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SIGN-CHANGING SOLUTIONS OF p-LAPLACIAN EQUATION WITH A SUB-LINEAR NONLINEARITY AT INFINITY

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ABSTRACT. In this article we obtain some existence and multiplicity results for sign-changing solutions of a p-Laplacian equation. We use the method of lower and upper solutions and Leray-Schauder degree theory. Moreover, the sign-changing solutions are located by using lower and upper solutions.

1. INTRODUCTION

In this article we present existence and multiplicity results for sign-changing solutions for the problem

$$\begin{gathered}(\varphi_p(u'(t)))' + f(t,u(t),u'(t)) = 0 \quad \text{a.e. } t \in (0,1),\\ u(0) = u(1) = 0,\end{gathered} \tag{1.1}$$

where $\varphi_p(s) = |s|^{p-2}s$, $s \in \mathbb{R}^1$, $p > 1$, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^1$.

In recent years there have been many studies on the existence of non-zero solutions of p-Laplacian differential boundary value problems, especially the existence of positive solutions of the p-Laplacian differential boundary value problems; see [1, 2, 4, 5, 6, 7, 8, 9, 15, 16, 17, 18] and the references therein. Recently, there were some papers considered the existence of sign-changing solutions of p-Laplacian differential boundary value problems by using the Leray-Schauder degree method, or the global bifurcation theorem or the variation method. For instance, in [6] the authors studied the p-Laplacian differential boundary value problems of the form

$$\begin{gathered}(\varphi_p(u'(t))' + \lambda h(t)f(u(t)) = 0 \quad \text{a.e. } t \in (0,1),\\ u(0) = u(1) = 0,\end{gathered} \tag{1.2}$$

where λ is a positive parameter, h a nonnegative measurable function on $(0,1)$ and $f \in C(\mathbb{R}^1, \mathbb{R}^1)$. By applying the global bifurcation theorem, the authors in paper [6] obtained existence results for positive solutions as well as sign-changing solutions of (1.2).

Zhang and Li [16] studied the problem

$$\begin{gathered}-\Delta_p u = h(u) \quad \text{in } \Omega,\\ u|_{\partial\Omega} = 0,\end{gathered} \tag{1.3}$$

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where $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2})\nabla u$ is the p-laplacian operator, Ω a smooth bounded domain in $\mathbb{R}^N$. The authors of [16] assumed that the boundary value problems (1.3) are with jumping nonlinearities at zero or infinity, then they get sign-changing solutions theorems of the p-Laplacian boundary value problems (1.3).

The main purpose of this paper is to obtain some existence and multiplicity results for sign-changing solutions of (1.1). We will employ the lower and upper solutions method and the Leary-Schauder degree method to show the existence and multiplicity results of sign-changing solutions of (1.1). Some sub-linear conditions on the nonlinearity f at infinity will be assumed. To show the multiplicity results for sign-changing solutions a pair of well ordered strict lower and upper solutions also be assumed. Then we will first construct another pair of well ordered (or non-well ordered) strict lower and upper solutions near the zero element θ of the Banach space $C_0^1[0,1]$. Next by computing the Leray-Schauder degree on different areas defined by the strict lower and upper solutions, we obtain the existence and multiplicity results for sign-changing solutions as well as positive and negative solutions of (1.1). The main feature of our results is that we not only obtain multiplicity results for sign-changing solutions of (1.1), but also give clear description of the locations of the sign-changing solutions of (1.1) through the strict lower and upper solutions.

In recent years, by using the method of invariant sets of the descending flow corresponding to the functional of the nonlinear problems some authors studied the existence results of sign-changing solutions of some partial differential boundary value problems, see [16, 17, 18] and the references therein. To show their results the authors always assumed the nonlinearities satisfy some kinds of monotony properties and therefore they always assumed the nonlinearities are without the gradient terms. For instance, Li and Li [7] considered the elliptic equation with Neumann boundary condition

$$
\begin{gathered}
-\Delta u + au = f(u), \quad x \in \Omega; \\
\frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega,
\end{gathered}
\tag{1.4}
$$

where Ω is a bounded domain with smooth boundary. The authors of [7] assumed that the nonlinearity f satisfies some increasing properties and obtained some multiplicity results for sign-changing solutions of (1.4) by using the method of invariant sets of the descending flow as well as the method of lower and upper solutions. Since we allow the nonlinearity f in (1.1) are with u', generally speaking, any monotony type conditions can not be assumed in (1.1) and therefore our main results can not be obtained by the method in [16, 17, 18].

This paper is organized in the following way. In the section 2, we give general hypothesis and technical results about the p-Laplacian differential boundary value problems. Then, we give degree information in terms of the lower and upper solutions. In the section 3, we will give existence and multiplicity results for sign-changing solutions of (1.1).

2. Some Lemmas

Let N^+ denote the set of natural numbers. Let $C[0,1]$ and $C^1[0,1]$ be the usual Banach spaces with the norms $\|\cdot\|_0$ and $\|\cdot\|$, respectively. Let $C_0^1[0,1] = \{x \in C^1[0,1] | x(0) = x(1) = 0\}$, $P_0 = \{x \in C[0,1] | x(t) \geqslant 0,\ t \in [0,1]\}$ and $P = P_0 \cap C_0^1[0,1]$. Then $C_0^1[0,1]$ is also a real Banach space with the norm $\|\cdot\|$,

P and P_0 are cones of $C_0^1[0,1]$ and $C[0,1]$, respectively. Let $\leqslant$ denote both the orderings induced by P in $C_0^1[0,1]$ and P_0 in $C[0,1]$. We write $x < y$ if $x \leqslant y$ and $x \neq y$. Let $e(t) = t(1-t)$ for all $t \in [0,1]$. For each $x, y \in C[0,1]$, we denote by $x \prec y$ or $y \succ x$ if $y - x \geqslant \delta_0 e$ for some $\delta_0 > 0$. For any $x_0 \in C[0,1]$, let $\Omega_1 = \{x \in C_0^1[0,1] | x \succ x_0\}$ and $\Omega_2 = \{x \in C_0^1[0,1] | x \prec x_0\}$. Then Ω_1 and Ω_2 are open subsets of $C_0^1[0,1]$.

Now we define the concepts of strict lower and upper solutions of (1.1) in a manner as that of [3]; see [3, Definition 5.4.47 and 5.4.48].

Definition 2.1. A function $u_0 \in C^1[0,1]$ with $\varphi_p(u_0'(t))$ absolutely continuous is called a lower solution of (1.1) if

$$u_0(0) \leqslant 0, \quad u_0(1) \leqslant 0$$

and

$$-(\varphi_p(u_0'(t))' \leqslant f(t, u_0(t), u_0'(t)) \quad \text{for a. e. } t \in (0,1).$$

In an analogous way we define an upper solution of (1.1).

Definition 2.2. A lower solution u_0 is said to be strict if every possible solution x of (1.1) such that $u_0 \leqslant x$ satisfies $u_0 \prec x$. In an analogous way we define a strict upper solution of (1.1).

Remark 2.3. Obviously, if f has the form of $f(t,x)$ and satisfies

$$f(t, x_2) - f(t, x_1) \geqslant -M(x_2 - x_1), \quad \forall x_2 \geqslant x_1$$

for some $M > 0$, $u_0 \in C^2[0,1]$ satisfies $u_0(0) \leqslant 0$, $u_0(1) \leqslant 0$ and

$$u_0'' + f(t, u_0(t)) > 0, \quad t \in (0,1),$$

then u_0 will be a strict lower solution in the Definition 2.2 for $p = 2$ by the Maximum Principle.

Definition 2.4. Let u_0 and v_0 be strict lower and upper solutions of (1.1), respectively. Then u_0 and v_0 are called a pair of well-ordered strict lower and upper solutions of (1.1) if $u_0 \prec v_0$.

Definition 2.5. A function $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^1$ is said to be a Carathéodory function, if $f(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^2$ for almost all $t \in [0,1]$; $f(\cdot, x, y)$ is a measurable function on $[0,1]$ for all $(x,y) \in \mathbb{R}^2$; for every $R > 0$ there exists a real-valued function $\Psi \equiv \Psi_R \in L^1(0,1)$ such that

$$|f(t,x,y)| \leqslant \Psi(t)$$

for a.e. $t \in [0,1]$ and for every $(x,y) \in \mathbb{R}^2$ with $|x| + |y| \leqslant R$.

Let $\alpha \in C^1[0,1]$. The function $p : [0,1] \times \mathbb{R}^1 \to \mathbb{R}^1$ be defined by

$$p(t,x) = \max\{\alpha(t), x\}, \ \forall (t,x) \in [0,1] \times \mathbb{R}^1.$$

The first result is Lemma 2.6, for which we omit the proof. A similar result and its proof can be found in [11].

Lemma 2.6. *For each $u \in C^1[0,1]$, the next two properties hold:*
- (i) $\frac{d}{dt} p(t, u(t))$ *exists for a.e.* $t \in I$.
- (ii) *If* $u, u_m \in C^1[0,1]$ *and* $u_m \to u$ *in* $C^1[0,1]$, *then*

$$\frac{d}{dt} p(t, u_m(t)) \to \frac{d}{dt} p(t, u(t)) \quad \text{for a.e. } t \in [0,1].$$

Lemma 2.7. *Let $\alpha_1,\ \alpha_2 \in C^1[0,1]$ and $\bar\alpha(t) = \max\{\alpha_1(t), \alpha_2(t)\}$ for all $t \in [0,1]$. Then the following conclusions hold.*

1. *$\bar\alpha'(t) = \alpha_1'(t)$ when $\alpha_1(t) > \alpha_2(t)$;*
2. *$\bar\alpha'(t) = \alpha_2'(t)$ when $\alpha_2(t) > \alpha_1(t)$;*
3. *$\bar\alpha'(t) = \alpha_1'(t) = \alpha_2'(t)$ when $\alpha_1(t) = \alpha_2(t)$ and $\alpha_1'(t) = \alpha_2'(t)$;*
4. *$\bar\alpha'_-(t) = \min\{\alpha_1'(t), \alpha_2'(t)\}$ and $\bar\alpha'_+(t) = \max\{\alpha_1'(t), \alpha_2'(t)\}$ when $\alpha_1(t) = \alpha_2(t)$ and $\alpha_1'(t) \neq \alpha_2'(t)$;*
5. *$\lim_{\tau\to t^-} \bar\alpha'(\tau) = \bar\alpha'_-(t)$ and $\lim_{\tau\to t^+} \bar\alpha'(\tau) = \bar\alpha'_+(t)$ when $\alpha_1(t) = \alpha_2(t)$ and $\alpha_1'(t) \neq \alpha_2'(t)$;*
6. $$|\bar\alpha'(t)| \leqslant \max\{\|\alpha_1'\|_0, \|\alpha_2'\|_0\} \ a.e.\ t \in [0,1]. \tag{2.1}$$

