

STOCHASTIC DYNAMIC EQUATIONS ON GENERAL TIME SCALES

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ABSTRACT. In this article, we construct stochastic integral and stochastic differential equations on general time scales. We call these equations stochastic dynamic equations. We provide the existence and uniqueness theorem for solutions of stochastic dynamic equations. The crucial tool of our construction is a result about a connection between the time scales Lebesgue integral and the Lebesgue integral in the common sense.

1. INTRODUCTION

This article is dedicated to the investigation of stochastic dynamic equations on general time scales. Dynamic equations on time scales offer a new direction in the study of dynamic systems which involve differential equations and difference equations as special cases. Their origin is connected with Stefan Hilger's work [13, 14]. In 1988, Stefan Hilger introduced the definition of a Δ -derivative. The common derivative and the common forward difference are special cases of the Δ -derivative. Various mathematical results of time scales theory are presented in the works of mathematicians such as Agarwal, Bohner, Peterson, and Guseinov (see [1, 2, 3, 4, 5, 6, 7]). Nowadays the monograph "Dynamic Equations on Time Scales" (see [6]) serves as a comprehensive treatment of results in this area of mathematics.

The theory of stochastic dynamic equations on time scales is only in its infancy. We note the paper [15], where the authors investigated dynamic systems whose evolutions depend on a process defined on a time scale. We also emphasize the work of Suman Sanyal [8, 17], where, in the case of isolated time scales, the stochastic integral was constructed and stochastic dynamic equations were studied. The Itô-type stochastic integral was given for isolated time scales. Stochastic integrals on time scales except the above mentioned ones were not constructed. For this reason, there currently does not exist a concept of stochastic dynamic equations on general time scales. In order to fill this gap, we will build the Lebesgue integral on general time scales. There are two possible ways to construct the Lebesgue measure and the Lebesgue integral on time scales. The first one consists in defining the Lebesgue measure over giving the Lebesgue integral first [16]. The second way

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consists in applying the standard Carathéodory extension scheme for time scales and the construction of the corresponding Lebesgue-type integral [12]. Herewith, the natural question arises about a connection between such integral (Δ -integral) and the Lebesgue integral with respect to Lebesgue measure on the line. In the paper [9], this connection was investigated by using a connection between corresponding measures. In the paper [10], the authors observed that such connection could be obtained by using the formula of the change of variable. In our paper, we investigate a connection between corresponding Riemann-type integrals as well. As we obtained the connections between the Δ -Riemann integral and the common Riemann integral, and between the Δ -Lebesgue integral and the common Lebesgue integral, it enabled us to construct the Itô-type stochastic integral for general time scales.

In this article, we introduce the construction of the stochastic integral on general time scales. We define the concept of stochastic dynamic equations on general time scales and study the properties of its solutions. The outline of the paper is as follows. In Section 2, the statement of the problem is made and some auxiliary results are given. Also in Section 2, we investigate the connection between Δ -Riemann integrable functions and Riemann integrable functions as well as the connection between Δ -Lebesgue integrable functions and Lebesgue integrable functions. In Section 3, we construct the stochastic integral and the stochastic differential on general time scales and study its properties. In Section 4, we consider stochastic dynamic equations on general time scales and prove the existence and uniqueness result for solutions of stochastic dynamic equations. The Markov property theorem is given as well.

2. PRELIMINARY DEFINITIONS AND AUXILIARY RESULTS

To build stochastic dynamic equations on general time scales, in the first place we construct the Itô-type stochastic integral on a general time scale. As noted in Section 1, such an integral was constructed in [17, 8] only for isolated time scales.

2.1. Time scales essentials.

Definition 2.1. A *time scale* \mathbb{T} is an arbitrary nonempty closed subset of the real numbers \mathbb{R} , where we assume \mathbb{T} has the topology that it inherits from the real numbers \mathbb{R} with the standard topology.

Example 2.2. Some “continuous” time scales are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = [0, 1]$, some “discrete” time scales are $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = \mathbb{N}$, and $\mathbb{T} = \mathbb{N}_0$, and some examples of “isolated” time scales are $\mathbb{T} = h\mathbb{Z}$ for $h > 0$ and $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, also called a “quantum” time scale. An example of a “hybrid” time scale is the finite union of closed subintervals of \mathbb{R} . Another example of a time scale is the Cantor set.

Obviously, a time scale \mathbb{T} may or may not be connected. That is why we introduce the concept of forward and backward jump operators.

Definition 2.3. We define the *forward jump operator* by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{for all } t \in \mathbb{T} \text{ such that this set is not empty}$$

and the *backward jump operator* by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{for all } t \in \mathbb{T} \text{ such that this set is not empty.}$$

Let $t \in \mathbb{T}$. If $\sigma(t) > t$, then t is called *right-scattered*. If $\sigma(t) = t$, then t is called *right-dense*. If $\rho(t) < t$, then t is called *left-scattered*. If $\rho(t) = t$, then t is called *left-dense*. Moreover, the sets \mathbb{T}^κ and \mathbb{T}_κ are derived from \mathbb{T} as follows: If \mathbb{T} has a left-scattered maximum, then \mathbb{T}^κ is the set \mathbb{T} without that left-scattered maximum; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. If \mathbb{T} has a right-scattered minimum, then \mathbb{T}_κ is the set \mathbb{T} without that right-scattered minimum; otherwise, $\mathbb{T}_\kappa = \mathbb{T}$.

The distance from a point $t \in \mathbb{T}^\kappa$ to the next point is introduced next.

Definition 2.4. We define the *graininess function* by

$$\mu(t) = \sigma(t) - t \quad \text{for all } t \in \mathbb{T}^\kappa.$$

Example 2.5. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t = \rho(t)$ and $\mu(t) = 0$ for all $t \in \mathbb{R}$. If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\rho(t) = t - 1$, and $\mu(t) = 1$ for all $t \in \mathbb{Z}$. If $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, then $\sigma(t) = t + h$, $\rho(t) = t - h$, and $\mu(t) = h$ for all $t \in h\mathbb{Z}$. If $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, then $\sigma(t) = qt$ and $\mu(t) = (q - 1)t$ for all $t \in q^{\mathbb{N}_0}$ and $\rho(t) = t/q$ for all $t \in q^{\mathbb{N}_0} \setminus \{1\}$.

