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OSCILLATION CRITERIA FOR THIRD-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH FUNCTIONAL ARGUMENTS

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Abstract. In this article, we consider the third-order nonlinear differential equations with functional arguments. By using the Riccati inequality, we find conditions for all solutions to be oscillatory.

1. Introduction

We are concerned with the oscillation of solutions to the nonlinear third-order functional differential equation

$$y'''(t) + a(t)y''(t) + b(t)y'(t) + \sum_{i=1}^{m} c_i(t)\varphi_i(y(\sigma_i(t))) = 0, \quad t > 0. \tag{1.1}$$

Throughout this paper we assume the following conditions:

- (H1) $a(t), b(t), c_i(t) \in C((0,\infty);[0,\infty))$, $(i = 1, 2, \dots, m)$;
- (H2) $\sigma_i(t) \in C([0,\infty);\mathbb{R})$, $\lim_{t\to\infty} \sigma_i(t) = \infty$ $(i = 1, 2, \dots, m)$, there exists a positive constant σ such that
$$\sigma_j'(t) \ge \sigma \quad \text{and} \quad t \ge \sigma_j(t)$$
for some $j \in \{1, 2, \dots, m\}$;
- (H3) $\varphi_i(s) \in C^1(\mathbb{R};\mathbb{R})$ $(i = 1, 2, \dots, m)$, $\varphi_i(-s) = -\varphi_i(s)$ for $s \ge 0$, $\varphi_j'(s) > 0$, $\varphi_j'(s)$ is nondecreasing for $s > 0$ and some $j \in \{1, 2, \dots, m\}$.

Definition 1.1. By a solution of (1.1) we mean a function $y(t) \in C^3([T_y,\infty);\mathbb{R})$ satisfying $\sup\{|y(t)| : t > T_y\} > 0$ for any $T_y \ge t_y$, where

$$t_y = \min\Big\{0, \min_{1\le i\le m} \big\{ \inf_{t\ge 0} \sigma_i(t)\big\}\Big\}.$$

A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros, otherwise it is non-oscillatory.

Definition 1.2. A function H belongs to the class $\mathbb{H}$, if H is in $C(D;[0,\infty))$; H satisfies

$$H(t,t) = 0, \quad H(t,s) > 0 \quad \text{for } t > s > t_1,$$

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where $D = \{(t,s) \in \mathbb{R}^2 : t \geq s \geq t_1\}$; there exists a constant $k_0 > 0$ such that

$$\lim_{t\to\infty} \frac{H(t,s)}{H(t,t_1)} = k_0 \quad \text{for all } t \geq t_1;$$

and the partial derivative $\partial H/\partial s$ exists on $D_0 = \{(t,s) \in \mathbb{R}^2; t > s \geq t_1\}$ and satisfies

$$\frac{\partial H}{\partial s}(t,s) = -h(t,s)H(t,s),$$

for some function h in $C(D_0;\mathbb{R})$.

Since the work by Hanan [5] was published, oscillation of solutions to third-order differential equations in special cases have been widely studied by many authors [1, 2, 3, 4, 5, 6, 8, 9, 10]. This maybe because third-order differential equations have applications in mechanical, physical and biological problems [8], and because (1.1) plays an important role in control theory.

In the mid-nineteenth century, Maxwell analyzed the stability problem of the Watt's governor, and obtained conditions for stability which are based on third-order linear differential equations with constant coefficients. Later, Routh and Hurwitz derived more general stability conditions which are known today as the Routh-Hurwitz stability criteria. In 1976, Erbe [4] studied the oscillatory and asymptotic behavior of solutions of the equation

$$y'''(t) + a(t)y''(t) + b(t)y'(t) + c(t)y^\alpha(t) = 0, \tag{1.2}$$

where α is the quotient of positive odd integers.