Proof. Let $I_1 = \{t \in [0,1] | \alpha_1(t) > \alpha_2(t)\}$, $I_2 = \{t \in [0,1] | \alpha_2(t) > \alpha_1(t)\}$ and $I_3 = [0,1]\backslash(I_1 \cup I_2)$. Assume without loss of generality that $I_i \neq \emptyset$ for $i = 1,2,3$. Obviously, we have $\bar\alpha'(t) = \alpha_1'(t)$ for each $t \in I_1$, and $\bar\alpha'(t) = \alpha_2'(t)$ for each $t \in I_2$. Let $I_{3,1} = \{t \in I | \alpha_1(t) = \alpha_2(t), \alpha_1'(t) = \alpha_2'(t)\}$ and $I_{3,2} = \{t \in I | \alpha_1(t) = \alpha_2(t), \alpha_1'(t) \neq \alpha_2'(t)\}$. Then we have $I_3 = I_{3,1} \cup I_{3,2}$. Obviously, the conclusions (1) and (2) hold. Let $t_0 \in I_3$. Now for each $t > t_0$, by the Mean-Value Theorem, there exists ξ_t and η_t with $t_0 < \xi_t < t$ and $t_0 < \eta_t < t$ such that

$$\begin{aligned} \alpha_1(t) &= \alpha_1(t_0) + \alpha_1'(\xi_t)(t - t_0), \\ \alpha_2(t) &= \alpha_2(t_0) + \alpha_2'(\eta_t)(t - t_0). \end{aligned}$$

Then, we have for each $t > t_0$,

$$\begin{aligned} \bar\alpha(t) &= \bar\alpha(t_0) + [\alpha_1'(\xi_t) \vee \alpha_2'(\eta_t)](t - t_0) \\ &= \bar\alpha(t_0) + \frac{\alpha_1'(\xi_t) + \alpha_2'(\eta_t) + |\alpha_1'(\xi_t) - \alpha_2'(\eta_t)|}{2}(t - t_0). \end{aligned}$$

Consequently,

$$\begin{aligned} \bar\alpha'_+(t_0) &= \lim_{t\to t_0^+} \frac{\bar\alpha(t) - \bar\alpha(t_0)}{t - t_0} \\ &= \lim_{t\to t_0^+} \frac{\alpha_1'(\xi_t) + \alpha_2'(\eta_t) + |\alpha_1'(\xi_t) - \alpha_2'(\eta_t)|}{2} \\ &= \frac{\alpha_1'(t_0) + \alpha_2'(t_0) + |\alpha_1'(t_0) - \alpha_2'(t_0)|}{2} \\ &= \begin{cases} \alpha_1'(t_0) = \alpha_2'(t_0), & \text{when } t_0 \in I_{3,1}; \\ \max\{\alpha_1'(t_0), \alpha_2'(t_0)\}, & \text{when } t_0 \in I_{3,2}. \end{cases} \end{aligned}$$

Similarly, we have

$$\bar\alpha'_-(t_0) = \lim_{t\to t_0^-} \frac{\bar\alpha(t) - \bar\alpha(t_0)}{t - t_0} = \begin{cases} \alpha_1'(t_0) = \alpha_2'(t_0), & \text{when } t_0 \in I_{3,1}; \\ \min\{\alpha_1'(t_0), \alpha_2'(t_0)\}, & \text{when } t_0 \in I_{3,2}. \end{cases}$$

Therefore, $\bar\alpha$ is differentiable at $t_0 \in I_{3,1}$ and $\bar\alpha'(t_0) = \alpha_1'(t_0) = \alpha_2'(t_0)$ for each $t_0 \in I_{3,1}$. Thus, the conclusion (3) and (4) hold.

Let $t_0 \in I_{3,2}$. Assume without loss of generality that $t_0 \in (0,1)$ and $\alpha_1'(t_0) < \alpha_2'(t_0)$. Then there exists $\delta_0 > 0$ small enough such that $\alpha_1(t) > \alpha_2(t)$ for all $t \in (t_0 - \delta_0, t_0)$. Thus, we have $\bar\alpha(t) = \alpha_1(t)$ for all $t \in (t_0 - \delta_0, t_0]$, and thus $\bar\alpha'(t) = \alpha_1'(t)$ for $t \in (t_0 - \delta_0, t_0]$. Therefore, $\lim_{t\to t_0^-} \bar\alpha'(t) = \alpha_1'(t_0) = \bar\alpha'_-(t_0)$.

Similarly, we have $\lim_{t\to t_0^+} \bar\alpha'(t) = \alpha_2'(t_0) = \bar\alpha'_+(t_0)$. This means that the conclusion (5) holds. The conclusion (6) follows from (1)-(4). The proof is complete. $\square$

Let $h \in L^1(0,1)$. Consider the boundary-value problem

$$\begin{gathered} (\varphi_p(u'(t)))' = h \quad \text{a.e. } t \in (0,1), \\ u(0) = u(1) = 0. \end{gathered} \tag{2.2}$$

A function $u \in C_0^1[0,1]$ is called a solution of (2.2), if $\varphi_p(u'(t))$ is absolutely continuous and satisfies (2.2). It is easy to see that (2.2) is equivalent to

$$u(t) = G_p(h)(t) := \int_0^t \varphi_p^{-1}\Big(a(h) + \int_0^s h(\tau)d\tau\Big)ds, \tag{2.3}$$

where $a : L^1(0,1) \to \mathbb{R}^1$ is a continuous functional satisfying

$$\int_0^1 \varphi_p^{-1}\Big(a(h) + \int_0^s h(\tau)d\tau\Big)ds = 0.$$

From [10], we see that $G_p : L^1(0,1) \to C_0^1[0,1]$ is continuous and maps equi-integrable sets of $L^1(0,1)$ into relatively compact sets of $C_0^1[0,1]$. One may refer to Manásevich and Mawhin [9] for more details.

Next we consider the eigenvalues problem of the form (2.4)

$$\begin{gathered} (\varphi_p(u'(t)))' + \lambda\varphi_p(u(t)) = 0 \quad \text{a.e. } t \in (0,1), \\ u(0) = u(1) = 0. \end{gathered} \tag{2.4}$$

Define the operator $T_\lambda^p : C_0^1[0,1] \to C_0^1[0,1]$ by

$$(T_\lambda^p u)(t) = G_p(-\lambda\varphi_p(u))(t) = \int_0^t \varphi_p^{-1}\Big(a(-\lambda\varphi_p(u)) - \int_0^s \lambda\varphi_p(u(\tau))d\tau\Big)ds.$$

Then T_λ^p is completely continuous and problem (2.4) is equivalent to equation $u = T_\lambda^p u$.

From [6, Proposition 2.6, Lemmas 2.7 and 2.8], we have the following Lemmas

Lemma 2.8. *The following conditions hold:*

(i) *the set of all eigenvalues of* (2.4) *is a countable set* $\{\mu_k(p) | k \in N^+\}$ *satisfying*
$$0 < \mu_1(p) < \mu_2(p) < \dots < \mu_k(p) < \dots \to \infty;$$
(ii) *for each* k, $\ker(I - T^p_{\mu_k(p)})$ *is a subspace of* $C^1[0,1]$ *and its dimension is 1;*
(iii) *let* ϕ_k *be a corresponding eigenfunction to* $\mu_k(p)$, *then the number of interior zeros of* ϕ_k *is* $k-1$.

Lemma 2.9. *For each* $k \in N^+$, $\mu_k(p)$ *as a function of* $p \in (1,\infty)$ *is continuous.*

By Lemma 2.9 and the method of homotopy along p which developed in [10], we have the following Lemma.

Lemma 2.10. *For fixed* $p > 1$ *and all* $r > 0$, *we have*

$$\deg(I - T_\lambda^p, B(\theta, r), \theta) = \begin{cases} 1, & \text{when } \lambda < \mu_1(p); \\ (-1)^k, & \text{when } \lambda \in (\mu_k(p), \mu_{k+1}(p)). \end{cases}$$

Now let us define the operator $F : C_0^1[0,1] \to L^1[0,1]$ and $T_p : C_0^1[0,1] \to C_0^1[0,1]$ by

$$(Fx)(t) = f(t, x(t), x'(t)), t \in [0,1]$$

and $(T_p x)(t) = (G_p F x)(t)$ for all $t \in [0,1]$. Then, T_p is completely continuous.

For convenience, we make the following assumptions.

(H1) $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^1$ is a Carathéodory function such that $xf(t,x,y) > 0$ for all $(t,y) \in [0,1] \times \mathbb{R}^1$ and $x \neq 0$, and there exists $\beta_\infty \geqslant 0$ with $(2^p \beta_\infty)^{\frac{1}{p-1}} < 1$ such that

$$\lim_{|x|+|y| \to \infty} \frac{|f(t,x,y)|}{\varphi_p(|x|+|y|)} = \beta_\infty \quad \text{uniformly for } t \in [0,1].$$

(H2) There exists $R_* > 0$ and $\beta_0 > 0$ such that

$$\lim_{x \to 0} \frac{f(t,x,y)}{\varphi_p(x)} = \beta_0 \quad \text{uniformly for} \quad t \in [0,1] \text{ and } y \in [-R_*, R_*].$$

(H3) There exist sign-changing functions u_1, v_1 such that u_1 and v_1 are a pair of strict lower and upper solutions of (1.1).

Let f and g be defined by

$$f(t,x,y) = \begin{cases} \beta_0 \varphi_p(x) + [\varphi_p(x)]^2 y^2, & x^2 + y^2 \leqslant 1; \\ 10, & x^2 + y^2 \geqslant 2; \\ 10(\sqrt{x^2+y^2} - 1) + (2 - \sqrt{x^2+y^2}) g(x,y), & 1 < x^2 + y^2 < 2, \end{cases}$$

$$g(x,y) = \beta_0 \varphi_p\Big(\frac{x}{\sqrt{x^2+y^2}}\Big) + \Big[\varphi_p\Big(\frac{x}{\sqrt{x^2+y^2}}\Big)\Big]^2 \Big(\frac{y}{\sqrt{x^2+y^2}}\Big)^2, \ 1 < x^2 + y^2 < 2.$$

Obviously, f satisfies the conditions (H1) and (H2).