2.2. Connections among integrable functions. Now we introduce the connection between Δ -Riemann integrable functions and Riemann integrable functions as well as the connection between Δ -Lebesgue integrable functions and Lebesgue integrable functions. These results constitute a key point in our construction of Itô-type stochastic integrals on general time scales.

Let \mathbb{T} be a time scale. We choose a couple of finite points $a, b \in \mathbb{T}$ such that $a < b$. Let us consider the segment $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$ and the real-valued function $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$. We now present the definition of the Δ -Riemann integral of the function f on $[a, b]_{\mathbb{T}}$ (see [7, page 118]). Let

$$P = \{a = t_0 < t_1 < \dots < t_n = b : t_i \in \mathbb{T} \text{ for } 1 \leq i \leq n - 1\}$$

be some partition of the interval $[a, b]_{\mathbb{T}}$. We denote the set of all partitions of $[a, b]_{\mathbb{T}}$ by $\mathcal{P}(a, b)$.

Definition 2.6. Let $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ be a function. The *upper Δ -Darboux sum* $U(f, P)$ and the *lower Δ -Darboux sum* $L(f, P)$ of f with respect to the partition

$$P = \{a = t_0 < t_1 < \dots < t_n = b : t_i \in \mathbb{T} \text{ for all } 1 \leq i \leq n - 1\} \in \mathcal{P}(a, b)$$

are given by

$$U(f, P) = \sum_{i=1}^n M_i(t_i - t_{i-1}) \quad \text{and} \quad L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1}),$$

respectively, where

$$M_i = \sup_{t \in [t_{i-1}, t_i]_{\mathbb{T}}} f(t) \quad \text{and} \quad m_i = \inf_{t \in [t_{i-1}, t_i]_{\mathbb{T}}} f(t) \quad \text{for all } 1 \leq i \leq n.$$

The *upper Δ -Darboux integral* $U(f)$ and the *lower Δ -Darboux integral* $L(f)$ of f on the interval $[a, b]_{\mathbb{T}}$ are defined by

$$U(f) = \inf_{P \in \mathcal{P}(a, b)} U(f, P) \quad \text{and} \quad L(f) = \sup_{P \in \mathcal{P}(a, b)} L(f, P),$$

respectively. Moreover, if $U(f) = L(f)$, then we say that f is Δ -integrable on the interval $[a, b]_{\mathbb{T}}$ and set

$$\int_a^b f \Delta t = U(f) = L(f),$$

the common value of the upper and lower Δ -Darboux integrals.

For the remainder of this section, without loss of generality, we may assume that $0, 1 \in \mathbb{T}$ and we put $a = 0$ and $b = 1$. As before, we denote $[0, 1]_{\mathbb{T}} = [0, 1] \cap \mathbb{T}$. Let a function $f : [0, 1]_{\mathbb{T}} \rightarrow \mathbb{R}$ be given. We will show that an integral

$$\int_0^1 f \Delta t \quad (2.1)$$

on the time scale $[0, 1]_{\mathbb{T}}$ could be defined in another way (different from [7]), and the value of this integral coincides with the value of (2.1).

Definition 2.7. For a function $f : [0, 1]_{\mathbb{T}} \rightarrow \mathbb{R}$, we define the *extension* $\tilde{f} : [0, 1] \rightarrow \mathbb{R}$ by

$$\tilde{f}(t) = f(\sup[0, t]_{\mathbb{T}}) \quad \text{for all } t \in [0, 1].$$

Remark 2.8. Note that $\tilde{f}(t) = f(t)$ if $t \in [0, 1]_{\mathbb{T}}$. If $t \in [0, 1] \setminus [0, 1]_{\mathbb{T}}$, then $\tilde{f}(t) = f(s)$, where s is the nearest left-hand point to the point t such that $s \in \mathbb{T}$. The function \tilde{f} is well defined by Definition 2.7 as can be seen as follows. For $t \in [0, 1]$, let us define the set $S = [0, 1]_{\mathbb{T}} \cap [0, t]$. Evidently, the set S is compact. We consider the function $\phi(s) = t - s$, $s \in S$. It is easy to see that the function ϕ is continuous, monotone, and defined on the compact set. Hence, using these facts, we derive that there exists a unique element $s^* \in S$ satisfying $\phi(s^*) = \inf_{s \in S} \phi(s)$. The point s^* is the nearest left-hand point to the point t such that $s^* \in \mathbb{T}$, so $\tilde{f}(t) = f(s^*)$. This shows our claim.

Next we give the result about the connection between the Δ -Riemann integral of f on the interval $[0, 1]_{\mathbb{T}}$ and the Riemann integral of \tilde{f} on the interval $[0, 1]$.

Theorem 2.9. *The function $f : [0, 1]_{\mathbb{T}} \rightarrow \mathbb{R}$ is Δ -Riemann integrable on the interval $[0, 1]_{\mathbb{T}}$ if and only if the function $\tilde{f} : [0, 1] \rightarrow \mathbb{R}$ is Riemann integrable on the interval $[0, 1]$, and then the following integrals are equal:*

$$\int_0^1 f(t) \Delta t = \int_0^1 \tilde{f}(t) dt. \quad (2.2)$$

Proof. We prove this statement using Darboux sums. Let $P \in \mathcal{P}(0, 1)$ be some partition of the interval $[0, 1]_{\mathbb{T}}$:

$$P = \{0 = t_0 < t_1 < \dots < t_n = 1 : t_i \in \mathbb{T} \text{ for all } 1 \leq i \leq n-1\}.$$

The lower Δ -Darboux sum for this partition is

$$L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1}), \quad \text{where } m_i = \inf_{t \in [t_{i-1}, t_i]_{\mathbb{T}}} f(t), \quad 1 \leq i \leq n.$$

