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EXISTENCE OF NON-OSCILLATORY SOLUTIONS FOR SECOND-ORDER ADVANCED HALF-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this article, we establish the necessary and sufficient conditions for existence of non-oscillatory solutions for the second-order advanced half-linear differential equation

$$\big(r(t)|x'(t)|^{\alpha-1}x'(t)\big)' + p(t)|x(h(t))|^{\alpha-1}x(h(t)) = 0, \quad t \ge t_0.$$

The obtained results generalize some well-known theorems in the literature

1. Introduction

Consider the second-order advanced half-linear differential equation

$$\Big(r(t)|x'(t)|^{\alpha-1}x'(t)\Big)' + p(t)|x(h(t))|^{\alpha-1}x(h(t)) = 0, \quad t \ge t_0, \tag{1.1}$$

where $\alpha > 0$ is a constant, $r \in C^1([t_0,\infty),\mathbb{R}^+)$ with $\int_{t_0}^\infty r^{-1/\alpha}(t)dt = \infty$, $p \in C([t_0,\infty),\ [0,\infty))$ with $p(t) \not\equiv 0$, and $h \in C([t_0,\infty),\mathbb{R})$ with $t \le h(t)$.

By a solution to (1.1) we mean a function $x \in C^1([T_x,\infty),\mathbb{R})$, $T_x \ge t_0$, such that $r|x'|^{\alpha-1}x' \in C^1([T_x,\infty),\mathbb{R})$ and x satisfies (1.1) for all $t \ge T_x$. Solutions of (1.1) vanishing in some neighborhood of infinity will be excluded from our consideration. A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is said to be non-oscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory. Similarly, it is called non-oscillatory if all its solutions are non-oscillatory.

Equation (1.1) can be considered as the natural generalization of the linear differential equation

$$\big(r(t)x'(t)\big)' + p(t)x(t) = 0, \tag{1.2}$$

or of the half-linear differential equation

$$\Big(r(t)|x'(t)|^{\alpha-1}x'(t)\Big)' + p(t)|x(t)|^{\alpha-1}x(t) = 0 \tag{1.3}$$

and of the advanced differential equation

$$\big(r(t)x'(t)\big)' + p(t)x(h(t)) = 0, \quad t \le h(t), \tag{1.4}$$

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The oscillation and nonoscillation of (1.2)-(1.3) has been extensively investigated from various viewpoints during the previous 60 years, see for example the monographs [1, 2] and the references therein. To motivate the formulation of our main results, we wish to quote the following known non-oscillation results.

Theorem 1.1 ([8, p. 379])**.** *Equation* (1.2) *has a nonoscillatory solution if and only if there is a positive differentiable function $\varphi(t)$ defined on $[t_1,\infty)$, $t_1 \ge t_0$, such that*

$$\varphi'(t) + \frac{\varphi^2(t)}{r(t)} \le -p(t), \quad t \ge t_1.$$

Theorem 1.2 ([9, Theorem 2.1])**.** *Assume that*

$$\int_t^\infty \frac{ds}{r(s)} = \infty \quad \text{and} \quad 0 \le \int_t^\infty p(s)ds < \infty, \quad t \in [t_0,\infty)$$

hold. Define a sequence of function $\{\upsilon_n(t)\}_0^\infty$ as follows:

$$\upsilon_0(t) = \int_t^\infty p(s)ds, \quad \upsilon_1(t) = \int_t^\infty \frac{\upsilon_0^2(s)}{r(s)}ds,$$
$$\upsilon_{n+1}(t) = \int_t^\infty \frac{[\upsilon_0(s)+\upsilon_n(s)]^2}{r(s)}ds, \quad t \in [t_0,\infty), \quad n = 1,2,\dots.$$

Then (1.2) *is non-oscillatory if and only if there exists $t_1 \ge t_0$ such that*

$$\lim_{n\to\infty} \upsilon_n(t) = \upsilon(t) < \infty \quad \text{for } t \ge t_1.$$

Recently, Yang and Lo [10] extended Theorem 1.2 to (1.3), see [10, Theorem 1]. On the other hand, in 1991, Lu [6] extended Theorem 1.1 to (1.4). More precisely, Lu proved the following theorem.