Lemma 2.11. *Suppose that* (H1) *holds,* α_1, α_2 *are strict lower solutions of* (1.1) *such that* $\alpha_1(t) \equiv \alpha_2(t)$ *or the set* $\{t \in [0,1] | \alpha_1(t) = \alpha_2(t), \alpha_1'(t) \neq \alpha_2'(t)\}$ *contains at most finite elements. Then there exists* $R_0 > 0$ *such that for each* $R_1 \geqslant R_0$, $T_p(\bar{B}(\theta, R_1)) \subset B(\theta, R_1)$, $\alpha_1, \alpha_2 \in B(\theta, R_1)$ *and*

$$\deg(I - T_p, \Omega, \theta) = 1,$$

where $\Omega = \{x \in B(\theta, R_1) | x \succ \alpha_1, x \succ \alpha_2\}$ *and* $B(\theta, R_1) = \{x \in C_0^1[0,1] : \|x\| < R_1\}$.

Proof. We consider only the case of $\alpha_1(t) \not\equiv \alpha_2(t)$. Let $\bar{\alpha}(t) = \max\{\alpha_1(t), \alpha_2(t)\}$ for each $t \in [0,1]$. Let β'_∞ be such that $\beta'_\infty > \beta_\infty$ and $(2^p \beta'_\infty)^{\frac{1}{p-1}} < 1$. From (H1), there exists $R' > 0$ such that

$$|f(t,x,y)| \leqslant \beta'_\infty \varphi_p(|x|+|y|), \forall t \in [0,1], |x| + |y| \geqslant R'.$$

Since $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^1$ is a Carathéodory function, then there exists $\Psi_{R'} \in L^1(0,1)$ such that

$$|f(t,x,y)| \leqslant \Psi_{R'}(t) \quad \text{a.e. } t \in [0,1], |x| + |y| \leqslant R'.$$

Consequently, we have

$$|f(t,x,y)| \leqslant \beta'_\infty \varphi_p(|x|+|y|) + \Psi_{R'}(t), \quad \text{a.e. } t \in [0,1], (x,y) \in \mathbb{R}^2.$$

Let

$$R_0 > \max\Big\{\|\alpha_1\|, \|\alpha_2\|, \frac{\varphi_p^{-1}(2^p M_{R'}) + 2(\|\alpha_1\| + \|\alpha_2\|)}{1 - (2^p \beta'_\infty)^{\frac{1}{p-1}}}\Big\}.$$

and $R_1 \geqslant R_0$, where $M_{R'} = \|\Psi_{R'}\|_{L^1(0,1)}$. For any $x \in \bar{B}(\theta, R_1)$, by the Rolle's Theorem there exists $t_x \in (0,1)$ such that $(T_p x)'(t_x) = 0$ and for all $t \in [0,1]$

$$|(T_p x)'(t)| = \Big|\varphi_p^{-1}\Big(\int_t^{t_x} f(\tau, x(\tau), x'(\tau))d\tau\Big)\Big|$$

and

$$|(T_p x)(t)| = \Big|\int_0^t \varphi_p^{-1}\Big(\int_s^{t_x} f(\tau, x(\tau), x'(\tau))d\tau\Big)ds\Big|.$$

Then we have for all $t \in [0,1]$

$$\begin{aligned}
|(T_p x)'(t)| &\leqslant \varphi_p^{-1}\Big(\int_0^1 |f(\tau, x(\tau), x'(\tau))|d\tau\Big) \\
&\leqslant \varphi_p^{-1}\Big(\int_0^1 [\beta'_\infty \varphi_p(|x(\tau)| + |x'(\tau)|) + \Psi_{R'}(\tau)]d\tau\Big) \\
&\leqslant \varphi_p^{-1}(\beta'_\infty \varphi_p(\|x\|) + M_{R'}) \\
&= \varphi_p^{-1}(\varphi_p((\beta'_\infty)^{\frac{1}{p-1}}\|x\|) + \varphi_p(\varphi_p^{-1}(M_{R'}))) \\
&\leqslant \varphi_p^{-1}(2\varphi_p((\beta'_\infty)^{\frac{1}{p-1}}\|x\| + \varphi_p^{-1}(M_{R'}))) \\
&= (2\beta'_\infty)^{\frac{1}{p-1}}\|x\| + \varphi_p^{-1}(2M_{R'})
\end{aligned}$$

Similarly, we have that for all $t \in [0,1]$,

$$\begin{aligned}
|(T_p x)(t)| &\leqslant \int_0^t \varphi_p^{-1}\Big(\int_0^1 |f(\tau, x(\tau), x'(\tau))|d\tau\Big)ds \\
&\leqslant \varphi_p^{-1}\Big(\int_0^1 |f(\tau, x(\tau), x'(\tau))|d\tau\Big) \\
&\leqslant (2\beta'_\infty)^{\frac{1}{p-1}}\|x\| + \varphi_p^{-1}(2M_{R'}).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|T_p x\| &= \|(T_p x)'\|_0 + \|T_p x\|_0 \\
&\leqslant 2[(2\beta'_\infty)^{\frac{1}{p-1}}\|x\| + \varphi_p^{-1}(2M_{R'})] \\
&= (2^p\beta'_\infty)^{\frac{1}{p-1}}\|x\| + \varphi_p^{-1}(2^p M_{R'})) \\
&\leqslant (2^p\beta'_\infty)^{\frac{1}{p-1}} R_1 + \varphi_p^{-1}(2^p M_{R'}) < R_1.
\end{aligned}$$

This implies that $T_p(\bar{B}(\theta, R_1)) \subset B(\theta, R_1)$.

Let the function $g : [0,1] \times \mathbb{R}^1 \to \mathbb{R}^1$ be defined by

$$g(t,x) = \max\{\bar{\alpha}(t), x\}, \forall (t,x) \in [0,1] \times \mathbb{R}^1.$$

We denote by $\widetilde{T}_p : C_0^1[0,1] \to C_0^1[0,1]$ the solution operator of

$$\begin{gathered}
(\varphi_p(y'(t)))' + f(t, g(t, x(t)), \frac{d}{dt} g(t, x(t)) = 0 \quad \text{a.e. } t \in (0,1); \\
y(0) = y(1) = 0;
\end{gathered} \tag{2.5}$$

that is, for $x, y \in C_0^1[0,1]$,

$$y = \widetilde{T}_p x$$

if and only if (2.5) holds. For any $x \in C_0^1[0,1]$ it follows by integration of (2.5) and the injectivity of $\varphi(s) = |s|^{p-2}s$ that the operator $\widetilde{T}$ is well defined. In fact, $\widetilde{T}_p = G_p\widetilde{F}$, where $\widetilde{F} : C_0^1[0,1] \to L^1[0,1]$ is defined by

$$(\widetilde{F}x)(t) = f(t, g(t, x(t)), \frac{d}{dt}g(t, x(t))$$

for a.e. $t \in [0,1]$. It follows from Lemma 2.6 and 2.7 that $\widetilde{F} : C_0^1[0,1] \to L^1[0,1]$ is bounded and continuous, and so $\widetilde{T}_p : C_0^1[0,1] \to C_0^1[0,1]$ is completely continuous.

For any $x \in \bar{B}(\theta, R_1)$ there exists $\widetilde{t}_x \in (0,1)$ such that $(\widetilde{T}_p x)'(\widetilde{t}_x) = 0$ and for all $t \in [0,1]$,

$$|(\widetilde{T}_p x)'(t)| = \Big|\varphi_p^{-1}\Big(\int_t^{\widetilde{t}_x} f(\tau, x(\tau), x'(\tau))d\tau\Big)\Big| \tag{2.6}$$

and

$$|(\widetilde{T}_p x)(t)| = \Big|\int_0^t \varphi_p^{-1}\Big(\int_s^{\widetilde{t}_x} f(\tau, x(\tau), x'(\tau))d\tau\Big)ds\Big|. \tag{2.7}$$

It follows from Lemma 2.7 that for each $x \in C_0^1[0,1]$,

$$\Big|\frac{d}{dt}g(t, x(t))\Big| \leqslant \max\{\|\alpha_1'\|_0, \|\alpha_2'\|_0, \|x'\|_0\} \quad \text{a.e. } t \in (0,1) \tag{2.8}$$

From (2.6)-(2.8) we have that for $t \in [0,1]$,

$$\begin{aligned}
|(\widetilde{T}_p x)'(t)| &\leqslant \varphi_p^{-1}\Big(\int_0^1 |f(\tau, g(\tau, x(\tau)), \frac{d}{d\tau}g(\tau, x(\tau)))|d\tau\Big) \\
&\leqslant \varphi_p^{-1}\Big(\int_0^1 \big[\beta_\infty'\varphi_p(|g(\tau, x(\tau))| + |\frac{d}{d\tau}g(\tau, x(\tau))|) + \Psi_{R'}(\tau)\big]d\tau\Big) \\
&\leqslant \varphi_p^{-1}\Big(\int_0^1 \big(\beta_\infty'\varphi_p(\max\{\|x\|_0, \|\alpha_1\|_0, \|\alpha_2\|_0\} \\
&\quad + \max\{\|x'\|_0, \|\alpha_1'\|_0, \|\alpha_2'\|_0\}) + \Psi_{R'}(\tau)\big)d\tau\Big) \\
&\leqslant \varphi_p^{-1}(\beta_\infty'\varphi_p(\|x\| + \|\alpha_1\| + \|\alpha_2\|) + M_{R'}) \\
&\leqslant (2\beta_\infty')^{\frac{1}{p-1}}(\|x\| + \|\alpha_1\| + \|\alpha_2\|) + \varphi_p^{-1}(2M_{R'}) \\
&\leqslant (2\beta_\infty')^{\frac{1}{p-1}}\|x\| + \|\alpha_1\| + \|\alpha_2\| + \varphi_p^{-1}(2M_{R'}).
\end{aligned}$$

Similarly, we have that for $t \in [0,1]$,

$$|(\widetilde{T}_p x)(t)| \leqslant (2\beta_\infty')^{\frac{1}{p-1}}\|x\| + \|\alpha_1\| + \|\alpha_2\| + \varphi_p^{-1}(2M_{R'}).$$

