by the $C^{1,\alpha}$ regularity results for quasilinear elliptic equations with p-growth condition, see e.g. [12, 14, 19], we have $u \in C^{1,\alpha}(\overline{\Omega})$ for any positive weak solution of (2.1). It follows that there exists $b_* > 0$ such that $|u|_\infty < b_*$ for any positive weak solution of (2.1). $\square$

Now we need the following lemma.

Lemma 2.2. *There is a global branch $S(\varepsilon) \subset \Sigma(\varepsilon)$ such that $(\varepsilon, 0) \in S(\varepsilon)$ and if $(a, 0) \in S(\varepsilon)$ then $a = \varepsilon$.*

Proof. We claim that

$$\operatorname{ind}(\Phi(\varepsilon, a, \cdot), 0) = 1 \quad \forall a > \varepsilon, \tag{2.2}$$

$$\operatorname{ind}(\Phi(\varepsilon, a, \cdot), 0) = 0 \quad \forall a < \varepsilon, \tag{2.3}$$

where ind denotes the Leray-Schauder index.

To show (2.2), we take $0 < \varepsilon < a$ and consider the homotopy $H(t, u) = u - tGf_\varepsilon(u - a)$, $t \in [0, 1]$. H is admissible on $V(r)$, $r > 0$ small enough, because, otherwise there are sequences $|u_n|_\infty \to 0$ and $t_n \in [0, 1]$ such that $H(t_n, u_n) = 0$. Letting $v_n = u_n/|u_n|_\infty$ it follows that

$$-\Delta_p v_n = t_n \frac{f_\varepsilon(u_n - a)}{u_n^{p-1}} v_n^{p-1}. \tag{2.4}$$

This equality and $|v_n|_\infty = 1$ imply $v_n \to \overline{v}$ in E and $|\overline{v}|_\infty = 1$. But for n large $u_n(x) < a - \varepsilon$, hence $f_\varepsilon(u_n - a) \equiv 0$; therefore, passing to the limit into (2.4) one finds $-\Delta_p \overline{v} = 0$, a contradiction, and (2.2) follows.

Next let $a < \varepsilon$. We prove (2.3) showing that there is $b > 0$ such that $\Phi(\varepsilon, a, u) \neq 0$ for all $u \in V(b)$. In fact, otherwise, there is a sequence $u_n \in E$, $u_n > 0$, such that $u_n \to 0$ and satisfies $-\Delta_p u_n = f_\varepsilon(u_n - a)$. Letting $u_n = t_n \phi_1 + w_n$ with $\int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla \phi_1 \, dx$, it follows, for n large

$$-\Delta_p u_n = f_\varepsilon(u_n - a) = f(0+)\Big(1 - \frac{(a - u_n)^{p-1}}{\varepsilon^{p-1}}\Big).$$

Multiplying by ϕ_1 integrating one finds

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega (t_n |\nabla u_n|^{p-2} |\nabla \phi_1|^2 + |t_n \nabla \phi_1 + \nabla w_n|^{p-2} \nabla w_n \nabla \varphi_1) \, dx \\
&= \int_\Omega f(0+)\Big(1 - (\frac{a}{\varepsilon})^{p-1}\Big) \phi_1 \, dx.
\end{aligned}$$

It follows that

$$0 = f(0+)\Big(1 - (\frac{a}{\varepsilon})^{p-1}\Big) \int_\Omega \phi_1 \, dx,$$

which is a contradiction, proving (2.3).

By standard arguments in global bifurcation theory (see [3, Proposition 3.5] or [16, Theorem 1.3]), (2.2) and (2.3) yield the existence of a global branch of positive solutions of (2.1) emanating from $(\varepsilon, 0)$. □

Step 2. (Limit as $\varepsilon \to 0$). To obtain the branch S of solutions of (1.1) we shall let $\varepsilon \to 0$ and show that $S(\varepsilon)$ converges (in a suitable sense) to S. This will be obtained, as in [5]], by means of the following topological lemma.