Theorem 1.3 ([6, Lemma 2])**.** *Equation* (1.4) *has a nonoscillatory solution if and only if there is a positive differentiable function $\varphi(t)$ defined on $[t_1,\infty)$, $t_1 \ge t_0$, such that*

$$\varphi'(t) + \frac{\varphi^2(t)}{r(t)} \le -p(t)\exp\Big(\int_t^{h(t)} \frac{\varphi(s)}{r(s)}ds\Big), \quad t \ge t_1.$$

For related works for (1.2), see. e.g., [3, 4, 5, 7].

Inspired by [6, 8, 9, 10], in this article, we extend the results by Lu [6], Wintner [8], Yan [9], and Yang and Lo [10] to the Equation (1.1). We establish necessary and sufficient conditions for existence of non-oscillatory solutions to (1.1). Using these results, we further establish oscillation criteria for (1.1). The obtained results generalize some well-known theorems in the literature.

2. Main results

Theorem 2.1. *If*

$$\int_{t_0}^\infty p(s)ds = \infty, \tag{2.1}$$

then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that (1.1) has a non-oscillatory solution $x(t)$. We assume that $x(t) > 0$ and $x(h(t)) > 0$ for $t \ge t_1 \ge t_0$. A similar proof is done if we assume $x(t) < 0$ on $[t_1,\infty)$. Since $p(t) \ge 0$ on $[t_1,\infty)$, $\big(r(t)|x'(t)|^{\alpha-1}x'(t)\big)' \le 0$, hence, $r(t)|x'(t)|^{\alpha-1}x'(t)$ is non-increasing on $[t_1,\infty)$, therefore, $x'(t)$ is eventually of constant sign. If $x'(t)<0$ for $t\geq t_1$, then

$$r(t)|x'(t)|^{\alpha-1}x'(t)\leq r(t_1)(-x'(t_1))^{\alpha-1}x'(t_1)=:-c<0\,.$$

It follows that

$$x(t)\leq x(t_1)-c^{1/\alpha}\int_{t_1}^{t}\frac{ds}{r^{1/\alpha}(s)}\to-\infty\quad\text{as } t\to\infty,$$

which contradicts $x(t)>0$. Thus, $x'(t)>0$ for $t\geq t_1$. Let

$$w(t)=\frac{r(t)|x'(t)|^{\alpha-1}x'(t)}{|x(t)|^{\alpha-1}x(t)}. \tag{2.2}$$

Obviously, $w(t)>0$, and $r(t)(x'(t))^{\alpha}=w(t)(x(t))^{\alpha}$; i.e.,

$$\frac{x(h(t))}{x(t)}=\exp\Big(\int_t^{h(t)}\big(\frac{w(s)}{r(s)}\big)^{1/\alpha}ds\Big). \tag{2.3}$$

Then, from (1.1) and (2.3), we obtain

$$w'(t)+\alpha\frac{(w(t))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)}+p(t)\exp\Big(\alpha\int_t^{h(t)}\big(\frac{w(s)}{r(s)}\big)^{1/\alpha}ds\Big)=0, \tag{2.4}$$

consequently,

$$w'(t)+p(t)\leq 0.$$

Integrating the above inequality from t_1 to t ($t>t_1$), we have

$$w(t)\leq w(t_1)-\int_{t_1}^{t}p(s)ds\to-\infty\quad\text{as } t\to+\infty,$$

which contradicts $w(t)>0$. □

According to Theorem 2.1, we can furthermore restrict our attention to the case:

$$\int_t^{\infty}p(s)ds<\infty. \tag{2.5}$$

For convenience, we define $P(t)=\int_t^{\infty}p(s)ds$ for $t\geq t_0$. Firstly, we give the following Lemma.

Lemma 2.2. *Let* (2.5) *hold. Suppose that* (1.1) *has a nonoscillatory solution* $x(t)\neq 0$ *for* $t\geq t_1\geq t_0$*, and let* $w(t)$ *be defined by* (2.2)*. Then the following statements hold for* $t\geq t_1$*:*

$$w(t)>0,\quad \lim_{t\to\infty}w(t)=0, \tag{2.6}$$

$$\int_t^{\infty}\frac{(w(s))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(s)}ds<\infty, \tag{2.7}$$

$$I(t)=\int_t^{\infty}p(s)\exp\Big(\alpha\int_s^{h(s)}\big(\frac{w(\tau)}{r(\tau)}\big)^{1/\alpha}d\tau\Big)ds<\infty, \tag{2.8}$$

$$w(t)=\alpha\int_t^{\infty}\frac{(w(s))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(s)}ds+I(t). \tag{2.9}$$