This partition is also a partition for the whole interval $[0, 1]$. Thus

$$L(\tilde{f}, P) = \sum_{i=1}^n \tilde{m}_i(t_i - t_{i-1}), \quad \text{where } \tilde{m}_i = \inf_{t \in [t_{i-1}, t_i]} \tilde{f}(t), \quad 1 \leq i \leq n.$$

Since the set of \tilde{f} values on $[t_{i-1}, t_i]$ coincides with the set of f values on $[t_{i-1}, t_i]_{\mathbb{T}}$, we obtain $m_i = \tilde{m}_i$ for all $1 \leq i \leq n$ and hence $L(f, P) = L(\tilde{f}, P)$. Similarly, we can show $U(f, P) = U(\tilde{f}, P)$. This proves that any Δ -Darboux sum for f with respect to the partition P is the same as the Darboux sum for \tilde{f} with respect to the same partition P . Now let $\tilde{\mathcal{P}}(0, 1)$ be the set of all partitions of the interval

$[0, 1]$, and as before denote by $\mathcal{P}(0, 1)$ the set of all partitions of the interval $[0, 1]_{\mathbb{T}}$. Obviously,

$$\mathcal{P}(0, 1) \subset \tilde{\mathcal{P}}(0, 1). \quad (2.3)$$

From (2.3), we obtain

$$\begin{aligned} L(f) &= \sup_{P \in \mathcal{P}(0,1)} L(f, P) \leq \sup_{\tilde{P} \in \tilde{\mathcal{P}}(0,1)} L(\tilde{f}, \tilde{P}) = L(\tilde{f}) \\ &\leq U(\tilde{f}) \leq \inf_{\tilde{P} \in \tilde{\mathcal{P}}(0,1)} U(\tilde{f}, \tilde{P}) \leq \inf_{P \in \mathcal{P}(0,1)} U(f, P). \end{aligned}$$

Consequently,

$$L(f) \leq L(\tilde{f}) \leq U(\tilde{f}) \leq U(f). \quad (2.4)$$

From (2.4), we conclude that if the function f is Δ -Riemann integrable, then the function \tilde{f} is Riemann integrable and (2.2) holds.

On the other hand, let \tilde{f} be Riemann integrable. We will show that f is Δ -Riemann integrable. Let us use the criterion of Δ -integrability (see [7, Theorem 5.6]) which says that a function f is Δ -integrable if and only if for any $\varepsilon > 0$, there exists $P \in \mathcal{P}(0, 1)$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

We check this criterion. First we fix some $\varepsilon > 0$. Since the function \tilde{f} is Riemann integrable, there exists $\tilde{P} \in \tilde{\mathcal{P}}(0, 1)$ such that the inequality

$$U(\tilde{f}, \tilde{P}) - L(\tilde{f}, \tilde{P}) < \varepsilon \quad (2.5)$$

holds. Let

$$\tilde{P} = \{0 = t_0 < t_1 < \dots < t_n = 1 : t_i \in \mathbb{T} \text{ for } 1 \leq i \leq n-1\} \in \tilde{\mathcal{P}}(0, 1).$$

Using \tilde{P} , we construct a subpartition in the subsequent way. If $t_1 \in [0, 1]_{\mathbb{T}}$, then we pass to the next point of \tilde{P} . If $t_1 \notin [0, 1]_{\mathbb{T}}$, then there exist points $s_1, s_2 \in [0, 1]_{\mathbb{T}}$ (possibly $s_1 = 0, s_2 = 1$) such that $0 \leq s_1 < t_1 < s_2$ and subsequently

$$\tilde{f}(t) = f(s_1), \quad t \in [s_1, s_2].$$

Without loss of generality, we may choose $s_2 < t_2$. We add the points s_1, s_2 to the partition \tilde{P} . It is obvious that

$$\inf_{t \in [s_1, t_1]} \tilde{f}(t) = \inf_{t \in [t_1, s_2]} \tilde{f}(t) = f(s_1).$$

Consequently

$$m_1 s_1 + f(s_1)(t_1 - s_1) + f(s_1)(s_2 - t_1) = m_1 s_1 + f(s_1)(s_2 - s_1).$$

This shows that the “part” of the lower Darboux sum for \tilde{f} with respect to the points $0 < s_1 < t_1 < s_2$ of the partition of $[0, 1]$ coincides with the “part” of the lower Darboux sum for f with respect to the points $0 < s_1 < s_2$. From s_2 to t_2 , we repeat this procedure, and so on, until we go over all the points of the partition \tilde{P} . As a result, we obtain two partitions. First, $\tilde{Q} \in \tilde{\mathcal{P}}$ is a partition of the interval $[0, 1]$ which is constructed by adding the points $\{s_i\}$ to the points $\{t_i\}$. The second partition $Q \in \mathcal{P}(0, 1)$ consists of all points $\{s_i\}$. In view of these subpartitions’ constructions, it is easy to see that

$$L(\tilde{f}, \tilde{Q}) = L(f, Q). \quad (2.6)$$

Similarly,

$$U(\tilde{f}, \tilde{Q}) = U(f, Q). \quad (2.7)$$

Apparently, \tilde{Q} is a subpartition of \tilde{P} . It follows that

$$L(\tilde{f}, \tilde{P}) < L(\tilde{f}, \tilde{Q}) < U(\tilde{f}, \tilde{Q}) < U(\tilde{f}, \tilde{P}). \quad (2.8)$$

Due to (2.5), (2.6), (2.7), and (2.8), we obtain

$$U(f, Q) - L(f, Q) = U(\tilde{f}, \tilde{Q}) - L(\tilde{f}, \tilde{Q}) < U(\tilde{f}, \tilde{P}) - L(\tilde{f}, \tilde{P}) < \varepsilon,$$

which establishes the integrability criterion. Therefore, f is Δ -integrable so that $U(f) = L(f)$. From (2.3), we obtain (2.2). This completes the proof. \square

Finally we give the result about the connection between the Δ -Lebesgue integral of f on the interval $[0, 1]_{\mathbb{T}}$ and the Lebesgue integral of \tilde{f} on the interval $[0, 1]$. The proof of this result can be done using similar techniques as in the proof of Theorem 2.9. A proof can also be found in [9]. For related results, see [10, 16].