Lemma 2.3 ([21, Theorem.9.1]). *Let X be a metric space and let S_n be a sequence of connected subsets of X. Let*

(i) $\liminf_{n\to\infty}(S_n) \neq \emptyset$;
(ii) $\cup S_n$ *is precompact.*

Then $S =: \limsup_{n\to\infty}(S_n)$ is (nonempty) compact and connected.

In our case we take $X = \mathbb{R}^+ \times E$, $\varepsilon = 1/n$ and $S_n = S(1/n)$. Lemma 2.2 implies that S_n is connected and $(0,0) \in \liminf\limits_{n\to\infty}(S_n)$. Using the fact $S(\varepsilon) \subset [0, a_*] \times V(b_*)$ uniformly in ε, we can easily to see that (ii) holds. By Lemma 2.3, it follows that $S = S(\Omega) \neq \emptyset$ is compact and connected. In addition there results $S(\Omega) \subset [0, a_*] \times V(b_*)$.

The same argument used for (2.3) allows us to prove (iii). Roughly, if $-\Delta_p u_n = f(u_n)$ with $u_n \in S$ and $|u_n|_\infty \to 0$, it follows $u_n \to 0$. Let $u_n = t_n\phi_1 + w_n$ with $\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \nabla\phi_1\, dx$. Using the same argument used for (2.3), we have

$$0 = f(0+)\int_\Omega \phi_1\, dx,$$

a contradiction.

By (iii) and $S(\Omega) \subset [0, a_*] \times V(b_*)$ it follows that $S \cap \Sigma_0 \neq \emptyset$, yielding (ii). This completes the proof of Theorem 2.1. □

Set $r = |x|$ ($x \in \mathbb{R}^N$), $E = C(\overline{B(R)})$ with norm $|u|_{\infty,R}$, $V(b) = \{u \in E : |u|_{\infty,R} < b\}$,

$$\Sigma(R) = \big\{(a,u) \in \mathbb{R}^+ \times E : u \text{ is a positive solution of (1.1) with } \Omega = B(R).\big\},$$
$$\Sigma_0(R) = \{u : u \text{ is a positive solution of (1.1) with } \Omega = B(R) \text{ for } a = 0\}.$$

To study problem (1.2) on all $\mathbb{R}^N$, we need the following special results of $\Omega = B(R)$.

Theorem 2.4. *Let* (F1)–(F3) *hold. Then there exists a global branch $S(R) \subset cl(\Sigma(R))$ such that:*

(i) *$(0,0) \in S(R)$ and if $(a,0) \in S(R)$ then $a = 0$;*
(ii) *$S(R)$ is bounded in $\mathbb{R}^+ \times E$ and $S(R) \cap (\Sigma_0(R)) \neq \emptyset$;*
(iii) *if $(a,u) \in S(R)$, with $0 < |u|_{\infty,R}$ small, then $a > 0$. As a consequence, there is $a_* > 0$ such that for all $a \in (0, a_*)$ (1.1) with $\Omega = B(R)$ has at least two distinct positive solutions with $(a,u) \in S(R)$;*
(iv) *any $u \in S(R)$ is radial, $u'(r) < 0$ $\forall r > 0$ and $|T(a)| = \mathrm{meas}[T(a)] = 0$.*

Proof. We need to prove only (iv). By the well known result in [9, 10, 11], the positive solutions of (2.1) are radial: $u_\varepsilon = u_\varepsilon(r)$ and $u'_\varepsilon(r) < 0$, for all $r > 0$. Letting $\varepsilon \to 0$, one has that $u = u(r)$ and $u'(r) \le 0$, for all $r > 0$, for any $u \in S(R)$. To show that $u'(r) < 0$, for all $R > r > 0$ one uses the maximum principle applied to $u'(r)$, similarly with in [5]. Using the Hopf Lemma (see [20] or [15, Lema A.0.8]]), we also have $u'(R) < 0$. As a consequence $|T(a)| = 0$ and (iv) follows. □

As anticipated, Theorem 2.4 holds in greater generality. We suppose that Ω is Steiner symmetric with respect to the hyperplane $\{x_1 = 0\}$. For $u \in W_0^{1,p}(\Omega)$, $u(x) > 0$, we denote by $u^*(x)$ the Steiner symmetrization (with respect to $\{x_1 = 0\}$) of u. For definitions concerning symmetrization, see, for example, [13, §II.l-f].