Proof. Assume that $x(t) > 0$ on $[t_1, \infty)$. A similar argument holds if we assume $x(t) < 0$ on $[t_1, \infty)$. Proceeding as in the proof of Theorem 2.1, we know $x'(t) > 0$ for $t \geq t_1$. Hence, $w(t) > 0$ for $t \geq t_1$, and (2.4) holds and

$$w'(t) \leq -\alpha \frac{(w(t))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)}.$$

It follows that

$$\frac{1}{w^{1/\alpha}(t)} - \frac{1}{w^{1/\alpha}(t_1)} \geq \int_{t_1}^{t} \frac{1}{r^{1/\alpha}(s)} ds \to \infty \quad \text{as } t \to +\infty,$$

thus $\lim_{t\to\infty} w(t) = 0$. Integrating (2.4) from t to T ($T \geq t \geq t_1$), we have

$$w(T) - w(t) + \alpha \int_t^T \frac{(w(s))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(s)} ds + \int_t^T p(s) \exp\Big(\alpha \int_s^{h(s)} \big(\frac{w(\tau)}{r(\tau)}\big)^{1/\alpha} d\tau\Big) ds = 0. \tag{2.10}$$

Let $T \to \infty$, then from (2.10) it follows that

$$w(t) = \alpha \int_t^\infty \frac{(w(s))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(s)} ds + I(t), \quad t \geq t_1.$$

Hence, (2.9) holds. Furthermore, (2.7) and (2.8) hold. □

Theorem 2.3. *Let* (2.5) *hold. Equation* (1.1) *is non-oscillatory if and only if there exist* $t_1 \geq t_0$ *and* $\varphi(t) \in C^1([t_1, \infty), \mathbb{R}^+)$ *such that*

$$\varphi'(t) + \alpha \frac{(\varphi(t))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)} + p(t) \exp\Big(\alpha \int_t^{h(t)} \big(\frac{\varphi(s)}{r(s)}\big)^{1/\alpha} ds\Big) \leq 0, \quad t \geq t_1. \tag{2.11}$$

Proof. The "only if" part. Let $x(t)$ be a non-oscillatory solution of (1.1). Assume that $x(t) > 0$ and $x(h(t)) > 0$ for $t \geq t_1$. Then, by Lemma 2.2, the function $w(t) \in C^1([t_1, \infty), \mathbb{R}^+)$ defined by (2.2) satisfies (2.9). Differentiation of (2.9) shows that $w(t)$ is a solution of (2.11) on $[t_1, \infty)$.

The "if" part. It follows from (2.11) that $\varphi'(t) < 0$, hence $\varphi(t)$ is decreasing and is bounded from below; consequently, its limit exists, namely, $\lim_{t\to\infty} \varphi(t) = d \geq 0$. Next, we prove that $d = 0$. Indeed, it follows from (2.11) that

$$\varphi'(t) \leq -\alpha \frac{(\varphi(t))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)}.$$

Dividing both sides of the above inequality by $(\varphi(t))^{(\alpha+1)/\alpha}$, and integrating from t to T, then we obtain

$$\frac{1}{\varphi^{1/\alpha}(T)} - \frac{1}{\varphi^{1/\alpha}(t)} \geq \int_t^T \frac{ds}{r^{1/\alpha}(s)},$$

letting $T \to \infty$ in the above, we have $\lim_{T\to\infty} \varphi(T) = 0$. Then integrating (2.11) from t to ∞, we have

$$\alpha \int_t^\infty \frac{(\varphi(s))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(s)} ds + \int_t^\infty p(s) \exp\Big(\alpha \int_s^{h(s)} \big(\frac{\varphi(\tau)}{r(\tau)}\big)^{1/\alpha} d\tau\Big) ds \leq \varphi(t), \quad t \geq t_1,$$

which implies that for $t \geq t_1$,

$$\int_t^\infty \frac{(\varphi(s))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(s)} ds < \infty, \quad \int_t^\infty p(s) \exp\Big(\alpha \int_s^{h(s)} \big(\frac{\varphi(\tau)}{r(\tau)}\big)^{1/\alpha} d\tau\Big) ds < \infty.$$