Thus we have

$$\begin{aligned}
\|\widetilde{T}_p x\| &\leqslant (2^p\beta_\infty')^{\frac{1}{p-1}}\|x\| + 2(\|\alpha_1\| + \|\alpha_2\|) + \varphi_p^{-1}(2^p M_{R'}) \\
&\leqslant (2^p\beta_\infty')^{\frac{1}{p-1}}R_1 + 2(\|\alpha_1\| + \|\alpha_2\|) + \varphi_p^{-1}(2^p M_{R'}) < R_1.
\end{aligned}$$

This implies that $\widetilde{T}_p(\bar{B}(\theta, R_1)) \subset B(\theta, R_1)$. Consequently,

$$\deg(I - \widetilde{T}_p, B(\theta, R_1), \theta) = 1. \tag{2.9}$$

Then $\widetilde{T}_p$ has fixed points in $B(\theta, R_1)$. Now we show that $x_0 \in \Omega$ whenever $x_0 \in \bar{B}(\theta, R_1)$ with $\widetilde{T}_p x_0 = x_0$. We need only to show that $x_0 \succ \alpha_1$ and $x_0 \succ \alpha_2$. Note that α_1 and α_2 are strict lower solutions of (1.1), then we need to show

$$x_0(t) \geqslant \bar{\alpha}(t), \quad t \in [0,1]. \tag{2.10}$$

Assume on the contrary that (2.10) does not hold. Then there exists $t_0 \in [0,1]$ such that

$$\bar{\alpha}(t_0) - x_0(t_0) = \max_{t\in[0,1]} (\bar{\alpha}(t) - x_0(t)) > 0.$$

Since x_0 is a fixed point of $\widetilde{T}_p$ and α_1, α_2 are strict lower solutions of (1.1), we easily see that $t_0 \in (0,1)$. Thus, there exists an interval $I_+ \subset (0,1)$ such that $\bar{\alpha}(t) > x_0(t)$ for all $t \in I_+$ and $\bar{\alpha}(t) = x_0(t)$ ($\forall t \in \partial I_+$). Let

$$(\bar{\alpha}(t) - x_0(t))^* = \begin{cases} \bar{\alpha}(t) - x_0(t), & \forall t \in I_+, \\ 0, & \forall t \in [0,1] \backslash I_+. \end{cases}$$

Then we have

$$\begin{aligned} &\int_{[0,1]} \varphi_p(x_0'(t)) \frac{d}{dt} (\bar{\alpha}(t) - x_0(t))^* dt \\ &= \int_{[0,1]} f(t, g(t, x_0(t)), \frac{d}{dt} g(t, x_0(t))(\bar{\alpha}(t) - x_0(t))^* dt \\ &= \int_{[0,1]} f(t, \bar{\alpha}(t), \bar{\alpha}'(t))(\bar{\alpha}(t) - x_0(t))^* dt. \end{aligned} \tag{2.11}$$

Since $\{t \in [0,1] | \alpha_1(t) = \alpha_2(t), \alpha_1'(t) \neq \alpha_2'(t)\}$ is a subset of $[0,1]$ which contains at most finite elements, for simplicity we assume that $\{t \in [0,1] | \alpha_1(t) = \alpha_2(t), \alpha_1'(t) \neq \alpha_2'(t)\} = \{t_1\}$, $t_1 \in (0,1)$ and $\alpha_1'(t_1) < \alpha_2'(t_1)$. Then we have $\bar{\alpha}(t) = \alpha_1(t)$ for all $t \in [0, t_1]$, and $\bar{\alpha}(t) = \alpha_2(t)$ for all $t \in [t_1, 1]$. From Lemma 2.7 we see that $\varphi_p(\bar{\alpha}'(t))$ is absolutely continuous on $[0, t_1]$ and $[t_1, 1]$, respectively. Using the formula of Integrating by part, we have

$$\begin{aligned} &\int_{[0,1]} \varphi_p(\bar{\alpha}'(t)) \frac{d}{dt} (\bar{\alpha}(t) - x_0(t))^* dt \\ &= \varphi_p(\bar{\alpha}'(t))(\bar{\alpha}(t) - x_0(t))^* \Big|_0^{t_1} + \varphi_p(\bar{\alpha}'(t))(\bar{\alpha}(t) - x_0(t))^* \Big|_{t_1}^1 \\ &\quad - \Big(\int_{[0,t_1)} + \int_{[t_1,1]} \Big) \frac{d}{dt} (\varphi_p(\bar{\alpha}'(t))(\bar{\alpha}(t) - x_0(t))^* dt \\ &= (\bar{\alpha}(t_1) - x_0(t_1))^* [\varphi_p(\alpha_1'(t_1)) - \varphi_p(\alpha_2'(t_1))] \\ &\quad - \Big(\int_{[0,t_1)} + \int_{[t_1,1]} \Big) \frac{d}{dt} (\varphi_p(\bar{\alpha}'(t))(\bar{\alpha}(t) - x_0(t))^* dt \\ &\leqslant -\Big(\int_{[0,t_1)} + \int_{[t_1,1]} \Big) \frac{d}{dt} (\varphi_p(\bar{\alpha}'(t))(\bar{\alpha}(t) - x_0(t))^* dt. \end{aligned} \tag{2.12}$$

Now since α_1 is a strict lower solution of (1.1), we have

$$\begin{aligned} \int_{[0,t_1)} -\frac{d}{dt} \varphi_p(\bar{\alpha}'(t))(\bar{\alpha}(t) - x_0(t))^* dt &= -\int_{[0,t_1)} \frac{d}{dt} \varphi_p(\alpha_1'(t))(\alpha_1(t) - x_0(t))^* dt \\ &\leqslant \int_{[0,t_1)} f(t, \alpha_1(t), \alpha_1'(t))(\alpha_1(t) - x_0(t))^* dt \\ &= \int_{[0,t_1)} f(t, \bar{\alpha}(t), \bar{\alpha}'(t))(\bar{\alpha}(t) - x_0(t))^* dt. \end{aligned} \tag{2.13}$$

In the same way, we have

$$\int_{[t_1,1]} -\frac{d}{dt}\varphi_p(\bar{\alpha}'(t))(\bar{\alpha}(t)-x_0(t))^* dt \leqslant \int_{[t_1,1]} f(t,\bar{\alpha}(t),\bar{\alpha}'(t))(\bar{\alpha}(t)-x_0(t))^* dt. \tag{2.14}$$

From (2.12)-(2.14) it follows that

$$\int_{[0,1]} \varphi_p(\bar{\alpha}'(t))\frac{d}{dt}(\bar{\alpha}(t)-x_0(t))^* dt \leqslant \int_{[0,1]} f(t,\bar{\alpha}(t),\bar{\alpha}'(t))(\bar{\alpha}(t)-x_0(t))^* dt. \tag{2.15}$$

By (2.11) and (2.15) we have

$$\int_{[0,1]} [\varphi_p(x_0'(t)) - \varphi_p(\bar{\alpha}'(t))](\bar{\alpha}'(t) - x_0'(t))dt \geqslant 0.$$

This is a contradiction to that $s \mapsto \varphi_p(s)$ is strictly increasing, which proves that $x_0 \succ \alpha_1$ and $x_0 \succ \alpha_2$. Now by the properties of the Leray-Schauder degree and (2.9) we have

$$\deg(I - \widetilde{T}_p, \Omega, \theta) = 1. \tag{2.16}$$

The assertion now follows from the fact that T_p and $\widetilde{T}_p$ coincides in $\bar{\Omega}$. The proof is complete.

□

As in the proof of Lemma 2.11 we have the following result.

Lemma 2.12. *Suppose that* (H1) *holds, β_1, β_2 are strict upper solutions of* (1.1) *such that $\beta_1(t) \equiv \beta_2(t)$ or the set $\{t \in [0,1] | \beta_1(t) = \beta_2(t), \beta_1'(t) \neq \beta_2'(t)\}$ contains at most finite elements. Then there exists $R_0 > 0$ such that for each $R_1 \geqslant R_0$, $T_p(\bar{B}(\theta, R_1)) \subset B(\theta, R_1)$, $\beta_1, \beta_2 \in B(\theta, R_1)$ and*

$$\deg(I - T_p, \Omega, \theta) = 1,$$

where $\Omega = \{x \in \bar{B}(\theta, R_1) | x \prec \beta_1, x \prec \beta_2\}$.

Lemma 2.13. *Suppose that* (H1) *and* (H2) *hold. Let $R_1 > 0$,*

$$S_{R_1} = \{x \in C_0^1[0,1] | x \text{ is a solution of } (1.1) \text{ and } \|x\| \leqslant R_1\},$$

$S_{R_1}^+ = \{x \in S_{R_1} | x > \theta\}$ and $S_{R_1}^- = \{x \in S_{R_1} | x < \theta\}$. Then there exists $\zeta_{R_1} > 0$ such that

$$S_{R_1}^+ \geqslant \zeta_{R_1} e, \quad S_{R_1}^- \leqslant -\zeta_{R_1} e.$$

Proof. Let $x_0 \in S_{R_1}^+$ be fixed at present. Take $t_0 \in (0,1)$ such that $x_0(t_0) = \|x_0\|_0$. Then we have

$$\begin{gathered} -(\varphi_p(x_0'(t)))' = f(t, x_0(t), x_0'(t)) \quad \text{a.e. } t \in (0,1), \\ x_0(1) = 0, \quad x_0(t_0) = \|x_0\|_0. \end{gathered} \tag{2.17}$$

Assume that $v \in C^1[0,1]$ satisfying

$$\begin{gathered} -(\varphi_p(v'(t)))' = 0 \quad \text{a.e. } t \in (0,1), \\ v(1) = 0, \quad v(t_0) = \|x_0\|_0. \end{gathered} \tag{2.18}$$

Now we show that

$$x_0(t) \geqslant v(t), \quad \forall t \in (t_0, 1). \tag{2.19}$$

Assume (2.19) is not true. Let $\omega(t) = x_0(t) - v(t)$ for all $t \in [t_0, 1]$. Then there exists $t^* \in (t_0, 1)$ such that $\omega(t^*) = \min_{t\in[t_0,1]} \omega(t) < 0$. Take $[t_1, t_2] \subset [t_0, 1]$ such that $t^* \in (t_1, t_2)$, $\omega(t_1) = \omega(t_2) = 0$, and

$$\omega(t) < 0, \quad \forall t \in (t_1, t_2). \tag{2.20}$$

By (2.17) and (2.18) we have

$$(\varphi_p(x_0'(t)))' - (\varphi_p(v'(t)))' = -f(t, x_0(t), x_0'(t)) \leqslant 0, \quad \text{a.e.} \quad t \in (t_1, t_2). \tag{2.21}$$