Theorem 2.10. *The function $f : [0, 1]_{\mathbb{T}} \rightarrow \mathbb{R}$ is Δ -Lebesgue integrable on the interval $[0, 1]_{\mathbb{T}}$ if and only if the function $\tilde{f} : [0, 1] \rightarrow \mathbb{R}$ is Lebesgue integrable on the interval $[0, 1]$, and then the following integrals are equal:*

$$\int_{[0, 1]_{\mathbb{T}}} f(t) \lambda_{\Delta}(dt) = \int_{[0, 1]} \tilde{f}(t) \lambda(dt),$$

where $\lambda(\cdot)$ is the Lebesgue measure and $\lambda_{\Delta}(\cdot)$ is the Δ -Lebesgue measure on \mathbb{T} .

3. STOCHASTIC INTEGRAL AND STOCHASTIC DIFFERENTIAL

Theorem 2.10 allows us to construct the stochastic integral on general time scale. Let (Ω, \mathcal{F}, P) be a probability space. Let W be a standard Wiener process and suppose

$$\{W(t+h) - W(t) : h \geq 0\} \text{ is independent of } \mathcal{F}_t := \sigma\{W(s) : 0 \leq s \leq t\},$$

where $\mathbb{F}_{\mathbb{R}} := \{\mathcal{F}_t : t \in \mathbb{R}\}$ is a filtration on \mathbb{R} , and with $\sigma\{\cdot\}$, we mean the σ -algebra generated by \cdot . We construct a Δ -stochastic integral on $[0, 1]_{\mathbb{T}}$, denoted by

$$\int_0^1 f(t) \Delta W(t).$$

First of all we determine the class $H^2([0, 1]_{\mathbb{T}})$ of random processes.

Definition 3.1. We say that the random process $f : \mathbb{T} \times \Omega \rightarrow \mathbb{R}$ belongs to class $H^2([0, 1]_{\mathbb{T}})$ if the following conditions hold:

- (i) f is adapted to $\mathbb{F}_{\mathbb{T}}$; i.e., $f(t, \cdot)$ is \mathcal{F}_t -measurable for all $t \in [0, 1]_{\mathbb{T}}$.
- (ii) $\mathbb{P}(\int_0^1 |f(t, \omega)|^2 \Delta t < \infty) = 1$.

Utilizing $f \in H^2([0, 1]_{\mathbb{T}})$, we define the random process \tilde{f} on $[0, 1]$ by

$$\tilde{f}(t, \omega) = f(\sup[0, t]_{\mathbb{T}}, \omega).$$

Note that \tilde{f} is well defined according to Subsection 2.2.

Proposition 3.2. *The process \tilde{f} has the following properties:*

- (i) \tilde{f} is adapted to $\mathbb{F}_{\mathbb{R}}$, i.e., $\tilde{f}(t, \cdot)$ is \mathcal{F}_t -measurable for all $t \in [0, 1]$.
- (ii) $\mathbb{P}(\int_0^1 |\tilde{f}(t, \omega)|^2 dt < \infty) = 1$.

Proof. The first property follows from the construction of \tilde{f} . The second property follows from Theorem 2.10. \square

By Proposition 3.2, starting with the processes $f \in \mathbb{H}^2([0, 1]_{\mathbb{T}})$ creates a family of processes \tilde{f} for which the Itô stochastic integral in the common sense is defined. Hence, the Δ -stochastic integral in the subsequent definition is well defined.

Definition 3.3. We say that the random process $f \in \mathbb{H}^2([0, 1]_{\mathbb{T}})$ has a Δ -stochastic integral on $[0, 1]_{\mathbb{T}}$ provided the corresponding process \tilde{f} has a stochastic integral in the common sense on $[0, 1]$, and then we set

$$\int_0^1 f(t) \Delta W(t) := \int_0^1 \tilde{f}(t) dW(t).$$

Remark 3.4. We note that in the case of $\mathbb{T} = \mathbb{R}$, the integral given as in Definition 3.3 coincides with the common Itô stochastic integral, and in the case of an isolated time scale, the integral given as in Definition 3.3 coincides with the integral introduced by Suman Sanyal [17, 8].

It follows from the Definition 3.3 that the Δ -stochastic integral has all the properties of the common Itô stochastic integral.

Proposition 3.5. *The Δ -stochastic integral has the following properties:*

(i) *If $f_1, f_2 \in \mathbb{H}^2([0, 1]_{\mathbb{T}})$ and $c_1, c_2 \in \mathbb{R}$, then*

$$\int_0^1 (c_1 f_1(t) + c_2 f_2(t)) \Delta W(t) = c_1 \int_0^1 f_1(t) \Delta W(t) + c_2 \int_0^1 f_2(t) \Delta W(t).$$

(ii) *If $\mathbb{E} \left(\int_0^1 |f(t)|^2 \Delta t \right) < \infty$, then*

$$\mathbb{E} \left(\int_0^1 f(t) \Delta W(t) \right) = 0$$

and the Itô-isometry holds:

$$\mathbb{E} \left(\left(\int_0^1 f(t) \Delta W(t) \right)^2 \right) = \mathbb{E} \left(\int_0^1 f^2(t) \Delta t \right).$$

The proof of the above proposition follows directly from Definition 3.3.

Now we may define the Δ -stochastic differential in a standard way (see for example [11]).

Definition 3.6. If for all $t_1, t_2 \in [0, 1]_{\mathbb{T}}$ such that $t_1 < t_2$ we have

$$X(t_2) - X(t_1) = \int_{t_1}^{t_2} b(t, X) \Delta t + \int_{t_1}^{t_2} B(t, X) \Delta W(t), \quad (3.1)$$

where b is Lebesgue integrable on $[0, 1]_{\mathbb{T}}$ and $B \in \mathbb{H}^2([0, 1]_{\mathbb{T}})$, then we say that the process X has a Δ -stochastic differential indicated by the notation

$$\Delta X(t) = b(t, X) \Delta t + B(t, X) \Delta W(t).$$

In the equality (3.1), the first integral is the common Lebesgue integral, and the second integral is the Δ -stochastic integral.