Define the following mapping

$$(Ly)(t) = \alpha \int_t^\infty \frac{(y(s))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(s)} ds + \int_t^\infty p(s) \exp\Big(\alpha \int_s^{h(s)} \big(\frac{y(\tau)}{r(\tau)}\big)^{1/\alpha} d\tau\Big) ds, \quad (2.12)$$

for $t \geq t_1$. Let

$$x_0(t) \equiv 0, \quad x_n(t) = L(x_{n-1}(t)), \quad n = 1, 2, 3, \dots.$$

It is easy to show that

$$x_0(t) \leq x_1(t) \leq \dots \leq x_n(t) \leq \dots \leq \varphi(t).$$

Hence

$$\lim_{n\to\infty} x_n(t) = u(t) \leq \varphi(t).$$

By (2.12), we have

$$x_n(t) = \alpha \int_t^\infty \frac{(x_{n-1}(s))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(s)} ds + \int_t^\infty p(s) \exp\Big(\alpha \int_s^{h(s)} \big(\frac{x_{n-1}(\tau)}{r(\tau)}\big)^{1/\alpha} d\tau\Big) ds,$$

for $t \geq t_1$. By Levi's monotone convergence theorem, and letting $n \to \infty$ in the above equation, we obtain

$$u(t) = \alpha \int_t^\infty \frac{(u(s))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(s)} ds + \int_t^\infty p(s) \exp\Big(\alpha \int_s^{h(s)} \big(\frac{u(\tau)}{r(\tau)}\big)^{1/\alpha} d\tau\Big) ds, \quad t \geq t_1. \quad (2.13)$$

Set

$$x(t) = \exp\Big(\int_{t_1}^t \big(\frac{u(\tau)}{r(\tau)}\big)^{1/\alpha} d\tau\Big), \quad t \geq t_1 .$$

Then

$$u(t) = \frac{r(t)(x'(t))^\alpha}{(x(t))^\alpha} . \quad (2.14)$$

By (2.13) and (2.14), we have

$$(r(t)(x'(t))^\alpha)' + p(t)(x(h(t)))^\alpha = 0;$$

i.e.,

$$(r(t)|x'(t)|^{\alpha-1}x'(t))' + p(t)|x(h(t))|^{\alpha-1}x(h(t)) = 0, \quad t \geq t_1.$$

Thus, $x(t)$ is a non-oscillatory solution of (1.1). □

Corollary 2.4. *Let (2.5) hold. If $h(t) \equiv t$, then (1.1) is non-oscillatory if and only if there exist $t_1 \geq t_0$, and $\varphi(t) \in C^1([t_1,\infty), \mathbb{R}^+)$ such that*

$$\varphi'(t) + \alpha \frac{\varphi^{(\alpha+1)/\alpha}(t)}{r^{1/\alpha}(t)} + p(t) \leq 0, \quad t \geq t_1,$$

We remark that for (1.4), Theorem 2.3 and Corollary 2.4 reduce to [6, Lemma 2] and [6, Corollary 1], respectively.

Let (2.5) hold. Define a sequence of functions $\{\upsilon_n(t)\}_0^\infty$ as follows (if they exist):

$$\upsilon_0(t) = P(t) = \int_t^\infty p(s) ds,$$

$$\upsilon_{n+1}(t) = \alpha \int_t^\infty \frac{(\upsilon_n(s))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(s)} ds + \int_t^\infty p(s) \exp\Big(\alpha \int_s^{h(s)} \big(\frac{\upsilon_n(\tau)}{r(\tau)}\big)^{1/\alpha} d\tau\Big) ds, \quad (2.15)$$

for $n = 0, 1, 2, \dots,\ t \ge t_1$. Clearly, $\upsilon_0(t) \ge 0$ and $\upsilon_1(t) \ge \upsilon_0(t)$. By induction, we obtain

$$\upsilon_{n+1}(t) \ge \upsilon_n(t), \quad n = 0, 1, 2 \dots; \tag{2.16}$$

i.e., the sequence $\{\upsilon_n(t)\}_0^\infty$ is nondecreasing on $[t_0, \infty)$.