By (2.20) and (2.21), we have

$$\int_{t_1}^{t_2} [(\varphi_p(x_0'(t)))' - (\varphi_p(v'(t)))']\omega(t)dt > 0. \tag{2.22}$$

On the other hand, by (2.20) and the inequality

$$(\varphi_p(b) - \varphi_p(a))(b - a) \geqslant 0, \forall b, a \in \mathbb{R}^1, \tag{2.23}$$

we have

$$\int_{t_1}^{t_2} [(\varphi_p(x_0'(t)))' - (\varphi_p(v'(t)))']\omega(t)dt = -\int_{t_1}^{t_2} [(\varphi_p(x_0'(t))) - (\varphi_p(v'(t)))]\omega'(t)dt \leqslant 0,$$

which contradicts to (2.22). This implies that (2.19) holds. Obviously, we have

$$v(t) = \frac{\|x_0\|_0}{1 - t_0}(1 - t), \quad t \in [t_0, 1].$$

Thus, we have

$$x_0(t) \geqslant \frac{\|x_0\|_0}{1 - t_0}(1 - t) \geqslant \|x_0\|_0 t(1 - t), \quad t \in [t_0, 1]. \tag{2.24}$$

Similarly, we can show that

$$x_0(t) \geqslant \|x_0\|_0 e(t), \quad \forall t \in [0, t_0]. \tag{2.25}$$

By (2.24) and (2.25) we have $x_0 \geqslant \|x_0\|_0 e$. Thus, we have $x \geqslant \frac{\|x_0\|_0}{2} e$ for any $x \in B(x_0, \frac{\|x_0\|}{4})$. Obviously, $\left\{B(x, \frac{\|x\|}{4}) | x \in S^+_{R_1}\right\}$ is an open cover of the set $S^+_{R_1}$. Since $T_p(S^+_{R_1}) = S^+_{R_1}$ and $T_p : C^1_0[0, 1] \to C^1_0[0, 1]$ is completely continuous, then $S^+_{R_1}$ is a compact set. Therefore, there exist finite subsets of $\{B(x, \frac{\|x\|}{4}) : x \in S^+_{R_1}\}$, assume without loss of generality that

$$B\big(x_1, \frac{\|x_1\|}{4}\big), B\big(x_2, \frac{\|x_2\|}{4}\big), \dots, B\big(x_n, \frac{\|x_n\|}{4}\big)$$

such that

$$\cup_{i=1}^n B\big(x_i, \frac{\|x_i\|}{4}\big) \supset S^+_{R_1}.$$

Let

$$\varepsilon^+ = \min\big\{\frac{\|x_1\|_0}{2}, \frac{\|x_2\|_0}{2}, \dots, \frac{\|x_n\|_0}{2}\big\} > 0.$$

Then we have $S^+_{R_1} \geqslant \varepsilon^+ e$. Similarly, we can prove that there exists $\varepsilon^- > 0$ such that $S^-_{R_1} \leqslant -\varepsilon^- e$. Let $\zeta_{R_1} = \min\{\varepsilon^+, \varepsilon^-\}$. Then the conclusion holds. The proof is complete. □

3. Main Results

Theorem 3.1. *Suppose that* (H1) *and* (H2) *hold,* $\beta_0 \in (\mu_{2k_0}(p), \mu_{2k_0+1}(p))$ *for some positive integer* k_0*. Then* (1.1) *has at least one sign-changing solution. Moreover,* (1.1) *has at least one positive solution and one negative solution.*

Proof. By (H1), Lemma 2.11 and Lemma 2.12, there exists $R_1 > 0$ such that $T_p(\bar{B}(\theta, R_1)) \subset B(\theta, R_1)$, and so

$$\deg(I - T_p, B(\theta, R_1), \theta) = 1. \tag{3.1}$$

Let $\{\mu_k(p) | k \in N^+\}$ be the sequence of eigenvalues of the problem (2.4) and ϕ_k the eigenfunction of (2.4) corresponding to the eigenvalue $\mu_k(p)$. From (iii) of Lemma 2.8, ϕ_1 is a non-negative function on $[0, 1]$. Take $\varepsilon_0 > 0$ small enough such that $\beta_0 - \varepsilon_0 > \mu_1(p)$. By (H2), there exists $\delta_1 > 0$ such that

$$f(t, x, y) \geqslant (\beta_0 - \varepsilon_0)\varphi_p(x), \quad t \in [0, 1],\ |x| \leqslant \delta_1,\ |y| \leqslant R_*,\ x \geqslant 0, \tag{3.2}$$

$$f(t, x, y) \leqslant (\beta_0 - \varepsilon_0)\varphi_p(x), \quad t \in [0, 1],\ |x| \leqslant \delta_1,\ |y| \leqslant R_*,\ x \leqslant 0. \tag{3.3}$$

Assume that $R_* < R_1$, where R_* as in (H2). Take $\delta_2 > 0$ small enough such that $\|\delta\phi_1\| < \min\{\delta_1, R_*, R_1\}$ for each $\delta \in (0, \delta_2]$. Then from (3.2) we have for any $\delta \in (0, \delta_2)$,

$$\begin{aligned}(\varphi_p(\delta\phi_1'))' + f(t, \delta\phi_1, \delta\phi_1') &\geqslant (\varphi_p(\delta\phi_1'))' + (\beta_0 - \varepsilon_0)\varphi_p(\delta\phi_1)\\ &= \delta^{p-1}[(\varphi_p(\phi_1'))' + (\beta_0 - \varepsilon_0)\varphi_p(\phi_1)]\\ &= \delta^{p-1}(\beta_0 - \varepsilon_0 - \mu_1(p))\varphi_p(\phi_1) > 0,\ \text{a.e. } t \in (0, 1).\end{aligned} \tag{3.4}$$

and

$$\delta\phi_1(0) = \delta\phi_1(1) = 0. \tag{3.5}$$

From (3.4) and (3.5), we see that $\delta\phi_1$ is a lower solution of (1.1). Similarly, by (3.3) we can easily see that $-\delta\phi_1$ is an upper solution of (1.1) for each $\delta \in (0, \delta_2)$. Let

$$S_{R_1} = \{x \in C_0^1[0, 1] | x \text{ is a solution of (1.1) and } \|x\| < R_1\},$$

$S_{R_1}^+ = \{x \in S_{R_1} | x > \theta\}$ and $S_{R_1}^- = \{x \in S_{R_1} | x < \theta\}$. By Lemma 2.13, there exists $\zeta_{R_1} > 0$ such that

$$S_{R_1}^+ \geqslant \zeta_{R_1} e, \quad S_{R_1}^- \leqslant -\zeta_{R_1} e. \tag{3.6}$$

Since $\phi_1 \in C_0^1[0, 1]$ satisfies

$$\begin{gathered}(\varphi_p(\phi_1'))' + \mu_1(p)\varphi_p(\phi_1) = 0 \quad \text{a.e. } t \in (0, 1),\\ \phi_1(0) = \phi_1(1) = 0,\end{gathered} \tag{3.7}$$

by Rolle's Theorem, there exists $t^* \in (0, 1)$ such that $\phi_1'(t^*) = 0$ and

$$\begin{aligned}\phi_1(t) &= \int_t^1 \varphi_p^{-1}\Big(\int_{t^*}^s \mu_1(p)\varphi_p(\phi_1(\tau))d\tau\Big)ds\\ &\leqslant (1 - t)\varphi_p^{-1}\Big(\mu_1(p)\int_0^1 \varphi_p(\phi_1(\tau))d\tau\Big)\\ &\leqslant \frac{1}{t^*}e(t)\varphi_p^{-1}\Big(\mu_1(p)\int_0^1 \varphi_p(\phi_1(\tau))d\tau\Big), \forall t \in (t^*, 1).\end{aligned} \tag{3.8}$$

Similarly, we can show that

$$\phi_1(t) \leqslant \frac{1}{1 - t^*}e(t)\varphi_p^{-1}\Big(\mu_1(p)\int_0^1 \varphi_p(\phi_1(\tau))d\tau\Big), \quad \forall t \in (0, t^*). \tag{3.9}$$

By (3.8) and (3.9) we have

$$\phi_1(t) \leqslant \frac{1}{t^*(1-t^*)} e(t)\varphi_p^{-1}\Big(\mu_1(p)\int_0^1 \varphi_p(\phi_1(\tau))d\tau\Big), \quad \forall t \in [0,1]. \tag{3.10}$$

Take

$$0 < \delta_3 < \min\Big\{\delta_2, t^*(1-t^*)\Big[\varphi_p^{-1}\big(\mu_1(p)\int_0^1 \varphi_p(\phi_1(\tau))d\tau\big)\Big]^{-1}\zeta_{R_1}\Big\}.$$

Let $u_0 = \delta_3\phi_1$ and $v_0 = -\delta_3\phi_1$. Then by (3.6) and (3.10), we see that $u_0, v_0 \in \bar{B}(\theta, R_1)$, u_0 and v_0 are strict lower and upper solutions of (1.1) in $\bar{B}(\theta, R_1)$, respectively. Moreover, we have $S^+_{R_1} \succ u_0$ and $S^-_{R_1} \prec v_0$. Let $\Omega_1 = \{x \in \bar{B}(\theta, R_1) | x \succ u_0\}$ and $\Omega_2 = \{x \in \bar{B}(\theta, R_1) | x \prec v_0\}$. By Lemmas 2.11 and 2.12 we have

$$\deg(I - T_p, \Omega_1, \theta) = 1, \tag{3.11}$$

$$\deg(I - T_p, \Omega_2, \theta) = 1. \tag{3.12}$$

Let $h(t,x,y) = f(t,x,y) - \beta_0\varphi_p(x)$ for all $(t,x,y) \in [0,1]\times\mathbb{R}^2$. By (H2) we have

$$\lim_{x\to 0} \frac{h(t,x,y)}{\varphi_p(x)} = 0 \quad \text{uniformly for } t \in [0,1] \text{ and } y \in [-R_*, R_*]. \tag{3.13}$$