4. STOCHASTIC DYNAMIC EQUATIONS

Definition 3.6 gives us an opportunity to introduce the concept of stochastic dynamic equations on general time scales. Let \mathbb{T} be a general time scale such that $\sup \mathbb{T} = \infty$. We consider the stochastic dynamic equation on $[0, a]_{\mathbb{T}} \subset \mathbb{T}$

$$\Delta X(t) = b(X, t)\Delta t + B(X, t)\Delta W(t) \quad (4.1)$$

with initial condition

$$X(0) = X_0, \quad (4.2)$$

where X_0 is a random variable, independent of $W(t)$. In the following, under filtration, we understand $\mathcal{F}_t = \sigma\{X_0, W_s : s \leq t\}$.

Definition 4.1. We say that a random process $X(t)$ with $X(0) = X_0$ is a *solution* of the stochastic dynamic equation (4.1) on $[0, a]_{\mathbb{T}}$ if the following conditions hold:

- (i) X is adapted to the filtration \mathbb{F} .
- (ii) For all $t \in [0, a]_{\mathbb{T}}$, we have almost surely

$$X(t) = X_0 + \int_0^t b(X(s), s)\Delta s + \int_0^t B(X(s), s)\Delta W(s). \quad (4.3)$$

We now present an existence and uniqueness theorem. For this purpose, we first give the subsequent definition.

Definition 4.2. We say that the function $\varphi : [0, a] \rightarrow \mathbb{R}$ is continuous at the point $t_0 \in [0, a]_{\mathbb{T}}$ if for each sequence $\{t_n \in [0, a]_{\mathbb{T}}\}$ such that $t_n \rightarrow t_0$ as $n \rightarrow \infty$, we have $\varphi(t_n) \rightarrow \varphi(t_0)$ as $n \rightarrow \infty$.

Next we have a result on the existence and uniqueness of solutions of stochastic dynamic equations.

Theorem 4.3. *Let the following conditions hold:*

- (i) *The functions $b, B : \mathbb{R} \times [0, a]_{\mathbb{T}} \rightarrow \mathbb{R}$ are continuous.*
- (ii) *There exists a constant $L > 0$ such that for each $x_1, x_2 \in \mathbb{R}$ and all $t \in [0, a]_{\mathbb{T}}$, we have*

$$|b(x_1, t) - b(x_2, t)| \leq L|x_1 - x_2|, \quad |B(x_1, t) - B(x_2, t)| \leq L|x_1 - x_2|, \quad (4.4)$$

and

$$|b(x, t)| \leq L(1 + |x|), \quad |B(x, t)| \leq L(1 + |x|). \quad (4.5)$$

- (iii) *The real-valued random variable X_0 satisfies the inequality $\mathbb{E}(|X_0|^2) < \infty$ and is independent of $W(t)$ for $t > 0$.*

Then there exists with probability $\mathbb{P} = 1$ a continuous solution X on $[0, a]_{\mathbb{T}}$ of the stochastic dynamic equation (4.1) with initial condition (4.2) such that

$$\mathbb{E}\left(\int_0^t X^2(\tau)\Delta\tau\right) < \infty. \quad (4.6)$$

Moreover, if X and \tilde{X} are both such solutions, then

$$\mathbb{P}\left(\sup_{t \in [0, a]_{\mathbb{T}}} |X(t) - \tilde{X}(t)| = 0\right) = 1.$$

Proof. First we show uniqueness. Obviously, the set $[0, a]_{\mathbb{T}}$ is closed and bounded and thus is compact. Due to compactness of the set $[0, a]_{\mathbb{T}}$, for all $\varepsilon > 0$, there exists a finite ε -net $\{t_1, t_2, \dots, t_n\}$. Hence we can choose the sequence $\{\varepsilon_j\}$ such that for each $\varepsilon_j = (1/2)^{j-1}$, there exists a finite ε_j -net $S_{\varepsilon_j} = \{t_1^j, \dots, t_n^j\}$ for $[0, a]_{\mathbb{T}}$. Now let $S_\varepsilon = \cup_j S_{\varepsilon_j}$. In view of the construction, the set S_ε is everywhere dense on $[0, a]_{\mathbb{T}}$. Let the processes X and \tilde{X} be solutions of equation (4.1) with initial condition (4.2). On the basis of the a.s. continuity of the solutions and the fact that the set S_ε is everywhere dense, it follows that

$$\mathbb{P}\left(\sup_{t \in S_\varepsilon} |X(t) - \tilde{X}(t)| = 0\right) = \mathbb{P}\left(\sup_{t \in [0, a]_{\mathbb{T}}} |X(t) - \tilde{X}(t)| = 0\right) = 1.$$

Now we show existence. We will employ the method of iteration. For this purpose, we define

$$\begin{aligned} X^0(t) &:= X_0, \\ X^n(t) &:= X_0 + \int_0^t b(X^{n-1}(s), s) \Delta s + \int_0^t B(X^{n-1}(s), s) \Delta W(s), \end{aligned}$$

where $n \in \mathbb{N}$ and $t \in [0, a]_{\mathbb{T}}$. Also we introduce

$$\delta^n(t) := \mathbb{E}(|X^{n+1}(s) - X^n(s)|).$$

Let us show that the inequality

$$\delta^n(t) \leq M^{n+1} h_{n+1}(t, 0) \tag{4.7}$$

holds for some constant M , which depends on L , a and X_0 , and for all $n \in \mathbb{N}$, $t \in [0, a]_{\mathbb{T}}$, where h_n are the generalized monomials [6, Section 1.6]. Let us check (4.7) for $n = 0$:

$$\begin{aligned} \delta^0(t) &= \mathbb{E}\left(|X^1(s) - X^0(s)|^2\right) \\ &= \mathbb{E}\left(\left|\int_0^t b(X_0, s) \Delta s + \int_0^t B(X_0, s) \Delta W(s)\right|^2\right) \\ &\leq 2\mathbb{E}\left(\left|\int_0^t L(1 + |X_0|) \Delta s\right|^2\right) + 2\mathbb{E}\left(\int_0^t L^2(1 + |X_0|)^2 \Delta s\right) \\ &\leq tM = Mh_1(t, 0), \end{aligned}$$