Theorem 2.5. *Let* (2.5) *hold. Then* (1.1) *is non-oscillatory if and only if there exists $t_1 \ge t_0$ such that $\{\upsilon_n(t)\}_0^\infty$ exists and converges; i.e.,*

$$\lim_{n\to\infty} \upsilon_n(t) = \upsilon(t) < \infty, \quad t \ge t_1. \tag{2.17}$$

Proof. The "only if" part. Suppose that $x(t)$ is a non-oscillatory solution of (1.1). Without loss of generality, we assume that $x(t) > 0$ and $x(h(t)) > 0$ on $[t_1, \infty)$. Let $w(t)$ be defined by (2.2), by Lemma 2.2, we obtain (2.9), which follows

$$w(t) \ge \int_t^\infty p(s) \exp\Big(\alpha \int_s^{h(s)} \big(\frac{w(\tau)}{r(\tau)}\big)^{1/\alpha} d\tau\Big) ds \ge \int_t^\infty p(s) ds = \upsilon_0(t) \ge 0.$$

By (2.9) again, we have

$$w(t) \ge \alpha \int_t^\infty \frac{(\upsilon_0(s))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(s)} ds + \int_t^\infty p(s) \exp\Big(\alpha \int_s^{h(s)} \big(\frac{\upsilon_0(\tau)}{r(\tau)}\big)^{1/\alpha} d\tau\Big) ds = \upsilon_1(t).$$

By induction, we obtain

$$w(t) \ge \upsilon_n(t) \ge 0,\ n = 0, 1, 2 \dots, \quad t \ge t_1. \tag{2.18}$$

It follows from (2.16) and (2.18) that (2.17) holds.

The "if" part. Assume that the function sequence $\{\upsilon_n(t)\}_0^\infty$ exists and converges. It follows from (2.16) and (2.17) that

$$0 \le \upsilon_n(t) \le \upsilon(t), \quad n = 1, 2, \dots,\ t \ge t_1.$$

By Levi's monotone convergence theorem for (2.15), we obtain

$$\upsilon(t) = \alpha \int_t^\infty \frac{(\upsilon(s))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(s)} ds + \int_t^\infty p(s) \exp\Big(\alpha \int_s^{h(s)} \big(\frac{\upsilon(\tau)}{r(\tau)}\big)^{1/\alpha} d\tau\Big) ds.$$

Consequently,

$$\upsilon'(t) + \alpha \frac{(\upsilon(t))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)} + p(t) \exp\Big(\alpha \int_t^{h(t)} \big(\frac{\upsilon(s)}{r(s)}\big)^{1/\alpha} ds\Big) = 0,\ \ t \ge t_1.$$

Then, by Theorem 2.3, (1.1) is non-oscillatory. □

As a consequence of Theorem 2.5, we have the following result.

Theorem 2.6. *Let* (2.5) *hold. Then* (1.1) *is oscillatory if one of the following conditions holds:*

1. *There exists an integer m such that $\upsilon_n(t)$ is defined for $n = 1, 2, \dots, m-1$, but $\upsilon_m(t)$ does not exist;*
2. *$\{\upsilon_n(t)\}_0^\infty$ is defined for $n = 1, 2, \dots,$ but for arbitrarily large $T \ge t_0$, there exists $t^* > T$ such that $\lim_{n\to\infty} \upsilon_n(t^*) = \infty$.*

Corollary 2.7. *Let* (2.5) *hold. Assume that there exists $R(t) \in C^1([t_0, \infty), \mathbb{R}^+)$ with $R'(t) = r^{-1/\alpha}(t)$, and there exists $\lambda_0 > \alpha^\alpha/(\alpha+1)^{\alpha+1}$ such that for all sufficiently large t,*

$$R^\alpha(t) P(t) \ge \lambda_0. \tag{2.19}$$

Then (1.1) *is oscillatory .*

Proof. It follows from (2.19) that $\upsilon_0(t) \geq \lambda_0 R^{-\alpha}(t)$, which implies, by (2.15),

$$\upsilon_1(t) \geq \upsilon_0(t) + \alpha\lambda_0^{(\alpha+1)/\alpha} \int_t^\infty \frac{dR(s)}{R^{\alpha+1}(s)} \geq \frac{\lambda_1}{R^\alpha(t)}, \quad \lambda_1 = \lambda_0 + \lambda_0^{(\alpha+1)/\alpha} > \lambda_0.$$

By induction, we can show that

$$\upsilon_{n+1}(t) \geq \frac{\lambda_{n+1}}{R^\alpha(t)} \quad \text{and} \quad \lambda_{n+1} = \lambda_0 + \lambda_n^{(\alpha+1)/\alpha} > \lambda_n, \quad \text{for } n = 1, 2, \dots.$$