For each $\tau \in [0,1]$, denote by $H(\tau,\cdot) : C_0^1[0,1] \to C_0^1[0,1]$ the solution operator of

$$\begin{gathered} -(\varphi_p(y'(t)))' = \tau\beta_0\varphi_p(x(t)) + (1-\tau)f(t,x(t),x'(t)) \quad \text{a.e. } t \in (0,1) \\ y(0) = y(1) = 0; \end{gathered} \tag{3.14}$$

that is, for $x, y \in C_0^1[0,1]$,

$$y = H(\tau, x)$$

if and only if the equality in (3.14) holds. Then $H(\cdot,\cdot) : C_0^1[0,1] \to C_0^1[0,1]$ is completely continuous. Now we will show that there exists $0 < r_0 < \min\{\|u_0\|_0, \|v_0\|_0\}$ such that

$$H(s,x) \neq x, \quad s \in [0,1],\ x \in \partial B(\theta, r_0). \tag{3.15}$$

Assume that (3.15) does not holds, then there exists $\{\tau_n\} \subset [0,1]$, $\{x_n\} \subset C_0^1[0,1]$ with $\|x_n\| > 0$ for each $n = 1, 2, \dots$ and $\|x_n\| \to 0$ as $n \to \infty$ such that $H(\tau_n, x_n) = x_n$. Obviously, $\|x_n\|_0 > 0$ for each $n = 1, 2, \dots$. Assume without loss of generality that $\tau_n \to \tau_0$ as $n \to \infty$. Then we have for each $n = 1, 2, \dots$

$$\begin{aligned} -(\varphi_p(x_n'(t)))' &= \tau_n\beta_0\varphi_p(x_n(t)) + (1-\tau_n)f(t,x_n(t),x_n'(t)) \\ &= \beta_0\varphi_p(x_n(t)) + (1-\tau_n)h(t,x_n(t),x_n'(t)) \quad \text{a.e. } t \in (0,1) \end{aligned} \tag{3.16}$$

$$x_n(0) = x_n(1) = 0. \tag{3.17}$$

Let $v_n(t) = \frac{x_n(t)}{\varphi_p(\|x_n\|_0)}$. Then by (3.16) and (3.17) we have

$$\begin{gathered} -(\varphi_p(v_n'(t)))' = \beta_0(\varphi_p(v_n(t))) + (1-\tau_n)\frac{h(t,x_n(t),x_n'(t))}{\varphi_p(\|x_n\|_0)} \quad \text{a.e. } t \in (0,1), \\ v_n(0) = v_n(1) = 0. \end{gathered} \tag{3.18}$$

Let

$$u_n(t) = \beta_0\varphi_p(v_n(t)) + (1-\tau_n)\frac{h(t,x_n(t),x_n'(t))}{\varphi_p(\|x_n\|_0)}, \quad t \in [0,1].$$

By (3.13) and (H2) we see that $\{u_n|n=1,2,\dots\}\subset L^1[0,1]$. By (3.18) and Rolle's Theorem, there exists $t_n\in(0,1)$ such that $v_n'(t_n)=0$ for each $n=1,2,\dots$. Then we have by (3.18)

$$|v_n'(t)|=\Big|\varphi_p^{-1}(\int_t^{t_n}u_n(s)ds)\Big|\leqslant\varphi_p^{-1}(\int_0^1|u_n(s)|ds),\quad t\in[0,1].$$

Thus, $\{v_n'(t)|n=1,2,\dots\}$ is a bounded set. Consequently, $\{v_n:n=1,2,\dots\}$ is a relatively compact set of $C[0,1]$. Assume without loss of generality that $v_n\to\bar{v}_0$ in $C[0,1]$ as $n\to\infty$. From (3.18) we have

$$v_n(t)=\int_0^t\varphi_p^{-1}\Big(\alpha(u_n)+\int_s^1u_n(\tau)d\tau\Big)ds,\quad t\in[0,1]. \tag{3.19}$$

where the continuous functional $\alpha(u_n)\in(0,1)$ satisfies

$$\int_0^1\varphi_p^{-1}\Big(\alpha(u_n)+\int_s^1u_n(\tau)d\tau\Big)ds=0,\quad n=1,2,\dots.$$

Assume without loss of generality that $\alpha(u_n)\to a_0$ as $n\to\infty$. Letting $n\to\infty$ in (3.19), by Lebesgue dominated convergence theorem we have

$$\bar{v}_0(t)=\int_0^t\varphi_p^{-1}\Big(a_0+\int_s^1\beta_0\varphi_p(\bar{v}_0(\tau))d\tau\Big)ds,\quad t\in[0,1].$$

Consequently, $\bar{v}_0\in C^1[0,1]$. By direct computation we have

$$-(\varphi_p(\bar{v}_0'(t)))'=\beta_0\varphi_p(\bar{v}_0(t))\quad\text{a.e. } t\in(0,1). \tag{3.20}$$

Obviously,

$$\bar{v}_0(0)=\bar{v}_0(1)=0. \tag{3.21}$$

By (3.20) and (3.21) we see that β_0 is an eigenvalue of (2.4) and $\bar{v}_0$ is the corresponding eigenfunction, which is a contradiction. Therefore, there exists $r_0>0$ small enough such that (3.15) holds. Assume without loss of generality that $u_0,v_0\notin\bar{B}(\theta,r_0)$. By the properties of the Leray-Schauder degree and Lemma 2.10 we have

$$\begin{aligned}\deg(I-T_p,B(\theta,r_0),\theta)&=\deg(I-H(0,\cdot),B(\theta,r_0),\theta)\\&=\deg(I-H(1,\cdot),B(\theta,r_0),\theta)\\&=\deg(I-T^p_{\beta_0},B(\theta,r_0),\theta)\\&=(-1)^{2k_0}=1.\end{aligned} \tag{3.22}$$

By (3.1), (3.11), (3.12) and (3.22), we have

$$\deg(T_p,\bar{B}(\theta,R_1)\backslash(\bar{B}(\theta,r_0)\cup Cl_{\bar{B}(\theta,R_1)}\Omega_1\cup Cl_{\bar{B}(\theta,R_1)}\Omega_2),\theta)=-1. \tag{3.23}$$

It follows from (3.11), (3.12) and (3.23) that T_p has at least three fixed points $x_1\in\Omega_1$, $x_2\in\Omega_2$ and $x_3\in\bar{B}(\theta,R_1)\backslash(\bar{B}(\theta,r_0)\cup Cl_{\bar{B}(\theta,R_1)}\Omega_1\cup Cl_{\bar{B}(\theta,R_1)}\Omega_2)$. Obviously x_1 is a positive solution of (1.1), x_2 is a negative solution of (1.1). Since $S^+_{R_1}\succ u_0$ and $S^-_{R_1}\prec v_0$, then $S^+_{R_1}\subset\Omega_1$ and $S^-_{R_1}\subset\Omega_2$. Therefore, x_3 is a sign-changing solution of (1.1). The proof is complete. $\square$

Now we will give some multiplicity results for sign-changing solutions of (1.1).

Theorem 3.2. *Suppose that* (H1)–(H3) *hold,* $\beta_0 > \mu_1(p)$*,* $\beta_0 \neq \mu_k(p)$ *for each* $k = 1, 2, \dots$. *Moreover, there exists* $\bar{\delta}_0 > 0$ *such that both* $\{t \in [0,1] | \delta\phi_1(t) = u_1(t)\}$ *and* $\{t \in [0,1] | -\delta\phi_1(t) = v_1(t)\}$ *contain at most finite elements for each* $\delta \in (0, \bar{\delta}_0)$*. Then* (1.1) *has at least four sign-changing solutions. Moreover,* (1.1) *has at least one positive solution and one negative solution.*

Proof. From (H1), there exists $R_1 > 0$ such that $T_p(\bar{B}(\theta, R_1)) \subset B(\theta, R_1)$ and so (3.1) holds. Since $\beta_0 > \mu_1(p)$ and $\beta_0 \neq \mu_k(p)$ for each $k = 1, 2, \dots$, in the same way as the proof of Theorem 3.1, we see that there exists $0 < \delta_2 < \bar{\delta}_0$ such that for any $\delta \in (0, \delta_2)$, $\delta\phi_1$ is a lower solution of (1.1) and $-\delta\phi_1$ is an upper solution of (1.1). Let $S^+_{R_1}$ and $S^-_{R_1}$ be defined as Theorem 3.1. Then by Lemma 2.13, there exists $\zeta_{R_1} > 0$ such that (3.6) holds. In the same way as the proof of Theorem 3.1, we can take $\delta_3 > 0$ small enough such that $u_0 \in \bar{B}(\theta, R_1)$ and $v_0 \in \bar{B}(\theta, R_1)$, where $u_0 := \delta_3\phi_1$ and $v_0 := -\delta_3\phi_1$. Moreover, u_0 and v_0 are strict lower and upper solutions of (1.1), respectively, and $S^+_{R_1} \succ u_0$, $S^-_{R_1} \prec v_0$. Also, assume $\delta_3 > 0$ small enough such that $u_0 \not\geq u_1$ and $v_0 \not\leq v_1$. Define the subsets $\Omega_1, \Omega_2, \Omega_3$ and Ω_4 of $C^1_0[0,1]$ by

$$\Omega_1 = \{x \in B(\theta, R_1) : x \succ u_0\}, \quad \Omega_2 = \{x \in B(\theta, R_1) : x \prec v_0\},$$
$$\Omega_3 = \{x \in B(\theta, R_1) : x \prec v_1\}, \quad \Omega_4 = \{x \in B(\theta, R_1) : x \succ u_1\}.$$

Then $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ are four closed convex subsets of $C^1_0[0,1]$. Let

$$O_{2,3} = \Omega_2 \cap \Omega_3, \quad \Omega_{3,4} = \Omega_3 \cap \Omega_4, \quad O_{4,1} = \Omega_4 \cap \Omega_1.$$

By Lemmas 2.11 and 2.12 we have

$$\deg(I - T_p, \Omega_i, \theta) = 1, \quad i = 1, 2, 3, 4, \tag{3.24}$$
$$\deg(I - T_p, O_{2,3}, \theta) = 1, \tag{3.25}$$
$$\deg(I - T_p, O_{3,4}, \theta) = 1, \tag{3.26}$$
$$\deg(I - T_p, O_{4,1}, \theta) = 1. \tag{3.27}$$