where we set $M = 4L^2(1 + |X_0|)^2$. This confirms (4.7) for $n = 0$. Suppose now that the inequality (4.10) holds for some $n - 1$. Then

$$\begin{aligned} \delta^n(t) &= \mathbb{E}\left(|X^{n+1}(s) - X^n(s)|^2\right) \\ &= \mathbb{E}\left(\left|\int_0^t (b(X^n(s), s) - b(X^{n-1}(s), s)) \Delta s \right. \right. \\ &\quad \left. \left. + \int_0^t (B(X^n(s), s) - B(X^{n-1}(s), s)) \Delta W(s)\right|^2\right) \\ &\leq 2\mathbb{E}\left(\left|\int_0^t (b(X^n(s), s) - b(X^{n-1}(s), s)) \Delta s\right|^2\right) \\ &\quad + 2\mathbb{E}\left(\left|\int_0^t (B(X^n(s), s) - B(X^{n-1}(s), s)) \Delta W(s)\right|^2\right) \end{aligned}$$

$$\begin{aligned}
&\leq 2\mathbb{E}\left(\left|\int_0^t (b(X^n(s), s) - b(X^{n-1}(s), s))\right|^2 \Delta s\right) \\
&\quad + 2\mathbb{E}\left(\left|\int_0^t (B(X^n(s), s) - B(X^{n-1}(s), s))\right|^2 \Delta s\right) \\
&\leq 2aL^2\mathbb{E}\left(\int_0^t |X^n(s) - X^{n-1}(s)|^2 \Delta s\right) \\
&\quad + 2L^2\mathbb{E}\left(\int_0^t |X^n(s) - X^{n-1}(s)|^2 \Delta s\right) \\
&\leq 2(a+1)L^2\mathbb{E}\left(\int_0^t |X^n(s) - X^{n-1}(s)| \Delta s\right) \\
&= 2(a+1)L^2 \int_0^t \delta^{n-1}(\tau) \Delta \tau \\
&\leq 2(a+1)L^2 \int_0^t M^n h^n(s, 0) \Delta s \\
&\leq M^{n+1} h^{n+1}(t, 0),
\end{aligned}$$

where we choose $M \geq 2(a+1)L^2$. This proves the inequality (4.7).

Using the Lipschitz property of the function $b(X, t)$, we have

$$\begin{aligned}
&\sup_{t \in [0, a]_{\mathbb{T}}} |X^{n+1}(t) - X^n(t)|^2 \\
&\leq 2aL^2 \int_0^t |X^n(s) - X^{n-1}(s)|^2 \Delta s \\
&\quad + 2 \sup_{t \in [0, a]_{\mathbb{T}}} \left| \int_0^t (B(X^n(s), s) - B(X^{n-1}(s), s)) \Delta W(s) \right|^2.
\end{aligned}$$

As a result, the martingale inequality [11] and the inequality (4.7) imply

$$\begin{aligned}
\mathbb{E}\left(\sup_{t \in [0, a]_{\mathbb{T}}} |X^{n+1}(t) - X^n(t)|^2\right) &\leq 2aL^2 \int_0^t \mathbb{E}\left(|X^n(s) - X^{n-1}(s)|^2\right) \Delta s \\
&\quad + 8L^2 \int_0^t \mathbb{E}\left(|X^n(s) - X^{n-1}(s)|^2\right) \Delta s \\
&\leq CM^n h^n(a, 0),
\end{aligned}$$

where $C = 2L^2(a+4)$. Therefore, let us apply the Borel–Cantelli lemma [11], since

$$\begin{aligned}
\mathbb{P}\left(\sup_{t \in [0, a]_{\mathbb{T}}} |X^{n+1}(t) - X^n(t)| > \frac{1}{2^n}\right) &\leq 4^n \mathbb{E}\left(\sup_{t \in [0, a]_{\mathbb{T}}} |X^{n+1}(t) - X^n(t)|^2\right) \\
&\leq 4^n CM^n h^n(a, 0)
\end{aligned}$$

and

$$\sum_{n=1}^{\infty} 4^n CM^n h^n(a, 0) < \infty$$

imply

$$\mathbb{P}\left(\sup_{t \in [0, a]_{\mathbb{T}}} |X^{n+1}(t) - X^n(t)| > \frac{1}{2^n}\right) = 0.$$

In view of this, for almost every ω , the series

$$X^0 + \sum_{j=0}^{n-1} (X^{j+1} - X^j)$$

converges uniformly. The partial sum of this series is an a.s. uniform bound of X^n and so $X^n \rightarrow X$ as $n \rightarrow \infty$. Thus we have

$$X(t) = X_0 + \int_0^t b(X(s), s) \Delta s + \int_0^t B(X(s), s) \Delta W(s) \quad \text{for all } t \in [0, a]_{\mathbb{T}}.$$

The process X is continuous being an a.s. uniform bound of a.s. continuous processes.

Let us show that (4.6) is valid. We observe

$$\begin{aligned} \mathbb{E} \left(|X^{n+1}(t)|^2 \right) &\leq C \mathbb{E}(|X_0|^2) + C \mathbb{E} \left(\left| \int_0^t b(X^n(s), s) \Delta s \right|^2 \right) \\ &\quad + C \mathbb{E} \left(\left| \int_0^t B(X^n(s), s) \Delta W(s) \right|^2 \right) \\ &\leq C (1 + \mathbb{E}(|X_0|^2)) + C \int_0^t \mathbb{E}(|X^n|^2) \Delta s, \end{aligned}$$

where by C we denote a constant. By induction, we get

$$\mathbb{E} \left(|X^{n+1}(t)|^2 \right) \leq (C + C^2 h_1(t, 0) + \dots + C^{n+2} h_{n+1}(t, 0)) (1 + \mathbb{E}(|X_0|^2)).$$