Now we claim that $\lim_{n\to\infty} \lambda_n = \infty$. Otherwise, as λ_n is monotone increasing, we must have $\lim_{n\to\infty} \lambda_n = \lambda < \infty$, and $\lambda > 0$ satisfies the equation $\lambda = \lambda_0 + \lambda^{(\alpha+1)/\alpha}$. Note that $\lambda_0 > \alpha^\alpha/(\alpha+1)^{\alpha+1}$, then, by Hölder inequality, we have

$$\begin{aligned} \lambda = \lambda_0 + \lambda^{(\alpha+1)/\alpha} &> \frac{\alpha+1}{\alpha}\Big[\frac{1}{\alpha+1}\big(\frac{\alpha}{\alpha+1}\big)^{\alpha+1} + \frac{\alpha}{\alpha+1}\lambda^{(\alpha+1)/\alpha}\Big] \\ &\geq \frac{\alpha+1}{\alpha}\frac{\alpha}{\alpha+1}\lambda = \lambda, \end{aligned}$$

which is impossible. Hence, the claim is true. Consequently, $\lim_{n\to\infty} \upsilon_n(t) = \infty$. Thus, by Theorem 2.6 (2), Equaton (1.1) is oscillatory. □

Corollary 2.8. *Let* (2.5) *hold. Assume that there exists* $\gamma_0 > (\alpha+1)^{-(\alpha+1)/\alpha}$ *such that for all sufficiently large* t,

$$\int_t^\infty \frac{P^{(\alpha+1)/\alpha}(s)}{r^{1/\alpha}(s)} ds \geq \gamma_0 P(t). \tag{2.20}$$

Then (1.1) *is oscillatory.*

Proof. It follows from (2.15) and (2.20) that

$$\upsilon_1(t) \geq \gamma_1 P(t), \quad \gamma_1 = 1 + \alpha\gamma_0 > 1.$$

Assume that $\upsilon_n(t) \geq \gamma_n P(t)$, then, by (2.15) again and induction, we have

$$\upsilon_{n+1}(t) \geq \gamma_{n+1} P(t), \quad \gamma_{n+1} = 1 + \alpha\gamma_0\gamma_n^{(\alpha+1)/\alpha}, \quad n = 1, 2, \dots.$$

We now claim that

$$\gamma_{n+1} > \gamma_n, \quad n = 1, 2, \dots. \tag{2.21}$$

Indeed, in view of the fact that $\gamma_1 > 1$ and $(\alpha+1)/\alpha > 1$, we have

$$r_2 = 1 + \alpha\gamma_0\gamma_1^{(\alpha+1)/\alpha} > 1 + \alpha\gamma_0 = \gamma_1.$$

Moreover, we have

$$r_3 = 1 + \alpha\gamma_0\gamma_2^{(\alpha+1)/\alpha} > 1 + \alpha\gamma_0\gamma_1^{(\alpha+1)/\alpha} = \gamma_2.$$

Hence, by induction, we can show that (2.21) holds. Then, by an argument similar to the proof of Corollary 2.7, we can prove $\lim_{n\to\infty} \lambda_n = \infty$; consequently $\lim_{n\to\infty} \upsilon_n(t) = \infty$. It follows from Theorem 2.6 (2) that (1.1) is oscillatory. □

Theorem 2.9. *Let* (2.5) *hold. If* (1.1) *has a nonoscillatory solution, then*

$$\lim_{t\to\infty} \upsilon(t) \exp\Big(\alpha \int_{t_1}^t \big(\frac{P(s)}{r(s)}\big)^{1/\alpha} ds\Big) < \infty, \tag{2.22}$$

where $\upsilon(t)$ *satisfies* (2.17).

Proof. Suppose $x(t) \neq 0$ is a nonoscillatory solution of (1.1) for $t \geq t_1$. Let $w(t)$ be defined by (2.2), it follows from (2.4) and (2.9) that

$$\begin{aligned} -w'(t) &= \alpha \frac{(w(t))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)} + p(t) \exp\Big(\alpha \int_t^{h(t)} \big(\frac{w(s)}{r(s)}\big)^{1/\alpha} ds\Big) \\ &\geq \alpha \frac{(w(t))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)} = \alpha w(t) \big(\frac{w(t)}{r(t)}\big)^{1/\alpha} \\ &\geq \alpha w(t) \big(\frac{P(t)}{r(t)}\big)^{1/\alpha}, \end{aligned}$$

hence,

$$w(t) \leq w(t_1) \exp\Big(-\alpha \int_{t_1}^{t} \big(\frac{P(s)}{r(s)}\big)^{1/\alpha} ds\Big). \tag{2.23}$$

