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POSITIVE SOLUTIONS FOR A NONLOCAL MULTI-POINT BOUNDARY-VALUE PROBLEM OF FRACTIONAL AND SECOND ORDER

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Abstract. In this article we study the existence of positive solutions for the nonlocal multi-point boundary-value problem

$$u''(t) + f(t,{}^c D^\alpha u(t)) = 0, \quad \alpha \in (0,1), \text{ a.e. } t \in (0,1),$$

$$u(0) = 0, \quad u(1) = \sum_{k=1}^{m} a_k u(\tau_k), \quad \tau_k \in (a,b) \subset (0,1).$$

We also consider the corresponding integral condition, and the two special cases $\alpha = 0$ and $\alpha = 1$.

1. Introduction

Problems with non-local conditions have been extensively studied by several authors in the previous two decades; see for example [1]-[3], [6]-[20], and the references therein. In this work we show the existence of at least one solution for the nonlocal multi-point boundary-value problem consisting of second and fractional-orders differential equation

$$u''(t) + f(t,{}^c D^\alpha u(t)) = 0, \quad \alpha \in (0,1), \text{ a.e. } t \in (0,1) \tag{1.1}$$

with the nonlocal conditions

$$u(0) = 0, \quad u(1) = \sum_{k=1}^{m} a_k u(\tau_k), \quad \tau_k \in (a,b) \subset (0,1). \tag{1.2}$$

Also we deduce the same results for the two differential equations

$$u''(t) + f(t, u(t)) = 0, \quad \text{a.e. } t \in (0,1) \tag{1.3}$$

and

$$u''(t) + f(t, u'(t)) = 0, \quad \text{a.e. } t \in (0,1). \tag{1.4}$$

with the nonlocal conditions (1.2). Also we study problems (1.1), (1.3) and (1.4) with an integral condition.

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2. Preliminaries

Let $L^1(I)$ denote the class of Lebesgue integrable functions on the interval $I = [a, b]$,where $0 \leq a < b < \infty$ and let $\Gamma(.)$ denote the gamma function.

Definition 2.1 ([22])**.** The fractional-order integral of the function $f \in L^1[a, b]$ of order $\beta > 0$ is defined by

$$I_a^\beta f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds,$$

Definition 2.2 ([21, 22])**.** The Caputo fractional-order derivative of order $\alpha \in (0, 1]$ of the absolutely continuous function $f(t)$ is defined by

$$D_a^\alpha f(t) = I_a^{1-\alpha} \frac{d}{dt} f(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{ds} f(s)ds.$$

Theorem 2.3 (Schauder fixed point theorem [4])**.** *Let E be a Banach space and Q be a convex subset of E, and $T : Q \to Q$ a compact, continuous map. Then T has at least one fixed point in Q.*

Theorem 2.4 (Kolmogorov compactness criterion [5])**.** *Let $\Omega \subseteq L^p(0, 1)$, $1 \leq p < \infty$. If*

- (i) *Ω is bounded in $L^p(0, 1)$, and*
- (ii) *$u_h \to u$ as $h \to 0$ uniformly with respect to $u \in \Omega$, then Ω is relatively compact in $L^p(0, 1)$, where*

$$u_h(t) = \frac{1}{h} \int_t^{t+h} u(s)ds.$$

3. Main results

Consider the fractional-order functional integral equation

$$\begin{aligned} y(t) = f\Big(t, \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \big\{A \int_0^1 (1-s)y(s)ds - A \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s)y(s)ds\big\} \\ - \int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)} y(s)ds\Big). \end{aligned} \tag{3.1}$$

The function y is called a solution of the fractional-order functional integral equation (3.1), if y belongs to $L^1[0, 1]$ and satisfies (3.1).

We consider the following assumptions:

- (H1) $f : [0, 1] \times R \to R^+$ be a function with the following properties:
  - (a) $u \to f(t, u)$ is continuous for almost all $t \in [0, 1]$,
  - (b) $t \to f(t, u)$ is measurable for all $u \in R$,
- (H2) there exists an integrable function $a \in L^1[0, 1]$ and constant b, such that

$$|f(t, u)| \leq a(t) + b|u|, \text{a.e} t \in [0, 1],$$

Theorem 3.1. *Let the* (H1), (H2) *be satisfied. If*

$$B = \frac{b}{\Gamma(3-\alpha)} < 1, \tag{3.2}$$

then (3.1) *has at least one positive solution $y \in L^1[0, 1]$.*

Proof. Define the operator T associated with (3.1) by

$$\begin{aligned} Ty(t) = f\Big(t, \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\big\{A\int_0^1 (1-s)y(s)ds - A\sum_{k=1}^m a_k \int_0^{\tau_k}(\tau_k - s)y(s)ds\big\} \\ - \int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)} y(s)ds\Big) \end{aligned}$$