It follows that

$$\mathbb{E} \left(|X^{n+1}(t)|^2 \right) \leq C (1 + \mathbb{E}(|X_0|^2)) e_C(t, 0),$$

where $e_C(\cdot, 0)$ is a time scales exponential function [6, Section 2.2]. As $n \rightarrow \infty$, we obtain

$$\mathbb{E} \left(|X(t)|^2 \right) \leq C (1 + \mathbb{E}(|X_0|^2)) e_C(t, 0) \quad \text{for all } t \in [0, a]_{\mathbb{T}},$$

and thus

$$\begin{aligned} \mathbb{E} \left(\int_0^t |X(\tau)|^2 \Delta \tau \right) &= \int_0^t \mathbb{E}(|X(\tau)|^2) \Delta \tau \\ &\leq (1 + \mathbb{E}(|X_0|^2)) \int_0^t C e_C(\tau, 0) \Delta \tau \\ &= (1 + \mathbb{E}(|X_0|^2)) (e_C(t, 0) - 1) < \infty, \end{aligned}$$

which proves (4.6). \square

Now we turn to the Markov property of solutions of stochastic dynamic equations. Notice that Theorem 2.10 could be reformulated in an obvious way for the case when instead of $[0, a]_{\mathbb{T}}$, we consider $[u, a]_{\mathbb{T}} \subset \mathbb{T}$, where $u > 0$. Thus we consider the equation

$$Z(t) = \xi + \int_u^t b(Z(s), s) \Delta s + \int_u^t B(Z(s), s) \Delta W(s) \quad (4.8)$$

with initial condition

$$Z(u) = \xi \quad (4.9)$$

This solution Z we denote by $Z_\xi(t)$, $t \in [u, a]_{\mathbb{T}}$. For the process $X = \{X(t) : t \in [0, a]_{\mathbb{T}}\}$, which is a solution of equation (4.1), (4.2), we have

$$\begin{aligned} X(t) &= X_0 + \int_0^t b(X(s), s)\Delta s + \int_0^t B(X(s), s)\Delta W(s) \\ &= X_0 + \int_0^u b(X(s), s)\Delta s + \int_0^u B(X(s), s)\Delta W(s) \\ &\quad + \int_u^t b(X(s), s)\Delta s + \int_u^t B(X(s), s)\Delta W(s) \\ &= X(u) + \int_u^t b(X(s), s)\Delta s + \int_u^t B(X(s), s)\Delta W(s). \end{aligned}$$

We will need the following auxiliary results.

Lemma 4.4. *Let the process $Y = \{Y(s) : s \in \mathbb{T}\}$ be progressively measurable on the time scale \mathbb{T} with respect to the filtration $\mathbb{F}_{\mathbb{T}} = (\mathcal{F}_s)_{s \in \mathbb{T}}$ in \mathcal{F} . Let the real-valued function $a(s, x)$ be defined on $\mathbb{T} \times \mathbb{R}$ and suppose it is $\mathcal{B}(\mathbb{T}) \otimes \mathcal{B}(\mathbb{R})$ -measurable. Then the process $U = \{U(t) = a(Y(t), t) : t \in \mathbb{T}\}$ is progressively $\mathbb{F}_{\mathbb{T}}$ -measurable.*

Proof. First we show $\mathcal{B}([0, t]) \otimes \mathcal{F}_t \mid \mathcal{B}([0, t]) \otimes \mathcal{B}(\mathbb{R})$ -measurability of the mappings $(s, \omega) \mapsto (s, Y_s(\omega))$ for $0 \leq s \leq t$ and $\omega \in \Omega$. Let $0 \leq u \leq t$ and $B \in \mathcal{B}(\mathbb{R})$. Then

$$\begin{aligned} &\{(s, \omega) \in [0, t] \otimes \Omega : (s, Y_s(\omega)) \in [0, u] \otimes B\} \\ &= \{(s, \omega) \in [0, t \wedge u] \otimes \Omega : Y_s(\omega) \in B\} \\ &\subset \mathcal{B}([0, t \wedge u]) \otimes \mathcal{F}_{t \wedge u} \subset \mathcal{B}([0, t]) \otimes \mathcal{F}_t. \end{aligned}$$

For $(s, x) \in [0, t] \times \Omega$, the mapping $(s, x) \mapsto a(s, x)$ is $\mathcal{B}([0, t]) \otimes \mathcal{B}(\mathbb{R})$ -measurable as a superposition of measurable mappings. \square

Assume that the functions b and B satisfy conditions (4.4) and (4.5). Let $X^{(0)}(t) = Z$, $t \in \mathbb{T}$, and set for $n \in \mathbb{N}$

$$X^{(n)}(t) = Z + \int_0^t b(s, X^{(n-1)}(s))\Delta s + \int_0^t B(s, X^{(n-1)}(s))\Delta W(s). \quad (4.10)$$

The fact that such a procedure is well defined is provided by the subsequent auxiliary result.

Lemma 4.5. *Let the functions $b, B : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy conditions (4.4) and (4.5). Suppose that the value Z is \mathcal{F}_0 -measurable and that $\mathbb{E}(Z^2) < \infty$. Then the process $X^{(n)} = \{X^{(n)}(t) : t \in \mathbb{T}\}$ is well defined by expression (4.10), and it is also progressively measurable. Thus $\sup_{t \in \mathbb{T}} \mathbb{E}(|X^{(n)}(t)|^2) < \infty$ for all $n \in \mathbb{N}$ and the right-hand side of (4.10) can be chosen a.s. continuous on \mathbb{T} .*

Proof. The process $Y_s(\omega) = Z(\omega)$, where $s \in \mathbb{T}$ and $\omega \in \Omega$, is progressively measurable since $Z \in \mathcal{F}_0 \mid \mathcal{B}(\mathbb{R})$. Due to Lemma 4.4, the functions $b(s, Z(\omega))$ and $B(s, Z(\omega))$ are progressively measurable. In view of conditions (4.4) and (4.5) on the function B , we have

$$\sup_{s \in \mathbb{T}} \mathbb{E}(B^2(s, Z)) \leq c(1 + \mathbb{E}(Z^2)) < \infty.$$