Theorem 1.3 ([4, Theorem 4.9]). *Let $a(t)b(t) + b'(t) \leq 0$ and $y(t)$ be a nontrivial solution of (1.2) with $F[y(t_0)] \leq 0$ for some $t_0 > 0$, where*

$$F[y(t)] = e^{A(t)}[2y''(t)y(t) - y'^2(t) + b(t)y^2(t)].$$

If the equation

$$\left(e^{A(t)}z'(t)\right)' + e^{A(t)}\{b(t)z(t) + \lambda^\alpha t^\alpha c(t)z^\alpha(t)\} = 0 \tag{1.3}$$

is oscillatory (that is, all solutions of (1.3) are oscillatory) for some $0 < \lambda < \frac{1}{2}$, then $y(t)$ is oscillatory.

Tiryki and Aktas [10], Agarwal et al [1], and Aktas et al [2] studied third-order nonlinear differential equations of the form

$$\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)' + p(t)y'(t) + q(t)\varphi(y(\sigma(t))) = 0. \tag{1.4}$$

Aktas et al [2] established the following results which ensures that every solution is oscillatory or converges to zero.

Theorem 1.4 ([2, Theorem 3.1]). *Assume that*

$$R_1(t,t_0) = \int_{t_0}^t \frac{ds}{r_1(s)} \to \infty, \quad R_2(t,t_0) = \int_{t_0}^t \frac{ds}{r_2(s)} \to \infty \quad \textit{as } t \to \infty,$$

that there exist functions $\phi(t)$ and $\rho_1(t)$ in $C([0,\infty);(0,\infty))$ such that

$$\rho_1'(t) \geq 0, \quad \phi(t) = \left(r_2(t)\rho_1'(t)\right)' r_1(t) + \rho_1(t)p(t) \geq 0,$$

$$\phi'(t) \leq 0, \quad \int^\infty \rho_1(t)q(t)dt = \infty,$$

and that the equation

$$(r_2(t)z'(t))' + \frac{p(t)}{r_1(t)}z(t) = 0$$

is non-oscillatory. If there exists a function $\rho_2(t) \in C^1([0,\infty);(0,\infty))$ *such that*

$$\limsup_{t\to\infty} \int_T^t \{\rho_2(s)q(s) - \frac{\beta^2(s)}{4\alpha(s)}\}ds = \infty,$$

then every solution of (1.4) *either oscillates or converges to zero as* $t \to \infty$*. Here*

$$\alpha(t) = \frac{K_0 R_2(\sigma(t),t)\sigma'(t)}{r_1(\sigma(t))\rho_2(t)}, \quad \beta(t) = \frac{\rho_2'(t)}{\rho_2(t)} - p(t)\frac{R_2(\sigma(t),t)}{r_1(t)}.$$

For the case when (1.1) has constant coefficients, it is easy to see that neither $a(t)b(t) + b(t) \le 0$ in Theorem 1.3, nor $R_2(t,t_0) \to \infty$ in Theorem 1.4 is satisifed. The natural question to ask is:

> Is it possible to find oscillation criteria for equation (1.1), which include the case of constant coefficients?

In this article we obtain an affirmative answer to this question.

2. Preliminaries

First we sate an assumption to be used in the next lemma, which is needed for proving our main results.

(H4) $a(t) \ge b(t) + 1$.

Lemma 2.1. *Assume that* (A4) *holds,*

$$\int_0^\infty \pi(t)e^{A(t)}\sum_{i=1}^m c_i(t)dt = \infty, \tag{2.1}$$

where

$$A(t) = \int_0^t a(s)ds, \quad \pi(t) = \int_t^\infty e^{-A(s)}ds,$$

and $y(t)$ *is a non-oscillatory solution of* (1.1)*. Then there exists a* $t_0 > 0$ *such that*

$$y(t)y'(t) > 0, \quad \forall t \ge t_0. \tag{2.2}$$

Proof. Suppose that $y(t)$ is a non-oscillatory solution of (1.1). Without loss of generality, we assume that $y(t) > 0$ and $y(\sigma_i(t)) > 0$ $(i = 1,2,\dots,m)$. Note that if $y(t)$ is a negative solution, then $-y(t)$ is a positive solution of (1.1).