Let $Q_r^+ = \{y \in R^+ : \|y\| < r, r > 0\}$,

$$r = \frac{\|a\|}{1 - B(1 + A + A\sum_{k=1}^m a_k)}.$$

Let y be an arbitrary element in Q_r^+, then from assumptions (H1) and (H2), we obtain

$$\begin{aligned} \|Ty\|_{L^1} &= \int_0^1 |Ty(t)|dt \\ &= \int_0^1 \Big|f\Big(t, \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\big\{A\int_0^1 (1-s)y(s)ds - A\sum_{k=1}^m a_k \int_0^{\tau_k}(\tau_k - s)y(s)ds\big\} \\ &\quad - I^{2-\alpha}y(t)\Big)\Big|dt \\ &\le \int_0^1 |a(t)|dt + b\int_0^1 \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\,dt\Big\{A\int_0^1 (1-s)|y(s)|ds \\ &\quad + A\sum_{k=1}^m a_k \int_0^{\tau_k}(\tau_k - s)|y(s)|ds\Big\} + b\int_0^1\int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)}|y(s)|\,ds\,dt \\ &= \|a\|_{L^1} + b\int_0^1 \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\,dt\Big\{A\int_0^1 |y(s)|ds + A\sum_{k=1}^m a_k\int_0^1 |y(s)|ds\Big\} \\ &\quad + b\int_0^1\int_s^1 \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)}dt|y(s)|ds \\ &\le \|a\|_{L^1} + b(A + A\sum_{k=1}^m a_k)\|y\|_{L^1}\int_0^1 \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}dt \\ &\quad + b\int_0^1 \frac{(t-s)^{2-\alpha}}{(2-\alpha)\Gamma(2-\alpha)}\Big|_s^1 |y(s)|\,ds \\ &\le \|a\|_{L^1} + \frac{b(A + A\sum_{k=1}^m a_k)}{\Gamma(3-\alpha)}\|y\|_{L^1} + b\int_0^1 \frac{1}{\Gamma(3-\alpha)}|y(s)|\,ds \\ &\le \|a\|_{L^1} + \frac{b(A + A\sum_{k=1}^m a_k)}{\Gamma(3-\alpha)}\|y\|_{L^1} + \frac{b}{\Gamma(3-\alpha)}\|y\|_{L^1} \\ &\le \|a\|_{L^1} + \frac{b(1 + A + A\sum_{k=1}^m a_k)}{\Gamma(3-\alpha)}\|y\|_{L^1} = r\,, \end{aligned}$$

which implies that the operator T maps Q_r^+ into it self. Assumption (H1) implies that T is continuous.

Now let Ω be a bounded subset of Q_r^+, then $T(\Omega)$ is bounded in $L^1[0,1]$; i.e., condition (i) of Theorem 2.4 is satisfied. Let $y \in \Omega$. Then

$$\|(Ty)_h - Ty\| = \int_0^1 |(Ty)_h(t) - (Ty)(t)|dt$$

$$
\begin{aligned}
&= \int_0^1 |\frac{1}{h}\int_t^{t+h} (Ty)(s)ds - (Ty)(t)|dt \\
&\le \int_0^1 \Big(\frac{1}{h}\int_t^{t+h} |(Ty)(s) - (Ty)(t)|ds\Big)dt \\
&\le \int_0^1 \frac{1}{h}\int_t^{t+h} \Big| f\Big(s, \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\big\{A\int_0^1 (1-s)y(s)ds \\
&\quad - A\sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s)y(s)ds\big\} - I^{2-\alpha}y(t)\Big) \\
&\quad - f\Big(t, \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\big\{A\int_0^1 (1-s)y(s)ds \\
&\quad - A\sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s)y(s)ds\Big\} - I^{2-\alpha}y(t)\Big)\Big|\, ds\, dt.
\end{aligned}
$$

then assumption (H2) implies that $f \in L^1(0,1)$ and

$$
\begin{aligned}
&\frac{1}{h}\int_t^{t+h} \Big| f\Big(s, \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\big\{A\int_0^1 (1-s)y(s)ds - A\sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s)y(s)ds\big\} \\
&- I^{2-\alpha}y(t)\Big) - f\Big(t, \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\big\{A\int_0^1 (1-s)y(s)ds - A\sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s)y(s)ds\big\} \\
&- I^{2-\alpha}y(t)\Big)\Big|\, ds \to 0
\end{aligned}
$$

Therefore, by Theorem 2.4, we have that $T(\Omega)$ is relatively compact; that is, T is a compact operator. Then the operator T has a fixed point Q_r^+, which proves the existence of positive solution $y \in L^1[0,1]$ for (3.1). □

For the existence of solutions to the nonlocal problem (1.1)–(1.2), we have the following theorem.