As a result, we have $\{B(s, Z) : s \in \mathbb{T}\} \in \mathcal{A}_T \cap L_T^2$, and $\int_0^t B(s, Z)\Delta W(s)$ for $t \in \mathbb{T}$ can be chosen a.s. continuous. Taking into account the progressive measurability

of $\{b(s, Z) : s \in \mathbb{T}\}$ and conditions (4.4) and (4.5), we have

$$\mathbb{E}\left(\int_0^a |b(s, Z)| \Delta s\right) \leq (ac(1 + \mathbb{E}(Z^2)))^{1/2} < \infty.$$

In light of Fubini's theorem, the integral $\int_0^a b(s, Z) \Delta s$ is finite for almost all ω , and for such ω , the process $\int_0^t b(s, Z) \Delta s$ is continuous for $t \in \mathbb{T}$. This implies that $\int_0^t b(s, Z) \Delta s$ is progressively measurable. In the same way, the a.s. continuity and progressive measurability of the right-hand side of (4.10) are verified provided $\{X^{(n+1)}(t) : t \in \mathbb{T}\}$ is progressively measurable and $\sup_{s \in \mathbb{T}} \mathbb{E}(|X^{(n+1)}(s)|^2) < \infty$. Moreover,

$$\begin{aligned} \mathbb{E}\left((X^{(n)}(t))^2\right) &\leq 3\mathbb{E}(Z^2) + 3\mathbb{E}\left(\left(\int_0^t b(s, X^{(n+1)}(s)) \Delta s\right)^2\right) \\ &\quad + 3\mathbb{E}\left(\left(\int_0^t B(s, X^{(n+1)}(s)) \Delta W(s)\right)^2\right) \\ &\leq 3\mathbb{E}(Z^2) + 3aL \int_0^a \mathbb{E}\left(1 + |X^{(n+1)}(s)|^2\right) \Delta s \\ &\quad + 3L \int_0^a \mathbb{E}\left(1 + |X^{(n+1)}(s)|^2\right) \Delta s \\ &\leq 3\left(\mathbb{E}(Z^2) + (a+1)L\left(1 + \sup_{s \in \mathbb{T}} \mathbb{E}(|X^{(n+1)}(s)|^2)\right)\right) < \infty. \end{aligned}$$

It follows that $\sup_{t \in \mathbb{T}} \mathbb{E}(|X^{(n)}(t)|^2) < \infty$. This completes the proof. \square

Lemma 4.6. *Let the functions $b, B : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy conditions (4.4) and (4.5). Then for all $t \in [u, a]_{\mathbb{T}}$ and for all $\xi \in \mathcal{F}_u | \mathcal{B}(\mathbb{R})$ such that $\mathbb{E}(\xi^2) < \infty$, the value $Z_t(\xi, \omega)$ is measurable with respect to the σ -algebra*

$$\mathcal{A}_{[u, a]_{\mathbb{T}}} = \sigma\{\xi, W(s) + W(u) : s \in [u, a]_{\mathbb{T}}\},$$

which is extended by the class of zero measure events.

Proof. Using the proof of Theorem 4.3, we obtain that $Z_t(\xi, \omega)$ is an a.s. bound of the values $Z_t^{(n)}(\xi, \omega)$ as $n \rightarrow \infty$, where $Z_t^{(0)}(\xi, \omega) = \xi$ for $t \in [u, a]_{\mathbb{T}}$. If $n \in \mathbb{N}$, then for $t \in [u, a]_{\mathbb{T}}$, we have

$$Z_t^{(n)}(\xi, \omega) = \xi + \int_u^t b(s, Z_s^{(n-1)}(\xi, \omega)) \Delta s + \int_u^t B(s, Z_s^{(n-1)}(\xi, \omega)) \Delta W(s).$$

Obviously, $Z_t^{(0)} \in \mathcal{A}_{[u, a]_{\mathbb{T}}} | \mathcal{B}(\mathbb{R})$. By induction and due to Lemma 4.5, we get the $\mathcal{A}_{[u, a]_{\mathbb{T}}}$ -measurability of $Z_t^{(n)}(\xi, \omega)$ for all $n \in \mathbb{N}$. \square

Theorem 4.7. *Assume that all conditions of Theorem 4.3 hold. Then a solution of the stochastic dynamic equation (4.1) is a Markov process, and its transition probability is given by*

$$P(s, Y, t, B) = \mathbb{P}(\{X_{s, Y}(t) \in B\}).$$

Proof. It is sufficient to verify that for $0 \leq u \leq t \leq a$ and any Borel bounded function $f : \mathbb{R} \rightarrow \mathbb{R}$, we have

$$\mathbb{E}(f(X(t)) | \mathcal{F}_u) = \mathbb{E}(f(X(t)) | X(u)).$$

Since $X_t(\omega) = Z_t(X_u(\cdot), \omega)$ a.s., instead of $f(X(t))$, we consider $f(Z_t(X(u), \omega))$. Due to Lemma 4.6, the value $f(Z_t(X(u), \omega))$ is bounded and $\mathcal{A}_{[u, a]_{\mathbb{T}}}$ -measurable. Therefore, $f(Z_t(X(u), \omega))$ can be presented as an a.s. bound and as $L^2(\Omega, \mathcal{F}, P)$ -bound of a linear combination of random variables such that

$$\eta = g(X(u))h_1(W(s_1) - W(u)) \cdots h_m(W(s_m) - W(u)),$$

where the functions g, h_1, \dots, h_m are Borel and bounded, $u \leq s_1 < \dots < s_m \leq t$, $m \in \mathbb{N}$. Since $X(u)$ is \mathcal{F}_u -measurable and $W(s) - W(u)$ is independent of \mathcal{F}_u for $s \geq u$, we have

$$\mathbb{E}(\eta | \mathcal{F}_u) = g(X(u))\mathbb{E}(h_1(W(s_1) - W(u)) \cdots h_m(W(s_m) - W(u)))$$

and

$$\mathbb{E}(\eta | X(u)) = g(X(u))\mathbb{E}(h_1(W(s_1) - W(u)) \cdots h_m(W(s_m) - W(u))).$$

Consequently, passing to the limit, we obtain the required equality

$$\mathbb{E}(f(X(t)) | \mathcal{F}_u) = \mathbb{E}(f(X(t)) | X(u)).$$

This completes the proof. \square

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