On the other hand, by induction, we have $w(t) \geq \upsilon_n(t)$, $n = 0, 1, 2, \dots$. Combining this with (2.23), we obtain

$$\upsilon_n(t) \exp\Big(\alpha \int_{t_1}^{t} \big(\frac{P(s)}{r(s)}\big)^{1/\alpha} ds\Big) \leq w(t_1), \quad n = 1, 2, \dots. \tag{2.24}$$

Note that from Theorem 2.5, it follows that $\lim_{n\to\infty} \upsilon_n(t) = \upsilon(t)$, then by (2.24), we have

$$\lim_{n\to\infty} \upsilon_n(t) \exp\Big(\alpha \int_{t_1}^{t} \big(\frac{P(s)}{r(s)}\big)^{1/\alpha} ds\Big) = \upsilon(t) \exp\Big(\alpha \int_{t_1}^{t} \big(\frac{P(s)}{r(s)}\big)^{1/\alpha} ds\Big) \leq w(t_1),$$

and then we obtain the desired inequality (2.22). $\square$

As a direct consequence of Theorem 2.9, we obtain the following theorem.

Theorem 2.10. *Let* (2.5) *hold, and* $\upsilon_n(t)$ *be defined for* $n = 1, 2, \dots, m$. *If one of the following conditions holds:*

1. $\lim_{t\to\infty} \upsilon_m(t) \exp\Big(\alpha \int_{t_0}^{t} \big(\frac{P(s)}{r(s)}\big)^{1/\alpha} ds\Big) = \infty$,
2. *Condition* (2.17) *holds, and* $\lim_{t\to\infty} \upsilon(t) \exp\Big(\alpha \int_{t_0}^{t} \big(\frac{P(s)}{r(s)}\big)^{1/\alpha} ds\Big) = \infty$,

then (1.1) *is oscillatory.*

Theorem 2.11. *Let* (2.5) *hold and*

$$\lim_{t\to\infty} \int_{t_0}^{t} \exp\Big(-\alpha \int_{t_0}^{s} \big(\frac{P(\tau)}{r(\tau)}\big)^{1/\alpha} d\tau\Big) ds < \infty\,. \tag{2.25}$$

If there exists $m \geq 1$ *such that*

$$\lim_{t\to\infty} \int_{t_0}^{t} \upsilon_m(s) ds = \infty, \tag{2.26}$$

then (1.1) *is oscillatory.*

Proof. Assume that $x(t) \neq 0$ is a non-oscillatory solution of (1.1) for $t \geq t_1$. Let $w(t)$ be defined by (2.2), similar to the proof of Theorem 2.9, we have

$$\upsilon_m(t) \leq w(t_1) \exp\Big(-\alpha \int_{t_1}^{t} \big(\frac{P(s)}{r(s)}\big)^{1/\alpha} ds\Big), \quad m \geq 1. \tag{2.27}$$

Integrating (2.27) from t_1 to t, and then letting $t \to \infty$ makes (2.25) contradict (2.26). Hence, (1.1) is oscillatory. $\square$

3. Examples

In this section, we will give some examples to illustrate our main results.

Example 3.1. Consider the equation

$$\Big(\frac{1}{t}|x'(t)|^{-1/2}x'(t)\Big)' + \frac{3\lambda}{2t^{5/2}}|x(3t)|^{-1/2}x(3t) = 0, \quad t \ge t_0, \tag{3.1}$$

where

$$\alpha = \frac{1}{2}, \quad r(t) = \frac{1}{t}, \quad h(t) = 3t, \quad \lambda > 0.$$

Then $R^{1/2}(t)P(t) = \lambda\sqrt{3}/3$. By Corollary 2.7, if there exists $\lambda_0 > 2\sqrt{3}/3$ such that $\lambda \ge \sqrt{3}\lambda_0$, i.e., $\lambda > 2$, then (3.1) is oscillatory.