Letting $S(S)$, $\Sigma(S)$ and $\Sigma_0(S)$ denote the sets $S(R)$, $\Sigma(R)$ and $\Sigma_0(R)$, we have:

Theorem 2.5. *Let* (F1)–(F3) *hold. Then there exists a global branch $S(S) \subset cl(\Sigma(S))$ such that:*

(i) *$(0, 0) \in S(S)$ and if $(a, 0) \in S(S)$ then $a = 0$;*
(ii) *$S(S)$ is bounded in $\mathbb{R}^+ \times E$ and $S(S) \cap (\Sigma_0(S)) \neq \emptyset$;*
(iii) *if $(a, u) \in S(S)$, with $0 < |u|_\infty$ small, then $a > 0$. As a consequence, there is $a_* > 0$ such that for all $a \in (0, a_*)$* (1.1) *has at least two distinct positive solutions with $(a, u) \in S(S)$;*
(iv) *any $u \in S(S)$ is Steiner symmetric and $\frac{\partial u}{\partial x_1} < 0$ for all $x_1 > 0$ and $|T(a)| = \operatorname{meas}[T(a)] = 0$.*

The proof of the above theorem is the same as that of Theorem 2.4. and is omitted.

3. Global bifurcation for problems in $\mathbb{R}^N$

In this section we study global branching for problem (1.2). As usual, (1.2) is approximated by Dirichlet problems (1.1) with $\Omega = B(R)$. The existence of a global branch of positive solutions of (1.2) will be established with another application of Lemma 2.3, letting $R \to \infty$. The meaning of positive solution of (1.2) is the same given for those of (1.1).

Fixed $a, R > 0$, let $(a, u_R) \in S(R)$. Set $T_R(a) = \{x \in \mathbb{R}^N : u_R(r) = a\}$ and denote by $\rho = \rho(R, a)$ the radius of the sphere $T_R(a)$. For $v \in W_0^{1p}(B(R))$, respectively $u \in D^{1,p}(\mathbb{R}^N)$, we set

$$\|v\|_R^p = \int_{B(R)} |\nabla v|^p \, dx \quad \|u\|^p = \int_{\mathbb{R}^N} |\nabla u|^p \, dx.$$

To pass to the limit as $R \to \infty$ we need to estimate $\rho(R, a)$.

Lemma 3.1. *If* (F1)–(F3) *hold then $\rho = \rho(R, a)$ satisfies:*

$$kd^{\frac{p}{p-1}} \rho^{\frac{pN+p-N}{p-1}} \leq \int_{\{u_R > a\}} |\nabla u|^p \, dx \leq kc_2^{\frac{p}{p-1}} \rho^{\frac{pN+p-N}{p-1}} + k^* c_1^{\frac{p}{p-1}} \rho^{\frac{(p+N-1)p+1-N}{p-1}}, \tag{3.1}$$

where

$$k = \frac{\omega_{N-1}(p-1)}{(pN+p-N)N^{\frac{p}{p-1}}}, \quad k^* = \omega_{N-1}(\frac{1}{p+2-1})^{\frac{p}{p-1}}$$

and ω_{N-1} denotes the measure of the unit sphere $\partial B(1)$.