Theorem 3.2. *Under the assumptions of Theorem 3.1, if* $0 < \sum_{k=1}^m a_k\tau_k < 1$, *then nonlocal problem* (1.1)–(1.2) *has at least one positive solution* $u \in C[0,1]$, *with* $u' \in AC[0,1]$.

Proof. For the problem (1.1)-(1.2), let $-y(t) = u''(t)$. Then

$$u(t) = tu'(0) - I^2y(t), \tag{3.3}$$

where y is the solution of the fractional-order functional integral equation (3.1). Letting $t = \tau_k$ in (3.3), we obtain

$$u(\tau_k) = -\int_0^{\tau_k} (\tau_k - s)y(s)ds + \tau_k u'(0)$$

and

$$\sum_{k=1}^m a_k u(\tau_k) = -\sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s)y(s)ds + u'(0)\sum_{k=1}^m a_k\tau_k \tag{3.4}$$

From equation (1.2) and (3.4), we obtain

$$-\int_0^1(1-s)y(s)ds+u'(0)=-\sum_{k=1}^m a_k\int_0^{\tau_k}(\tau_k-s)y(s)ds+u'(0)\sum_{k=1}^m a_k\tau_k\,.$$

Then

$$u'(0)=A\Big(\int_0^1(1-s)y(s)ds-\sum_{k=1}^m a_k\int_0^{\tau_k}(\tau_k-s)y(s)ds\Big),$$
$$A=(1-\sum_{k=1}^m a_k\tau_k)^{-1}.$$

Then

$$\begin{aligned} u(t)&=At\int_0^1(1-s)y(s)ds-At\sum_{k=1}^m a_k\int_0^{\tau_k}(\tau_k-s)y(s)ds-\int_0^t(t-s)y(s)ds\\ &=t\Big\{\int_0^1(1-s)y(s)ds+A\sum_{k=1}^m a_k\tau_k\int_0^1(1-s)y(s)ds\\ &\quad-A\sum_{k=1}^m a_k\int_0^{\tau_k}(\tau_k-s)y(s)ds\Big\}-\int_0^t(t-s)y(s)ds. \end{aligned} \tag{3.5}$$

where y is the solution of the fractional-order functional integral equation (3.1). Hence, by Theorem 3.1, Equation (3.5) has at least one solution $u\in C(0,1)$.

Now, from equation (3.5), we have

$$u(0)=\lim_{t\to 0^+}u(t)=0,$$

$$\begin{aligned} u(1)&=\lim_{t\to 1^-}u(t)\\ &=A\sum_{k=1}^m a_k\int_0^{\tau_k}(\tau_k-s)y(s)ds-A\int_0^1(1-s)y(s)ds-\int_0^1(1-s)y(s)ds \end{aligned}$$

from which we deduce that (3.5) has at least one positive solution $u\in C[0,1]$. Now,

$$\begin{aligned} \sum_{k=1}^m a_k\int_0^{\tau_k}(\tau_k-s)y(s)ds&<\sum_{k=1}^m a_k\tau_k\int_0^1(1-\frac{s}{\tau_k})y(s)ds\\ &<\sum_{k=1}^m a_k\tau_k\int_0^1(1-s)y(s)ds \end{aligned}$$

and

$$\int_0^t(t-s)y(s)ds<t\int_0^1(1-\frac{s}{t})y(s)ds<\int_0^1(1-s)y(s)ds\,.$$

Then the solution of (3.5) is positive.

To complete the proof, we show that (3.5) satisfies the nonlocal problem (1.1)–(1.2). Differentiating (3.5), we obtain

$$\frac{d^2u}{dt^2}=-y(t),$$

and

$$D^\alpha u(t)=I^{1-\alpha}\frac{d}{dt}u(t)$$

$$
\begin{aligned}
&= -I^{1-\alpha}\Big(A\int_0^1 (1-s)y(s)ds - A\sum_{k=1}^m a_k\int_0^{\tau_k}(\tau_k-s)y(s)ds - \int_0^t y(s)ds\Big)\\
&= \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\Big\{A\int_0^1 (1-s)y(s)ds - A\sum_{k=1}^m a_k\int_0^{\tau_k}(\tau_k-s)y(s)ds\Big\}\\
&\quad - \int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)}y(s)ds,
\end{aligned}
$$