Example 3.2. Consider the equation

$$\big(t|x'(t)|x'(t)\big)' + \frac{k}{t^2}|x(2t)|x(2t) = 0, \quad t \ge t_0, \tag{3.2}$$

where $\alpha = 2$, $r(t) = t$, $h(t) = 2t$, $k > 0$. Then $P(t) = k/t$. If $k > 1/27$, then (3.2) is oscillatory. Indeed, note that there exists $\gamma_0 \in (\frac{\sqrt{3}}{9}, \sqrt{k})$, then

$$\int_t^\infty \frac{P^{1+1/\alpha}(s)}{r^{1/\alpha}(s)}ds = \frac{k^{3/2}}{t} = \frac{k}{t}\sqrt{k} \ge \gamma_0\frac{k}{t} > \frac{P(t)}{(\alpha+1)^{(\alpha+1)/\alpha}},$$

for all sufficiently large t. Hence, by Corollary 2.8, the conclusion holds.

Example 3.3. Consider the equation

$$\Big(\frac{1}{\sqrt{t}}|x'(t)|^{1/2}x'(t)\Big)' + \frac{k}{t^{5/2}}\Big(\frac{3}{2} + \frac{3}{2\ln t} + \frac{1}{\ln^2 t}\Big)|x(2t)|^{1/2}x(2t) = 0, \quad t \ge 1, \tag{3.3}$$

where

$$k > 0, \quad \alpha = \frac{3}{2}, \quad r(t) = \frac{1}{\sqrt{t}}, \quad h(t) = 2t, \quad p(t) = \frac{k}{t^{5/2}}(\frac{3}{2} + \frac{3}{2\ln t} + \frac{1}{\ln^2 t}).$$

Note that

$$\upsilon_0(t) = P(t) = \frac{k}{t^{3/2}}(1 + \frac{1}{\ln t}), \quad \upsilon_1(t) > \frac{9k^{5/3}}{7t^{7/6}} + \frac{k}{t^{3/2}}(1 + \frac{1}{\ln t}).$$

Then

$$\begin{aligned}
&\lim_{t\to\infty} \upsilon_1(t)\exp\Big(\alpha\int_1^t (\frac{P(s)}{r(s)})^{1/\alpha}ds\Big)\\
&\ge \lim_{t\to\infty}\Big(\frac{9k^{5/3}}{7t^{7/6}} + \frac{k}{t^{3/2}}(1+\frac{1}{\ln t})\Big)\exp\Big(\frac{3}{2}\int_1^t (\frac{k(1+\frac{1}{\ln s})}{s})^{2/3}ds\Big)\\
&\ge \lim_{t\to\infty}\Big(\frac{9k^{5/3}}{7t^{7/6}} + \frac{k}{t^{3/2}}\Big)\exp\Big(\frac{3}{2}\int_1^t (\frac{k}{s})^{2/3}ds\Big)\\
&\ge \lim_{t\to\infty}\frac{k_1}{t^{3/2}}e^{k_2t^{1/3}} = \infty,
\end{aligned}$$

where $k_1 = ke^{-9/2k^{2/3}}$ and $k_2 = 9k^{2/3}/2$. Thus, Theorem 2.10 (1) is satisfied for $m = 1$. Hence (3.3) is oscillatory.

References

[1] R. P. Agarwal, S. R. Grace, D. O'Regan; *Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations*, Kluwer, Dordrecht, 2002.
[2] O. Došlý, P. Řehák; *Half-linear Differential Equations*, Elsevier, Amsterdam, 2005.
[3] H. Hoshino, R. Imabayashi, T. Kusano, T. Tanigawa; *On second-order half-linear oscillations*, Adv. Math. Sci. Appl. 8 (1998) 199-210.
[4] T. Kusano, Y. Naito; *Oscillation and nonoscillation criteria for second order quasilinear differential equations*, Acta Math. Hungar. 76 (1-2) (1997) 81-99.
[5] H.J. Li, C.C. Yeh; *Nonoscillation criteria for second-order half-linear differential equations*, Appl. Math. Lett. 8 (1995) 63-70.
[6] W. D. Lu; *Oscillation property for second order advanced differential equations*, Chinese Ann. Math. 12 (1991) 133-138 (in Chinese).
[7] K. Takasi, N. Yoshida; *Nonoscillation theorems for a class of quasilinear differential equations of second order*, J. Math. Anal. Appl. 189 (1995) 115-127.
[8] A. Wintner; *On the non-existence of conjugate points*, Amer. J. Math. 73 (1951) 368-380.
[9] J. Yan; *Oscillation property for second order differential equations with an "integral small" coefficient*, Acta Math. Sinica. 30 (1987) 206-215 (in Chinese).
[10] X. Yang, K. Lo; *Nonoscillation criteria for quasilinear second order differential equations*, J. Math. Anal. Appl. 331 (2007) 1023-1032.

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