Proof. Let u be any radial solution of (1.1) with $\Omega = B(R)$. Then $u = u_R(r)$ solves the problem

$$\begin{gathered} \left(r^{N-1}(-u')^{p-1}\right)' = r^{N-1} f(u - a) \quad \text{in } (0, R), \\ u'(0) = 0, \quad u(R) = 0. \end{gathered}$$

By this and the fact that $f(s) = 0$ for $s \leq 0$, there results

$$u_R(r) = a + \int_r^\rho (s^{1-N} \int_0^s t^{N-1} f(u(t) - a) \, dt)^{\frac{1}{p-1}} \, ds \quad \text{for } 0 \leq r \leq \rho; \tag{3.2}$$

$$u_R(r) = \frac{a}{1-(\frac{\rho}{R})^{\frac{N-p}{p-1}}}\big[(\frac{\rho}{r})^{\frac{N-p}{p-1}} - (\frac{\rho}{R})^{\frac{N-p}{p-1}}\big] \quad \text{for } \rho \le r \le R \tag{3.3}$$

with $\rho = \rho(R,a)$ determined by

$$u'(\rho^-) = u'(\rho^+). \tag{3.4}$$

By (3.2) we deduce that

$$\int_{\{u_R>a\}} |\nabla u|^p\,dx = \omega_{N-1}\int_0^\rho s^{\frac{1-N}{p-1}}\Big[\int_0^s t^{N-1} f(u(t)-a)\,dt\Big]^{\frac{p}{p-1}}\,ds. \tag{3.5}$$

Using (3.5) and (F3), the lemma follows. □

We denote by Σ the set of pairs $(a,u) \in \mathbb{R}^+ \times D^{1,p}(\mathbb{R}^N)$ such that u is a positive solution of (1.2) and by $T(a)$ the set $\{x \in \mathbb{R}^N : u(x) = a\}$.

Theorem 3.2. *Let* (F1)–(F3) *hold. Then there is a global, unbounded branch* $S \subset cl(\Sigma)$ *such that:*

- (i) $(0,0) \in S$;
- (ii) *if* $(a,u) \in S$ *then* u *is a radial, positive solution of* (2.1), $u'(r) < 0$ $\forall r > 0$ *and* $|T(a)| = 0$.

Proof. Firstly, any solution $u_R \in S(R)$ can be extended on all $\mathbb{R}^N$ setting $u_R = 0$ for $r > R$. Fixed an integer $j \gg l$, let $X_j = \{(a,u) \in \mathbb{R}^+ \times D^{1,p}(\mathbb{R}^N) : a^p + \|u\|^p \le j^p\}$. Taken a sequence $R_n \to \infty$, we set $S_{n,j} = S(R_n) \cap X_j$. We claim that (i) and (ii) of Lemma 2.3 hold true for $S_n = S_{n,j}$. In fact, (i) is trivially verified. As for (ii), let us take a sequence $(a_h,u_h) \in \cup_{n\in\mathbb{N}} S_{n,j}$. This means that u_h is a solution of (1.1) with $a = a_h$ and $\Omega = B(R_{n(h)})$. Set $\rho(h) = \rho(R_{n(h)}, a_h)$. Since $a_h \le j$, and $\|u_h\| \le j$, the left hand side of (3.1) implies:

$$kd^{\frac{p}{p-1}}\rho^{\frac{pN+p-N}{p-1}} \le \int_{\{u_R>a\}} |\nabla u|^p\,dx \le \|u_h\|^p.$$

Hence there exists $\rho^* > 0$ such that $\rho(h) \le \rho^*$ for all h, and $\{u_h > a_h\} \subset \{r < \rho^*\}$. From this and since $-\Delta_p u = f(u-a) = 0$ on $\{u < a\}$, it follows as in [5] that u_h converges, up to a subsequence, in $D^{1,p}(\mathbb{R}^N)$. This shows that (ii) holds. Applying Lemma 2.3 we find a non-empty, closed, connected set $S_j = \limsup(S_{n,j})$.