where

$$u''(t) + f(t, D^\alpha u(t)) = 0$$

and

$$
\begin{aligned}
&\sum_{k=1}^m a_k u(\tau_k)\\
&= A\sum_{k=1}^m a_k\tau_k\sum_{k=1}^m a_k\int_0^{\tau_k}(\tau_k-s)y(s)ds\\
&\quad - A\sum_{k=1}^m a_k\tau_k\int_0^1(1-s)y(s)ds - \sum_{k=1}^m a_k\int_0^{\tau_k}(\tau_k-s)y(s)ds\\
&= \Big(A\sum_{k=1}^m a_k\tau_k - 1\Big)\sum_{k=1}^m a_k\int_0^{\tau_k}(\tau_k-s)y(s)ds - A\sum_{k=1}^m a_k\tau_k\int_0^1(1-s)y(s)ds\\
&= (A+1-1)\sum_{k=1}^m a_k\int_0^{\tau_k}(\tau_k-s)y(s)ds - (A+1)\int_0^1(1-s)y(s)ds\\
&= A\sum_{k=1}^m a_k\int_0^{\tau_k}(\tau_k-s)y(s)ds - A\int_0^1(1-s)y(s)ds - \int_0^1(1-s)y(s)ds\\
&= u(1)\,.
\end{aligned}
$$

This complete the proof of the equivalent between problem (1.1)–(1.2) and the integral equation (3.5). This also implies that there exists at least one positive solution $u \in C[0,1]$ of the nonlocal problem (1.1)–(1.2). □

Example. Our results can be applied to the nonlocal problem

$$
\begin{gathered}
u''(t) + f(t, D^\alpha u(t)) = 0, \quad \alpha \in (0,1), \text{ a.e. } t \in (0,1),\\
u(0) = 0, \quad u(1) = 2u(1/4) - 3u(1/2) + 3/2u(3/4).
\end{gathered}
$$

As an application, we have the following corollaries for the two cases $\alpha = 1$ and $\alpha = 0$.

Corollary 3.3. *Under the assumptions of Theorem 3.2, the nonlocal problem*

$$
\begin{gathered}
u''(t) + f(t, u'(t)) = 0 \quad \text{a.e. } t \in (0,1)\\
u(0) = 0, \quad u(1) = \sum_{k=1}^m a_k u(\tau_k), \quad \tau_k \in (a,b) \subset (0,1), \quad 0 < \sum_{k=1}^m a_k\tau_k < 1.
\end{gathered}
$$

has at least one positive solution.

The proof of the above corollary follows by letting $\alpha \to 1$ in Theorems 3.1 and 3.2 (see [21]).

Corollary 3.4. *Under the assumptions of Theorem 3.2, the nonlocal problem*

$$u''(t) + f(t, u(t)) = 0 \quad a.e. \ t \in (0, 1)$$

$$u(0) = 0, \quad u(1) = \sum_{k=1}^{m} a_k u(\tau_k), \quad \tau_k \in (a, b) \subset (0, 1), \quad 0 < \sum_{k=1}^{m} a_k \tau_k < 1.$$

has at least one positive solution.

The proof of the above corollary follows by letting $\alpha \to 0$ in Theorems 3.1 and 3.2 (see [21]).

4. Integral condition

Let$u \in C[0, 1]$ be the solution of the problem (1.1)–(1.2). Let $a_k = t_j - t_{k-1}$, $\eta_k \in (t_{k-1}, t_j)$, $a = t_0 < t_1 < t_2 < \dots < t_n = b$. Then condition (1.2) becomes

$$u(1) = \sum_{k=1}^{m} (t_j - t_{k-1}) u(\eta_k).$$

From the continuity of the solution u ,to (1.1)–(1.2), we can obtain

$$\lim_{m \to \infty} \sum_{k=1}^{m} (t_j - t_{k-1}) u(\eta_k) = \int_a^b u(s) ds.$$

and condition (1.2) is transformed into the integral condition

$$u(0) = 0, u(1) = \int_a^b u(s) ds.$$

Now, we have the following result.

Theorem 4.1. *Under the assumptions of Theorem 3.2, there exist at least one positive solution $u \in AC[0, 1]$ to the problem*

$$u''(t) + f(t, D^\alpha x(t)) = 0, \quad \alpha \in [0, 1], \ a.e. \ t \in (0, 1),$$

$$u(0) = 0, \quad u(1) = \int_a^b u(s)\, ds.$$

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Addendum posted by the editor on December 4, 2013

In March 2013, an anonymous reader informed us that the results in this article are incorrect:

> Theorem 3.2 is wrong, the example following it is not valid. Corollary 3.3 and Corollary 3.4 are not correct, under the given conditions it is possible that NO positive solution exist.

The authors tried to solve the problem, but the correction was not satisfactory¿ The authors were informed, but they have not sent any new corrections.

End of addendum.

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