We claim that $y'(t)$ is non-oscillatory. If $y'(t)$ is socillatory, then $x(t) = -y'(t)$ is oscillatory and satisfies

$$\left(e^{A(t)}x'(t)\right)' + b(t)e^{A(t)}x(t) = \sum_{i=1}^m c_i(t)e^{A(t)}\varphi_i(y(\sigma_i(t))) \ge 0. \tag{2.3}$$

Let $x(t)$ have consecutive zeros at α and β $(t_0 < \alpha < \beta)$ such that $x'(\alpha) \ge 0$, $x'(\beta) \le 0$ and $x(t) \ge 0$ for $t \in (\alpha,\beta)$. Multiplying (2.3) by e^{-t} and integrating over $[\alpha,\beta]$, we obtain

$$\int_\alpha^\beta e^{-t}\left(e^{A(t)}x'(t)\right)' dt + \int_\alpha^\beta b(t)e^{A(t)}x(t)e^{-t}dt \ge 0.$$

Integrating by parts,

$$e^{A(\beta)-\beta}x'(\beta) - e^{A(\alpha)-\alpha}x'(\alpha) + \int_\alpha^\beta e^{A(t)-t}x'(t)dt + \int_\alpha^\beta b(t)e^{A(t)-t}x(t)dt \ge 0.$$

Integrating by parts again and using that $x(\alpha) = x(\beta) = 0$,

$$e^{A(\beta)-\beta}x'(\beta) - e^{A(\alpha)-\alpha}x'(\alpha) \ge \int_\alpha^\beta e^{A(t)-t}\{a(t) - 1 - b(t)\}x(t)dt \ge 0.$$

Since $x'(\alpha) \ge 0$ and $x'(\beta) \le 0$, the above inequality is a contradiction. Therefore, $x(t)$ is non-oscillatory, and there are two possible cases:

Case 1: $x(t) < 0$ for all t large enough. By definition $x(t) = -y'(t)$. So $y'(t) > 0$ while $y(t) > 0$ for all t large enough. Therefore, (2.2) is satisfied.

Case 2: $x(t) > 0$ for all t large enough. Then $y'(t) < 0$ while $y(t) > 0$. From the continuity of φ_i, there is a positive constant K_0 such that

$$\varphi_i(y(\sigma_i(t))) \le K_0\,.$$

From (2.3),

$$\Big(e^{A(t)}x'(t)\Big)' + b(t)e^{A(t)}x(t) \le K_0 e^{A(t)} \sum_{i=1}^m c_i(t)\,. \tag{2.4}$$

Let

$$v(t) = x(t) + K_0 \int_t^\infty e^{-A(s)} \int_{t_0}^s e^{A(\xi)} \sum_{i=1}^m c_i(\xi)d\xi ds,\,. \tag{2.5}$$

Then (2.4) implies

$$\Big(e^{A(t)}v'(t)\Big)' \le -b(t)e^{A(t)}x(t) \le 0.$$

From this inequality, either $v'(t) \ge 0$ or $v'(t) < 0$ for all t large enough. Differentiating (2.5), we have

$$v'(t) = x'(t) - K_0 e^{-A(t)} \int_{t_0}^t e^{A(s)} \sum_{i=1}^m c_i(s)ds \le x'(t) = -y''(t).$$

If $v'(t) \ge 0$, then $y''(t) = -v'(t) \le 0$. Since $y'(t) < 0$ and $y''(t) \le 0$, we have $\lim_{t\to\infty} y(t) = -\infty$ which contradicts $y(t) \ge 0$. Therefore, $v'(t) < 0$. From $v(t) > 0$ and $v'(t) < 0$ it follows that there is a constant $K_1 > 0$ such that

$$\begin{aligned} K_1 > v(t) &> K_0 \int_t^\infty e^{-A(s)} \int_{t_0}^s e^{A(\xi)} \sum_{i=1}^m c_i(\xi)d\xi ds \\ &= K_0 \int_t^\infty (-\pi(s))' \Big(\int_{t_0}^s e^{A(\xi)} \sum_{i=1}^m c_i(\xi)d\xi \Big) ds \\ &\ge K_0 \int_{t_0}^t \pi(s)e^{A(s)} \sum_{i=1}^m c_i(s)ds, \end{aligned}$$

which contradicts the assumption (2.1). Therefore, Case 2 can not happen. The proof is complete. □