Next, note that for $a = 0$, problem (2.1) with $\Omega = B(R)$ has positive solutions $u \in \Sigma_0(R)$ and obviously one has $\rho(R,0) = R$. Hence, using the left-hand side of (3.1) one finds $\|u\|^p \ge kd^{\frac{p}{p-1}}R^{\frac{pN+p-N}{p-1}}$ for all $u \in \Sigma_0(R)$. Since each $S(R)$ is connected, it follows that $\forall j$ there is $n(j)$ such that, for all $n \ge n(j)$ there exists $(a_n,u_n) \in S(R_n)$ such that $a_n^p + \|u_n\|^p = j^p$. The preceding compactness argument shows that, up to a subsequence, (a_n,u_n) converges to some $(a,u) \in S_j$ with $a^p + \|u\|^p = j^p$. Therefore, the set $S = \cup_{j\in\mathbb{N}} S_j$ is unbounded, yielding the searched global branch.

The required properties of the solutions listed in (ii) follow by standard arguments as in [5]. □

To control the behavior of the branch S we have the following result.

Lemma 3.3. *Let* u *be any radial solution of* (1.2) *and let* $\rho(a) = \{r : u(r) = a\}$. *Then there results*

$$\Big(\frac{a(N-p)(N+p-1)^{\frac{1}{p-1}}}{(p-1)c_1{}^{\frac{1}{p-1}}}\Big)^{\frac{p-1}{2p-1}} + \Big(\frac{a(N-p)N^{\frac{1}{p-1}}}{(p-1)c_2{}^{\frac{1}{p-1}}}\Big)^{\frac{p-1}{p}}$$

$$\le \rho \le \Big(\frac{a(N-p)N^{\frac{1}{p-1}}}{(p-1)d^{\frac{1}{p-1}}}\Big)^{\frac{p-1}{p}};$$

in particular,

$$\Big(\frac{a(N-p)N^{\frac{1}{p-1}}}{(p-1){c_2}^{\frac{1}{p-1}}}\Big)^{\frac{p-1}{p}} \le \rho \le \Big(\frac{a(N-p)N^{\frac{1}{p-1}}}{(p-1)d^{\frac{1}{p-1}}}\Big)^{\frac{p-1}{p}}.$$

Proof. First of all we remark that formulas (3.3)–(3.4) hold for $u(r)$ with 0 instead of ρ/R. Using (3.4) one finds

$$(\rho^{1-N}\int_0^\rho t^{N-1}f(u(t)-a)\,dt)^{\frac{1}{p-1}} = \frac{(N-p)a}{(p-1)\rho}. \tag{3.6}$$

Using (3.6) and (F3), (3.7) follows. □

Theorem 3.4. *Let* (F1)–(F3) *hold. Then for any positive value all a, a radial solution u of* (1.2) *satisfies*

$$\begin{aligned}
&\frac{Ka^{\frac{pN+p-N}{p}}}{{c_2}^{\frac{N-p}{p}}}\Big[\big(\frac{d}{c_2}\big)^{\frac{p}{p-1}}\frac{N-p}{pN+p-N}+1\Big]\\
&\le \|u\|^p\\
&\le \frac{Ka^{\frac{pN+p-N}{p}}}{d^{\frac{N-p}{p}}}\Big[\big(\frac{c_2}{d}\big)^{\frac{p}{p-1}}\frac{N-p}{pN+p-N}+1\Big]\\
&\quad + k^*\big(\frac{c_1}{d}\big)^{\frac{1}{p-1}}\Big(\frac{a(N-p)N^{\frac{1}{p-1}}}{p-1}\Big)^{\frac{(p+N-1)p+1-N}{p}},
\end{aligned} \tag{3.7}$$

where

$$K = \omega_{N-1}N^{\frac{N-p}{p}}\big(\frac{N-p}{p-1}\big)^{\frac{N(p-1)}{p}}.$$

Hence for all $a > 0$, Equation (1.2) *possesses a positive, radial solution u, with $(a,u) \in S$.*