Since $\beta_0 > \mu_1(p), \beta_0 \neq \mu_k(p), k = 1, 2, \dots$, then by a similar way as that of the proof of Theorem 3.1 we see that, there exists $r_0 > 0$ small enough such that $B(\theta, r_0) \cap \Omega_i = \emptyset (i = 1, 2, 3, 4)$ and

$$\deg(I - T_p, B(\theta, r_0), \theta) = (-1)^{k_0} = \pm 1, \tag{3.28}$$

where k_0 is the sum of all algebraic multiplicities of all eigenvalues $\mu_k(p)$ of (E^p_λ) with $\beta_0 > \mu_k(p)$. Let

$$O_1 = \Omega_3 \backslash (Cl_{\bar{B}(\theta,R_1)} O_{2,3} \cup Cl_{\bar{B}(\theta,R_1)} O_{3,4}),$$
$$O_2 = \Omega_4 \backslash (Cl_{\bar{B}(\theta,R_1)} O_{3,4} \cup Cl_{\bar{B}(\theta,R_1)} O_{4,1}).$$

Then, by (3.24)-(3.27) we have

$$\deg(I - T_p, O_1, \theta) = 1 - 1 - 1 = -1, \tag{3.29}$$
$$\deg(I - T_p, O_2, \theta) = 1 - 1 - 1 = -1. \tag{3.30}$$

It follows from (3.1), (3.24), (3.28)-(3.30) that

$$\begin{aligned} \deg\Big(&I - T_p, \bar{B}(\theta, R_1) \backslash (Cl_{\bar{B}(\theta,R_1)} \Omega_1 \cup Cl_{\bar{B}(\theta,R_1)} O_1 \cup Cl_{\bar{B}(\theta,R_1)} \Omega_{3,4} \cup Cl_{\bar{B}(\theta,R_1)} O_2 \\ &\cup Cl_{\bar{B}(\theta,R_1)} \Omega_2 \cup \bar{B}(\theta, r_0)), \theta\Big) = 1 - 1 - (-1) - 1 - (-1) - 1 - (\pm 1) = \mp 1. \end{aligned} \tag{3.31}$$

It follows from (3.24), (3.26), (3.29), (3.30) and (3.31) that T_p has fixed points $x_1 \in \Omega_1$, $x_2 \in \Omega_2$, $x_3 \in O_1$ $x_4 \in O_2$, $x_5 \in \Omega_{3,4}$ and $x_6 \in \bar{B}(\theta, R_1)\backslash(Cl_{\bar{B}(\theta,R_1)}\Omega_1 \cup Cl_{\bar{B}(\theta,R_1)}O_1 \cup Cl_{\bar{B}(\theta,R_1)}\Omega_{3,4} \cup Cl_{\bar{B}(\theta,R_1)}O_2 \cup Cl_{\bar{B}(\theta,R_1)}\Omega_2 \cup \bar{B}(\theta, r_0))$. It is easy to see that x_1 is a positive solution of (1.1), x_2 is a negative solution of (1.1), x_3, x_4, x_5, x_6 are four sign-changing solutions of (1.1). The proof is complete. □

Remark 3.3. To show multiplicity results for sign-changing solutions of (1.1) in Theorem 3.2 we constructed a pair of lower and upper solutions u_0 and v_0 which satisfy $u_0 \nleqslant v_0$. We call this pair of lower and upper solutions is non-well ordered. For other discussions concerning the non-well ordered upper and lower solutions, the reader is refereed to [3, 5.4B].

Remark 3.4. In Theorem 3.2 we obtained not only multiplicity results for sign-changing solutions of (1.1) but also the existence results for positive solutions as well as negative solution of (1.1).

Theorem 3.5. *Suppose that* (H1)–(H3) *hold,* $\beta_0 < \mu_1(p)$. *Moreover, there exists* $\bar{\delta}_0 > 0$ *such that both* $\{t \in [0,1] | \delta\phi_1(t) = v_1(t)\}$ *and* $\{t \in [0,1] | -\delta\phi_1(t) = u_1(t)\}$ *contain at most finite elements for each* $\delta \in (0, \bar{\delta}_0)$. *Then* (1.1) *has at least four sign-changing solutions. Moreover,* (1.1) *has at least two positive solutions and two negative solutions.*

Proof. By (H1), there exists $R_1 > 0$ such that $T_p(\bar{B}(\theta, R_1)) \subset B(\theta, R_1)$ and so (3.1) holds. Let $S^+_{R_1}$ and $S^-_{R_1}$ be defined as Theorem 3.1. By Lemma 2.13, there exists $\zeta_{R_1} > 0$ such that (3.6) holds. Since $\beta_0 < \mu_1(p)$, in the same way as that of Theorem 3.1 we can show that, there exists $\bar{\delta}_0 > \delta_2 > 0$ such that for any $\delta \in (0, \delta_2)$, $-\delta\phi_1$ is a lower solution of (1.1) and $\delta\phi_1$ is an upper solution of (1.1). Also by a similar argument as the proof of (3.15) we can show that, there exists $r_0 > 0$ small enough such that θ is the unique fixed point of T_p in $\bar{B}(\theta, r_0)$, and for any $0 < r \leqslant r_0$,

$$\deg(I - T_p, B(\theta, r), \theta) = 1. \tag{3.32}$$

Let

$$S_i = \{x \in C^1_0[0,1] : x(t) \text{ has exactly } i-1 \text{ simple zeros on } (0,1)\},$$
$$S^+_i = \{x \in S_i : \lim_{t\to 0^+} \operatorname{sign} x(t) = 1\}, \quad S^-_i = S_i \backslash S^+_i, \quad i = 1, 2, \dots.$$

Then we have $S^+_{R_1} = S^+_1 \cap \bar{B}(\theta, R_1)$, $S^-_{R_1} = S^-_1 \cap \bar{B}(\theta, R_1)$ and $S_{R_1} \subset \big(\cup_{i=1}^\infty S_i\big) \cap \bar{B}(\theta, R_1)$. Moreover, for each $i = 1, 2, \dots$, S_i is an open subset of $C^1_0[0,1]$. We say that there exists $\delta_3 \in (0, \delta_2)$ small enough such that

$$\{x \in C^1_0[0,1] : -\delta_3\phi_1 \leqslant x, \|x\| \leqslant R_1\} \cap (S_{R_1} \backslash \{\theta\}) \cap \Big((\cup_{i=2}^\infty S_i) \cup S^-_{R_1}\Big) = \emptyset, \tag{3.33}$$

$$\{x \in C^1_0[0,1] : x \leqslant \delta_3\phi_1, \|x\| \leqslant R_1\} \cap (S_{R_1} \backslash \{\theta\}) \cap \Big((\cup_{i=2}^\infty S_i) \cup S^+_{R_1}\Big) = \emptyset. \tag{3.34}$$

We prove only (3.33). In a similar way we can prove (3.34). If (3.33) does not hold, then there exists a sequence of positive numbers $\{\bar{\delta}_n\}$ with $\bar{\delta}_n \to 0$ as $n \to \infty$ such that for each $n = 1, 2, \dots$,

$$\{x \in C^1_0[0,1] : -\delta_n\phi_1 \leqslant x, \|x\| \leqslant R_1\} \cap (S_{R_1} \backslash \{\theta\}) \cap \Big((\cup_{i=2}^\infty S_i) \cup S^-_{R_1}\Big) \neq \emptyset.$$

For each $n = 1, 2, \dots$, take

$$x_n \in \{x \in C_0^1[0,1] : -\delta_n\phi_1 \leqslant x, \|x\| \leqslant R_1\} \cap (S_{R_1} \backslash \{\theta\}) \cap \Big((\cup_{i=2}^{\infty} S_i) \cup S_{R_1}^- \Big).$$

Obviously, $\|x_n\| \geqslant r_0$ for each $n = 1, 2, \dots$. Let $D = \{x_n | n = 1, 2, \dots\}$. Then we have $D = T_p(D)$. Therefore, D is a relatively compact subset of $C_0^1[0,1]$. Assume without loss of generality that $x_n \to x_0$ as $n \to \infty$ for some $x_0 \in C_0^1[0,1]$. Obviously, x_0 is a solution of (1.1) and $\|x_0\| \geqslant r_0$, and thus $x_0 \in \big(\cup_{i=1}^{\infty} S_i \big) \cap \bar{B}(\theta, R_1)$. Note that $-\bar{\delta}_n\phi_1 \leqslant x_n$, letting $n \to \infty$ then we have $x_0 \in S_1^+ \cap \bar{B}(\theta, R_1)$. Since S_1^+ is an open subset of $C_0^1[0,1]$, then there exists $r_1 > 0$ such that $B(x_0, r_1) \subset S_1^+$. Now since $x_n \to x_0$ as $n \to +\infty$, then we can take n_0 large enough such that $x_{n_0} \in B(x_0, r_1) \subset S_1^+$, which contradicts to

$$x_n \in (\cup_{i=2}^{\infty} S_i) \cup S_{R_1}^-$$

for each $n = 1, 2, \dots$. Therefore, (3.33) and (3.34) hold. Take $0 < \delta_4 < \delta_3$. Then $-\delta_4\phi_1$ is a strict lower solution of (1.1) and $\delta_4\phi_1$ is a strict upper solution of (1.1). Also, assume that $\delta_4 > 0$ small enough such that $-\delta_4\phi_1 \nleqslant v_1$, $\delta_4\phi_1 \ngeqslant u_1$ and $-\delta_4\phi_1, \delta_4\phi_1 \in \bar{B}(\theta, R_1)$. Let $u_0 = -\delta_4\phi_1$ and $v_0 = \delta_4\phi_1$. Let the subsets $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ of $C_0^1[0,1]$ be defined by

$$\begin{aligned} &\Omega_1 = \{x \in \bar{B}(\theta, R_1) : x \succ u_0\}, \quad \Omega_2 = \{x \in \bar{B}(\theta, R_1) : x \prec v_0\}, \\ &\Omega_3 = \{x \in \bar{B}(\theta, R_1) : x \prec v_1\}, \quad \Omega_4 = \{x \in \bar{B}(\theta, R_1) : x \succ u_1\}. \end{aligned}$$

Let $O_{1,2} = \Omega_1 \cap \Omega_2$, $O_{2,3} = \Omega_2 \cap \Omega_3$, $O_{3,4} = \Omega_3 \cap \Omega_4$ and $O_{4,1} = \Omega_4 \cap \Omega_1$. Then $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ and $O_{1,2}, O_{2,3}, O_{3,4}, O_{4,1}$ are nonempty open subsets of $\bar{B}(\theta, R_1)$. It follows from Lemmas 2.11 and 2.12 that

$$\deg(I - T_p, \Omega_1, \theta) = 1, \tag{3.35}$$
$$\deg(I - T_p, \Omega_2, \theta) = 1, \tag{3.36}$$
$$\deg(I - T_p, \Omega_3, \theta) = 1, \tag{3.37}$$
$$\deg(I - T_p, \Omega_4, \theta) = 1, \tag{3.38}$$
$$\deg(I - T_p, O_{1,2}, \theta) = 1, \tag{3.39}$$
$$\deg(I - T_p, O_{2,3}, \theta) = 1, \tag{3.40}$$
$$\deg(I - T_p, O_{3,4}, \theta) = 1, \tag{3.41}$$
$$\deg(I - T_p, O_{4,1}, \theta) = 1. \tag{3.42}$$