For the next lemma we use the assumption

(H5) there exists $a(t) \in C^1((0,\infty);[0,\infty))$ such that

$$b(t) \ge a'(t).$$

Lemma 2.2. *Assume that* (H5) *and* (2.1) *hold. If* $y(t)$ *is a nonoscillatory solution of* (1.1)*, then there exists a* $t_0 > 0$ *such that* (2.2) *is satisfied.*

Proof. Suppose that $y(t)$ is a non-oscillatory solution of (1.1).

without loss of generality, we assume that $y(t) > 0$ and $y(\sigma_i(t)) > 0$ ($i = 1, 2, \dots, m$). Note that if $y(t)$ is a negative solution, then $-y(t)$ is a positive solution of (1.1).

We claim that $y'(t)$ is non-oscillatory. If $y'(t)$ is oscillatory, then $x(t) = -y'(t)$ is oscillatory and satisfies

$$x''(t) + a(t)x'(t) + b(t)x(t) \geq 0. \tag{2.6}$$

Let $x(t)$ be a consecutive zeros at α and β ($t_0 < \alpha < \beta$) such that $x'(\alpha) \geq 0$ and $x'(\beta) \leq 0$. Multiplying (2.6) by $\frac{1}{a(t)} e^{\int_{t_0}^t \frac{b(s)}{a(s)} ds}$ and integrating over $[\alpha, \beta]$, we obtain

$$\int_\alpha^\beta \Big\{ \frac{1}{a(t)} e^{\int_{t_0}^t \frac{b(s)}{a(s)} ds} x''(t) + \Big(e^{\int_{t_0}^t \frac{b(s)}{a(s)} ds} x(t) \Big)' \Big\} dt \geq 0.$$

Integrating by parts,

$$\frac{1}{a(\beta)} e^{\int_{t_0}^\beta \frac{b(s)}{a(s)} ds} x'(\beta) - \frac{1}{a(\alpha)} e^{\int_{t_0}^\alpha \frac{b(s)}{a(s)} ds} x'(\alpha) \geq \int_\alpha^\beta \Big(\frac{1}{a(t)} e^{\int_{t_0}^t \frac{b(s)}{a(s)} ds} \Big)' x'(t) dt.$$

Integrating by parts again and using that $x(\alpha) = x(\beta) = 0$,

$$0 \geq \frac{1}{a(\beta)} e^{\int_{t_0}^\beta \frac{b(s)}{a(s)} ds} x'(\beta) - \frac{1}{a(\alpha)} e^{\int_{t_0}^\alpha \frac{b(s)}{a(s)} ds} x'(\alpha) \geq \int_\alpha^\beta \frac{(b(t) - a'(t))}{a^2(t)} e^{\int_{t_0}^t \frac{b(s)}{a(s)} ds} x'(t) dt,$$

which implies that $x'(t) \leq 0$ on $[\alpha, \beta]$. The rest of the proof is the same as in Lemma 2.1, and hence is omitted. □

Theorem 2.3. *Assume that* (H1)– (H4) *or* (H1)–(H3), (H5) *are satisfied. If* (2.1) *holds and the Riccati inequalities*

$$z'(t) + \frac{1}{2} \frac{1}{P_i(t)} z^2(t) \leq -Q_i(t) \quad (i = 1, 2)$$

have no solution on intervals $[T, \infty)$ *for all large* $T > 0$*, then every solution of* (1.1) *is oscillatory. Here*

$$P_1(t) = 1, \quad Q_1(t) = -\frac{1}{2} a^2(t) + b(t) + K_1 c_j(t),$$

$$P_2(t) = \frac{1}{\sigma K_2 A_e(\sigma_j(t))}, \quad Q_2(t) = c_j(t) e^{A(t)} - \frac{1}{2} \Big(\frac{b^2(t) A_e(\sigma_j(t))}{\sigma K_2} \Big),$$

$$A_e(\sigma_j(t)) = \int_0^{\sigma_j(t)} e^{-A(s)} ds$$

for some $K_i > 0$ ($i = 1, 2$)*, and some* $i \in \{1, 2, \dots, m\}$.