Proof. As remarked in Lemma 3.3, $u(r)$ has the form (3.3)–(3.4) with 0 instead of ρ/R. In particular: (i) Lemma 3.1 holds with u instead of u_R; and (ii) one has

$$\begin{aligned}
\int_{\{u<a\}}|\nabla u|^p\,dx &= \omega_{N-1}\int_\rho^\infty \Big(a\rho^{\frac{N-p}{p-1}}\frac{N-p}{p-1}\Big)^p r^{\frac{1-N}{p-1}}\,dr\\
&= \omega_{N-1}\big(\frac{N-p}{p-1}\big)^{p-1}a^p\rho^{N-p}.
\end{aligned}$$

Then

$$\begin{aligned}
\|u\|^p &= \int_{\{u>a\}}|\nabla u|^p\,dx + \int_{\{u<a\}}|\nabla u|^p\,dx\\
&= \int_{\{u>a\}}|\nabla u|^p\,dx + \omega_{N-1}\big(\frac{N-p}{p-1}\big)^{p-1}a^p\rho^{N-p}.
\end{aligned}$$

By (3.1) it follows that

$$\begin{aligned}
&kd^{\frac{p}{p-1}}\rho^{\frac{N(p-1)+p}{p-1}} + \omega_{N-1}\Big(\frac{N-p}{p-1}\Big)^{p-1}a^p\rho^{N-p}\\
&\le \|u\|^p
\end{aligned}$$

$$\le kc_2^{\frac{p}{p-1}}\rho^{\frac{N(p-1)+p}{p-1}} + k^* c_1^{\frac{p}{p-1}}\rho^{\frac{(p+N-1)p+1-N}{p-1}} + \omega_{N-1}(\frac{N-p}{p-1})^{p-1}a^p\rho^{N-p}.$$

Using the estimates for ρ found in Lemma 3.3, we obtain (3.7).

Lastly, suppose that, for some $a>0$, (1.2) has no solutions u, with $(a,u)\in S$. Since, by Theorem 3.2, S is connected, unbounded and bifurcates from $(0,0)$, there would exist $A>0$ and sequences $(a_n,u_n)\in S$ with $a_n\uparrow A$ and $\|u_n\|\to+\infty$. This contradicts the right hand side of (3.7). $\square$

Remark 3.5. Using the same arguments as above, one can find the following estimate for the L^∞ norm $|u|_\infty$ of any radial solution of (1.2):

$$\begin{aligned}
&a\big[1+(\frac{d}{c_2})^{\frac{1}{p-1}}\frac{N-p}{p}\big]\\
&\le |u|_\infty = u(0)\\
&\le a\big[1+(\frac{c_2}{d})^{\frac{1}{p-1}}\frac{N-p}{p}\big] + \Big(\frac{c_1}{p+N-1}\Big)^{\frac{1}{p-1}}\frac{2p-1}{p-1}\Big(\frac{a(N-p)N^{\frac{1}{p-1}}}{(p-1)d^{\frac{1}{p-1}}}\Big)^{\frac{2p-1}{p}}.
\end{aligned}$$

Remark 3.6. Note that the results of Theorems 2.1, 2.4, 2.5 and 3.2 also hold when $p\ge N$. However, if $p\ge N$ problem (1.2) may have no radial positive solutions.

Remark 3.7. We also would like to point out that the results of Theorem 2.1, 2.4, 2.5 also hold when $N=1$ even in the case of $p=2$.

Remark 3.8. Note that the methods of this paper also can be used to deal with the case of the nonlinearity f changing its sign, non-decreasing and discontinuous at $u=a$. In this case, we can get a global branch of negative solutions as well as the global branches of positive solution bifurcating from $(0,0)$.

Remark 3.9. Note that more general nonlinearities like $f(x,s)$ could be considered, under suitable growth assumptions also with respect to the x-variable.

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