Let

$$\begin{aligned} O_1 &= \Omega_1 \backslash (Cl_{\bar{B}(\theta,R_1)} O_{1,2} \cup Cl_{\bar{B}(\theta,R_1)} O_{4,1}), \\ O_2 &= \Omega_2 \backslash (Cl_{\bar{B}(\theta,R_1)} O_{1,2} \cup Cl_{\bar{B}(\theta,R_1)} O_{2,3}), \\ O_3 &= \Omega_3 \backslash (Cl_{\bar{B}(\theta,R_1)} O_{2,3} \cup Cl_{\bar{B}(\theta,R_1)} O_{3,4}), \\ O_4 &= \Omega_4 \backslash (Cl_{\bar{B}(\theta,R_1)} O_{3,4} \cup Cl_{\bar{B}(\theta,R_1)} O_{4,1}). \end{aligned}$$

Then by (3.35)-(3.42) we have

$$\deg(I - T_p, O_1, \theta) = -1, \tag{3.43}$$
$$\deg(I - T_p, O_2, \theta) = -1, \tag{3.44}$$
$$\deg(I - T_p, O_3, \theta) = -1, \tag{3.45}$$
$$\deg(I - T_p, O_4, \theta) = -1. \tag{3.46}$$

It follows from (3.36), (3.38), (3.43),(3.45) that

$$\begin{aligned} &\deg\Big(I-T_p,\bar{B}(\theta,R_1)\backslash\Big(Cl_{\bar{B}(\theta,R_1)}O_1\cup Cl_{\bar{B}(\theta,R_1)}O_3\cup Cl_{\bar{B}(\theta,R_1)}\Omega_2\\ &\cup Cl_{\bar{B}(\theta,R_1)}\Omega_4\Big),\theta\Big)\\ &=1-(-1)-(-1)-1-1=1. \end{aligned}\tag{3.47}$$

From (3.35)-(3.47), T_p has fixed points $x_1\in O_{3,4}$, $x_2\in O_4$, $x_3\in O_3$,

$$x_4\in\bar{B}(\theta,R_1)\backslash(Cl_{\bar{B}(\theta,R_1)}O_1\cup Cl_{\bar{B}(\theta,R_1)}O_3\cup Cl_{\bar{B}(\theta,R_1)}\Omega_2\cup Cl_{\bar{B}(\theta,R_1)}\Omega_4).$$

Then $x_1,\dots,x_4$ are four sign-changing solutions of (1.1). From (3.42) and (3.43), T_p has fixed points $x_5\in O_{4,1}$, $x_6\in O_1$. Obviously, $x_5\geqslant u_0$, $x_6\geqslant u_0$ and $x_5\neq\theta$, $x_6\neq\theta$. Then we see from (3.33) that x_5 and x_6 are two positive solutions of (1.1). Similarly we can show that there exist $x_7\in O_{3,4}$ and $x_8\in O_2$, and x_7, x_8 are two negative solutions of (1.1). The proof is complete. □

Now we study the existence and multiplicity of sign-changing solutions of (1.1) when f has jumping nonlinearity at zero. Let us first introduce the following conditions.

(H4) There exist $R_*,\beta_+>0$ such that

$$\lim_{x\to0^+}\frac{f(t,x,y)}{\varphi_p(x)}=\beta_+\quad\text{uniformly for } t\in[0,1] \text{ and } y\in[-R_*,R_*].$$

(H5) There exist $R_*,\beta_->0$ such that

$$\lim_{x\to0^-,\ x<0}\frac{f(t,x,y)}{\varphi_p(x)}=\beta_-\quad\text{uniformly for } t\in[0,1] \text{ and } y\in[-R_*,R_*].$$

In the same way as the proof of Theorems 3.1, 3.2 and 3.5, we can prove the following Theorems 3.6–3.12. For brevity, we only give the sketch of the proof of Theorem 3.6.

Theorem 3.6. *Suppose that* (H1), (H3), (H4) *hold, and* $\beta_+>\mu_1(p)$. *Moreover, there exists* $\bar{\delta}_0>0$ *such that* $\{t\in[0,1]|\delta\phi_1(t)=u_1(t)\}$ *contains at most finite elements for each* $\delta\in(0,\bar{\delta}_0)$. *Then* (1.1) *has at least two sign-changing solutions. Moreover,* (1.1) *has at least one positive solution.*

Theorem 3.7. *Suppose that* (H1), (H3), (H4) *hold,* $\beta_+<\mu_1(p)$. *Moreover, there exists* $\bar{\delta}_0>0$ *such that* $\{t\in[0,1]|\delta\phi_1(t)=v_1(t)\}$ *contains at most finite elements for each* $\delta\in(0,\bar{\delta}_0)$. *Then* (1.1) *has at least two sign-changing solutions. Moreover,* (1.1) *has at least one negative solution.*

Theorem 3.8. *Suppose that* (H1), (H3), (H5) *hold,* $\beta_->\mu_1(p)$. *Moreover, there exists* $\bar{\delta}_0>0$ *such that* $\{t\in[0,1]|-\delta\phi_1(t)=v_1(t)\}$ *contains at most finite elements for each* $\delta\in(0,\bar{\delta}_0)$. *Then* (1.1) *has at least two sign-changing solutions. Moreover,* (1.1) *has at least one negative solution.*

Theorem 3.9. *Suppose that* (H1), (H3), (H5) *hold,* $\beta_-<\mu_1(p)$. *Moreover, there exists* $\bar{\delta}_0>0$ *such that* $\{t\in[0,1]|-\delta\phi_1(t)=u_1(t)\}$ *contains at most finite elements for each* $\delta\in(0,\bar{\delta}_0)$. *Then* (1.1) *has at least two sign-changing solutions. Moreover,* (1.1) *has at least one positive solution.*

Theorem 3.10. *Suppose that* (H1), (H3), (H4), (H5) *hold,* $\beta_- > \mu_1(p)$*, and* $\beta_+ > \mu_1(p)$*. Moreover, there exists* $\bar{\delta}_0 > 0$ *such that both* $\{t \in [0,1] : -\delta\phi_1(t) = v_1(t)\}$ *and* $\{t \in [0,1] : \delta\phi_1(t) = u_1(t)\}$ *contain at most finite elements for each* $\delta \in (0, \bar{\delta}_0)$*. Then* (1.1) *has at least three sign-changing solutions. Moreover,* (1.1) *has at least one positive solution and one negative solution.*

Theorem 3.11. *Suppose that* (H1), (H3), (H4), (H5) *hold,* $\beta_- < \mu_1(p)$, $\beta_+ < \mu_1(p)$*. Moreover, there exists* $\bar{\delta}_0 > 0$ *such that both* $\{t \in [0,1] | \delta\phi_1(t) = v_1(t)\}$ *and* $\{t \in [0,1] | -\delta\phi_1(t) = u_1(t)\}$ *contain at most finite elements for each* $\delta \in (0, \bar{\delta}_0)$*. Then* (1.1) *has at least four sign-changing solutions. Moreover,* (1.1) *has at least two positive solutions and two negative solutions.*

Theorem 3.12. *Suppose that* (H3) *holds,* f *is a Carathéodory function. Then* (1.1) *has at least one sign-changing solution.*

Sketch of the Proof of Theorem 3.6. By assumption (H1), there exists $R_1 > 0$ such that $T_p(\bar{B}(\theta, R_1)) \subset B(\theta, R_1)$. Let $S_{R_1}^+$ be defined as Theorem 3.1. By Lemma 2.13, there exists $\zeta_{R_1} > 0$ such that $S_{R_1}^+ \geqslant \zeta_{R_1} e$. Since $\beta_+ > \mu_1(p)$, there exists $\bar{\delta}_0 > \delta_2 > 0$ such that for any $\delta \in (0, \delta_2)$, $\delta\phi_1$ is a lower solution of (1.1). Take a $\delta_3 \in (0, \delta_2)$ small enough such that $u_0 := \delta_3\phi_1$ is a strict lower solution of (1.1) in $\bar{B}(\theta, R_1)$, $S_{R_1}^+ \geqslant u_0, u_1 \nleqslant u_0$. Let us define the sets $\Omega_1, \Omega_3, \Omega_4, O_{3,4}, O_{4,1}$ and O_4 as in Theorem 3.5. Then (3.38), (3.41), (3.42) and (3.46) hold. Therefore, T_p has fixed points $x_1 \in O_{3,4}, x_2 \in O_4$ and $x_3 \in \Omega_1$. Obviously, x_1 and x_2 are two sign-changing solutions of (1.1), and x_3 is a positive solution of (1.1). The proof is complete. $\square$

Remark 3.13. We should point out, the condition that f is sub-linear at infinity can be substituted by a pair of well ordered lower and upper solutions u_3 and v_3 such that u_1 and v_1 belongs to the ordered interval $[u_3, v_3]$. However, in those cases we need a condition of Nagumo type, see [12, 14]. Also, in those case we can study the multiplicity of sign-changing solutions when f both has jumping nonlinearity at zero and infinity.

Remark 3.14. In Theorem 3.5 the two pairs of well ordered lower and upper solutions u_0 and v_0, u_1 and v_1 satisfy

$$u_0 \nleqslant v_1, \quad u_1 \nleqslant v_0. \tag{3.48}$$

We say two pairs of well ordered lower and upper solutions u_0 and v_0, u_1 and v_1 are parallelled to each other when (3.48) holds. The concept of parallelled pairs of well ordered lower and upper solutions is put forward by Sun Jingxian. For other discussions concerning parallelled pairs of well ordered lower and upper solutions, the reader is refereed to [13].

Remark 3.15. In Theorems 3.2 and 3.5, we employed a pair of sign-changing strict lower and upper solutions. Generally speaking, it is difficult to construct a pair of sign-changing strict lower and upper solutions. However, we can use the method of [14] to give an example of this kind strict lower and upper solutions; see [14, Example 3.1].

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