Proof. Suppose that $y(t)$ is a nonoscillatory solution of (1.1) on $[t_0, \infty)$ for some $t_0 \geq T > 0$. Then there exists a $t_1 \geq t_0$ such that $y(t) > 0$ and $y(\sigma_i(t)) > 0$ ($i = 1, 2, \dots, m$) for $t \geq t_1$. We shall consider only this case, because the proof when $y(t) < 0$ is similar. From (1.1), for each $j \in \{1, 2, \dots, m\}$, we have

$$\Big(e^{A(t)} y''(t) \Big)' + b(t) e^{A(t)} y'(t) + c_j(t) e^{A(t)} \varphi_j(y(\sigma_j(t))) \leq 0.$$

According Lemma 2.1 or Lemma 2.2, $y'(t) \geq 0$. Then from the above inequality,
$$\Big(e^{A(t)}y''(t)\Big)' \leq 0.$$
Hence $y''(t) \geq 0$ or $y''(t) < 0$. First we assume that $y''(t) < 0$. Letting
$$w_1(t) = \frac{e^{A(t)}y''(t)}{y'(t)},$$
we have
$$\begin{aligned} w_1'(t) &= \frac{\big(e^{A(t)}y''(t)\big)'}{y'(t)} - e^{-A(t)}w_1^2(t) \\ &\leq -b(t)e^{A(t)} - c_j(t)e^{A(t)}\frac{\varphi_j(y(\sigma_j(t)))}{y'(t)} - e^{-A(t)}w_1^2(t). \end{aligned}$$
On the other hand, there exist constants K_0 and K_1 such that
$$y(t) \geq K_0 \quad \text{and} \quad y'(t) \leq K_1.$$
It is easy to see that
$$w_1'(t) \leq -\Big(b(t) + \frac{K_0}{K_1}c_j(t)\Big)e^{A(t)} - e^{-A(t)}w_1^2(t).$$
Multiplying this by $e^{-A(t)}$, we obtain
$$\Big(e^{-A(t)}w_1(t)\Big)' + a(t)e^{-A(t)}w_1(t) \leq -\Big(b(t) + \frac{K_0}{K_1}c_j(t)\Big) - \Big(e^{-A(t)}w_1(t)\Big)^2. \tag{2.7}$$
By Hölder's inequality we have
$$|a(t)e^{-A(t)}w_1(t)| \leq \frac{1}{2}\Big(a^2(t) + \Big(e^{-A(t)}w_1(t)\Big)^2\Big) \tag{2.8}$$
Substituting (2.8) into (2.7) yields
$$\Big(e^{-A(t)}w_1(t)\Big)' + \frac{1}{2}\Big(e^{-A(t)}w_1(t)\Big)^2 \leq -\Big(-\frac{1}{2}a^2(t) + b(t) + \frac{K_0}{K_1}c_j(t)\Big), \tag{2.9}$$
which clearly imply that $e^{-A(t)}w_1(t)$ is a solution of (2.9). Next we assume that $y''(t) \geq 0$. Setting
$$w_2(t) = \frac{e^{A(t)}y''(t)}{\varphi_j(y(\sigma_j(t)))},$$
we obtain
$$\begin{aligned} w_2'(t) &= \frac{\big(e^{A(t)}y''(t)\big)'}{\varphi_j(y(\sigma_j(t)))} - e^{A(t)}y''(t)\frac{\varphi'(y(\sigma_j(t)))y'(\sigma_j(t))\sigma_j'(t)}{\varphi_j^2(y(\sigma_j(t)))} \\ &\leq -\frac{b(t)e^{A(t)}y'(t)}{\varphi_j(y(\sigma_j(t)))} - c_j(t)e^{A(t)} - e^{A(t)}y''(t)\frac{\sigma K_2 y'(\sigma_j(t))}{\varphi_j^2(y(\sigma_j(t)))} \\ &\leq -\frac{b(t)y'(t)}{\varphi_j(y(\sigma_j(t)))} - c_j(t)e^{A(t)} - e^{A(t)}y''(t)\frac{\sigma K_2 y'(\sigma_j(t))}{\varphi_j^2(y(\sigma_j(t)))} \end{aligned} \tag{2.10}$$
Since $(e^{A(t)}y''(t))' \leq 0$, we see that
$$y'(t) \geq y'(\sigma_j(t)) \geq \int_{t_0}^{\sigma_j(t)} e^{-A(s)}\Big(e^{A(s)}y''(s)\Big)\,ds$$

$$\geq e^{A(\sigma_j(t))}y''(\sigma_j(t))\int_{t_0}^{\sigma_j(t)} e^{-A(s)}ds$$
$$\geq e^{A(t)}y''(t)\int_{t_0}^{\sigma_j(t)} e^{-A(s)}ds = e^{A(t)}y''(t)A_e(\sigma_j(t)).$$

By using this relation, (2.10) is rewritten as

$$w_2'(t) \leq -b(t)A_e(\sigma_j(t))w_2(t) - c_j(t)e^{A(t)} - \sigma K_2 A_e(\sigma_j(t))w_2^2(t).$$

Applying Hölder's inequality,

$$|b(t)A_e(\sigma_j(t))w_2(t)| \leq \frac{1}{2}\Big(\frac{(b(t)A_e(\sigma_j(t)))^2}{(\sigma K_2 A_e(\sigma_j(t)))} + \sigma K_2 A_e(\sigma_j(t))w_2^2(t)\Big).$$

It is easy to establish the inequality

$$w_2'(t) \leq \frac{1}{2}\left(\frac{b^2(t)A_e(\sigma_j(t))}{\sigma K_2}\right) - c_j(t)e^{A(t)} - \frac{1}{2}\sigma K_2 A_e(\sigma_j(t))w_2^2(t), \tag{2.11}$$

and then, $w_2(t)$ is a solution of (2.11). This contradicts the hypothesis and completes the proof. □

3. Main results

In this section, we establish some new oscillatory criteria for (1.1). First, we state following useful lemmas.

Lemma 3.1 ([11, Theorem 4]). *If there is a function $\phi(t) \in C^1([T_0,\infty);(0,\infty))$ such that*

$$\int_{T_1}^{\infty}\Big(\frac{\bar{p}(t)|\phi'(t)|^{\beta}}{\phi(t)}\Big)^{1/(\beta-1)}dt < \infty, \quad \int_{T_1}^{\infty}\frac{1}{\bar{p}(t)(\phi(t))^{\beta-1}}dt = \infty,$$
$$\int_{T_1}^{\infty}\phi(t)\bar{q}(t)dt = \infty$$

for some $T_1 \geq T_0$, then the Riccati inequality

$$x'(t) + \frac{1}{\beta}\frac{1}{\bar{p}(t)}|x(t)|^{\beta} \leq -\bar{q}(t), \tag{3.1}$$

where $\beta > 1$, $\bar{p}(t) \in C([T_0,\infty);(0,\infty))$ and $\bar{q}(t) \in C([T_0,\infty);\mathbb{R})$, has no solution on intervals $[T,\infty)$ for all large T.

Let $\rho(s) \in C^1([0,\infty);(0,\infty))$, and define an integral operator A_τ^ρ by

$$A_\tau^\rho(v;t) = \int_\tau^t H(t,s)v(s)\rho(s)ds, \quad t \geq \tau \geq T,$$

where $v \in ([\tau,\infty);\mathbb{R})$. It is easy to see that A_τ^ρ is linear and positive, and in fact satisfies the following conditions:

- (H6) $A_\tau^\rho(k_1v_1 + k_2v_2;r) = k_1A_\tau^\rho(v_1;r) + k_2A_\tau^\rho(v_2;r)$ for $k_1,k_2 \in \mathbb{R}$;
- (H7) $A_\tau^\rho \geq 0$ for $v \geq 0$;
- (H8) $A_\tau^\rho(v';r) = -H(r,\tau)v(\tau)\rho(\tau) + A_\tau^\rho((h - \frac{\rho'}{\rho})v;r)$.

Lemma 3.2 ([12, Theorem 1]). *If*

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}A_T^\rho\Big(\bar{q} - \frac{\beta-1}{\beta}\bar{p}^{\frac{1}{\beta-1}}|h - \frac{\rho'}{\rho}|^{\beta/(\beta-1)};t\Big) = \infty,$$

then (3.1) has no solution on $[T,\infty)$ for all large T.

Theorem 3.3. *Assume that* (H1)–(H4) *or* (H1)–(H3), (H5) *are satisfied. If* (2.1) *holds, and there exists functions* $\phi_i(t) \in C^1([T_0,\infty);(0,\infty))$ $(i = 1, 2)$ *such that*

$$\int_T^\infty \Big(\frac{P_i(t)\phi_i'(t)^2}{\phi_i(t)}\Big)dt < \infty, \quad \int_T^\infty \frac{1}{P_i(t)\phi_i(t)}dt = \infty,$$
$$\int_T^\infty \phi_i(t)Q_i(t)dt = \infty \quad (i = 1, 2),$$

then every solution $y(t)$ *of* (1.1) *is oscillatory.*

An application. The flow of chemically reacting mixtures of gases plays a functional role in studying such diverse problems as the solar atmosphere, the atmosphere of other stars, and the gas flow in the combustion chamber of a rocket engine. It can be shown that for certain types of gases the propagation of small disturbances through the gas as time t varies is described by the DE $y''' + ay'' + by' + cy = 0$, where the given constants a, b and c are all positive. The independent variable $y(t)$ is proportional to the gas pressure. The coefficient a, b and c are related to physical properties and the temperature of the gas. In particular, the constants b and c are usually called the frozen and equilibrium sound speeds of the gas, respectively. From the physical properties, it is known that $b > c$. If the DE is asymptotically stable, then all disturbances to the gas will eventually disappear because they are dissipated by the chemical reactions. If the DE is not asymptotically stable, then there are disturbances which do not decay as $t \to \infty$. Then shock waves may form in the gas (see, [7]). Thus we consider the equation

$$y'''(t) + \frac{3}{4}y''(t) + \frac{1}{4}y'(t) + \frac{3}{16}y(t) = 0, \quad t > 0. \tag{3.2}$$

Here $a(t) = 3/4$, $b(t) = 1/4$ and $c(t) = 3/16$. It is easy to check that $ab = c$, which implies that (3.2) is not asymptotically stable. Since $b(t) > c(t)$ and

$$a(t) = \frac{3}{4} < \frac{5}{4} = b(t) + 1,$$

Assumption (H4) is not satisfied, but (H5) is satisfied. A straightforward computation yields

$$\int^\infty \pi(t)e^{A(t)}c(t)dt = \int^\infty \Big(\frac{4}{3}e^{-3t/4}\Big)\Big(e^{3t/4}\Big)\Big(\frac{3}{16}\Big)dt = \infty.$$

By choosing $\phi_1(t) = t^{1/2}$ and $\phi_2(t) = e^{-t/2}$, we can show that

$$\int^\infty \frac{P_1(t)\phi_1'(t)^2}{\phi_1(t)}dt = \int^\infty \Big(\frac{1\cdot\big(\frac{1}{2}t^{-\frac{1}{2}}\big)^2}{t^{1/2}}\Big)dt < \infty,$$
$$\int^\infty \frac{1}{P_1(t)\phi_1(t)}dt = \int^\infty \Big(\frac{1}{1\cdot t^{1/2}}\Big)dt = \infty,$$
$$\int^\infty \phi_1(t)Q_1(t)dt = \int^\infty \Big(t^{1/2}\Big)\Big(\frac{5}{32}\Big)dt = \infty,$$

and

$$\int^\infty \frac{P_2(t)\phi_2'(t)^2}{\phi_2(t)}dt = \int^\infty \Big(\frac{\frac{3}{4(1-e^{-3t/4})}\big(-\frac{1}{2}e^{-t/2}\big)^2}{e^{-t/2}}\Big)dt < \infty,$$
$$\int^\infty \frac{1}{P_2(t)\phi_2(t)}dt = \int^\infty \Big(\frac{1}{\frac{3}{4(1-e^{-3t/4})}e^{-t/2}}\Big)dt = \infty,$$

$$\int^{\infty}\phi_2(t)Q_2(t)dt=\int^{\infty}e^{-t/2}\big\{\frac{3}{16}e^{3t/4}-\frac{1}{24}(1-e^{-3t/4})\big\}dt=\infty.$$

So every solution of (3.2) is oscillatory by Theorem 3.3. Moreover, we note that $y(t)=\sin\frac{t}{2}$ is a solution of (3.2), which is oscillatory.

Theorem 3.4. *Assume that* (H1)–(H4) *or* (H1)–(H3), (H5) *are satisfied. If*

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}A_T^{\rho}\Big(Q_i-P_i|h-\frac{\rho'}{\rho}|;t\Big)=\infty,$$

then every solution of (1.1) *is oscillatory.*

Now, we consider the linear case of equation (1.1):

$$y'''(t)+a(t)y''(t)+b(t)y'(t)+\sum_{i=1}^{m}c_i(t)y(\sigma_i(t))=0,\quad t>0, \tag{3.3}$$

where $\sigma_i(t)\ge t$ $(i=1,2,\dots,m)$.

Corollary 3.5. *Assume that* (H1)–(H4) *or* (H1)–(H3), (H5) *are satisfied. If* (2.1) *holds and*

$$\int^{\infty}\Big\{\frac{2}{27}a^3(t)-\frac{1}{3}a(t)b(t)+c_j(t)\ -\frac{2}{3\sqrt{3}}\Big(\frac{a^2(t)}{3}-(b(t)-a'(t))\Big)^{3/2}\Big\}dt=\infty,$$

then every solution of (3.3) *is oscillatory.*

Proof. Suppose that $y(t)$ is a nonoscillatory solution of (3.3). It follows from Lemma 2.1 or Lemma 2.2 that $y(t)y'(t)>0$ holds. Now we define

$$u(t)=\frac{y'(t)}{y(t)}>0,$$

then we see that

$$\begin{aligned}u''(t)&=\frac{y'''(t)}{y(t)}-\frac{y'(t)y''(t)}{y^2(t)}-2u'(t)u(t)\\&\le -a(t)u'(t)-3u'(t)u(t)-\{u^3(t)+a(t)u^2(t)+b(t)u(t)+c_j(t)\},\end{aligned}$$

and so,

$$\begin{aligned}&\big[u'(t)+\frac{3}{2}u^2(t)+a(t)u(t)\big]'\\&\le-\big\{u^3(t)+a(t)u^2(t)+(b(t)-a'(t))\,u(t)+c_j(t)\big\}\equiv -F(u(t),t).\end{aligned} \tag{3.4}$$

Clearly, $F(u(t),t)$ has a minimum value for $u(t)>0$ at

$$u(t)=\frac{-a(t)+\sqrt{a^2(t)-3(b(t)-a'(t))}}{3}.$$

This, together with (3.4), implies that

$$\begin{aligned}&\big[u'(t)+\frac{3}{2}u^2(t)+a(t)u(t)\big]'\\&\le-\Big\{\frac{2}{27}a^3(s)-\frac{1}{3}a(s)b(s)+c_j(s)-\frac{2}{3\sqrt{3}}\Big(\frac{a^2(s)}{3}-(b(s)-a'(s))\Big)^{3/2}\Big\}.\end{aligned}$$

Integrating this over $[t_0,t]$ yields

$$u'(t)\le u'(t_0)+\frac{3}{2}u^2(t_0)+a(t_0)u(t_0)-\int_{t_0}^{t}\Big\{\frac{2}{27}a^3(s)-\frac{1}{3}a(s)b(s)+c_j(s)$$

$$- \frac{2}{3\sqrt{3}}\Big(\frac{a^2(s)}{3} - (b(s) - a'(s))\Big)^{3/2}\Big\}ds,$$

which implies that $u(t) < 0$ for large t. This contradiction completes the proof. $\square$

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