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CONTROLLABILITY OF SEMILINEAR MATRIX LYAPUNOV SYSTEMS

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Abstract. In this article, we establish some sufficient conditions for the complete controllability of semilinear matrix Lyapunov systems involving Lipschitzian and non-Lipschitzian nonlinearities. In case of non-Lipschitzian nonlinearities, we assume that nonlinearities are of monotone type.

1. Introduction

The tools of applied mathematics have been explored extensively for tackling control problems in the literature. Many of the real world problems arising in mechanics, biological systems, finance industry and in space applications are control theoretic in nature. In control theory one looks for a control which can steer the system from any given state to any desired final state. Sometimes it is required that the control should also optimize the cost functional associated with the control system. Vast literature is available on the controllability of linear and non-linear systems, for example, [3, 4, 9, 10] and references there in. Recently . Murty et al [6] studied the controllability of the matrix Lyapunov systems

$$\dot{X}(t) = A(t)X(t) + X(t)B(t) + F(t)U(t). \tag{1.1}$$

Furthermore in [7] the stability of matrix Lyapunov systems of type (1.1) is investigated. Often the actual system can not be modelled by the linear system of the form (1.1) due to the presence of inherent non-linearities in the system. Therefore, our aim in this paper is to investigate the controllability of nonlinear matrix Lyapunov systems represented by:

$$\dot{X}(t) = A(t)X(t) + X(t)B(t) + F(t)U(t) + G(t, X(t)), \tag{1.2}$$

where $X(t)$ is an $n \times n$ real matrix called state matrix, $U(t)$ is an $m \times n$ real matrix called control matrix and $G(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is a nonlinear function. $A(t)$, $B(t)$, $F(t)$ are $n \times n$, $n \times n$ and $n \times m$ real matrices respectively. All of them are assumed to be piecewise continuous function of t in $[t_0, t_1](t_0 < t_1 < \infty)$. Furthermore entries in the state matrix $X(t)$ and the control matrix $U(t)$ belong to $L^2([t_0, t_1], \mathbb{R})$. The function G satisfies the 'Caratheodory conditions'; that is,

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$G(\cdot, x)$ is measurable with respect to t for all $x \in \mathbb{R}^{n\times n}$ and $G(t, \cdot)$ is continuous with respect to x for almost all $t \in [t_0, t_1]$.

Note that under the assumptions $G(t, x) \equiv 0$ the system (1.2) reduces to system (1.1) whose controllability is investigated in [6]. Furthermore if $G(t, x) \equiv 0$ and $B(t) \equiv 0$ then the system (1.2) reduces to linear time-varying control system whose controllability is well established in the literature, for example, [2],[9]. In this article we establish the complete controllability results for nonlinear matrix Lyapunov systems (1.2) using the tools of functional analysis and operator theory.

The organization of the paper is as follows. In Section 2 we state some of the basic properties of Kronecker products. In Section 3 we formulate a semi-linear system equivalent to the original non-linear matrix Lyapunov system. In Section 4 the controllability of the semi-linear system obtained in Section 3 is reduced to the solvability of a system of coupled operator equations. Finally, sufficient conditions for the controllability of non-linear matrix Lyapunov systems (1.2) with Lipschitzian and non-Lipschitzian nonlinearities are established.

2. Preliminaries

Throughout the paper $\mathbb{R}$ denotes the set of all real numbers. $\mathbb{R}^+$ denotes the set of all non-negative real numbers. $\mathbb{R}^{m\times n}$ and $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ real matrices and $m \times n$ complex matrices, respectively. Given any matrix $A = [a_{ij}] \in \mathbb{R}^{m\times n}$, $\|A\|_F$ denotes its Frobenius norm and is defined as

$$\|A\|_F := [\sum_{i,j=1}^{n} |a_{ij}|^2]^{\frac{1}{2}};$$

$\|A\|$ denotes the $2-$norm (spectral norm) of A. Given any vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the $2-$norm (Euclidean norm) of x. I_n denotes the $n \times n$ identity matrix. Given any matrix A, $\sum A$ denotes the sum of the absolute values of entries of A.

We start with some basic definitions related to Kronecker products which we shall use in this paper.

Definition 2.1. [1] Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$ then the Kronecker product of A and B is written as $A \otimes B$ and is defined to be the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

which is an $mp \times nq$ matrix and in $\mathbb{C}^{mp\times nq}$.

Definition 2.2. Let $A = [a_{ij}] \in \mathbb{C}^{m\times n}$. We denote

$$\hat{A} = \operatorname{Vec} A = \begin{bmatrix} A_{.1} \\ A_{.2} \\ \vdots \\ A_{.n} \end{bmatrix}_{mn\times 1}, \quad \text{where} A_{.j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}, \quad (1 \le j \le n).$$

The Kronecker product satisfies the following properties [1]:

(1) $(A \otimes B)^T = (A^T \otimes B^T)$
(2) $(A \otimes B)^{-1} = (A^{-1} \otimes B^{-1})$

(3) $(A\otimes B)(C\otimes D) = (AC\otimes BD)$, provided the dimensions of various matrices are compatible with matrix product.
(4) If $A(t)$ and $B(t)$ are matrices, then
$$\frac{d}{dt}(A(t)\otimes B(t)) = \frac{d}{dt}(A(t))\otimes B(t) + A(t)\otimes \frac{d}{dt}(B(t))$$
(5) $\operatorname{Vec}(AYB) = (B^T\otimes A)\operatorname{Vec}(Y)$
(6) If A and X are matrices of order $n\times n$, then
   (i) $\operatorname{Vec}(AX) = (I_n\otimes A)\operatorname{Vec}(X)$
   (ii) $\operatorname{Vec}(XA) = (A^T\otimes I_n)\operatorname{Vec}(X)$

3. Reduction of nonlinear matrix Lyapunov system to a semi-linear control system

By applying Vec operator to equation (1.2), we have the following system:
$$\dot\psi(t) = A_1(t)\psi(t) + B_1(t)\hat u(t) + G_1(t,\psi(t)), \tag{3.1}$$
where $\psi(t) = \operatorname{Vec}(X(t))$, $A_1(t) = (B^T\otimes I_n) + (I_n\otimes A)$, $B_1(t) = I_n\otimes F(t)$, $\hat u(t) = \operatorname{Vec}(U(t))$ and $G_1(t,\psi(t)) = \operatorname{Vec}(G(t,X(t)))$.

Definition 3.1. The nonlinear matrix Lyapunov system (1.2) is said to be controllable (completely controllable) on $[t_0,t_1]$ in the domain of controllability $D\subset\mathbb{R}^{n\times n}$ if for each pair of matrices $X_0,X_1\in D$, there exists a control $u\in L^2([t_0,t_1];\mathbb{R}^{m\times n})$ such that the solution of (1.2) together with $X(t_0)=X_0$ also satisfies $X(t_1)=X_1$.

Proposition 3.2. *The matrix Lyapunov system* (1.2) *is completely controllable if and only if the semi linear system given in* (3.1) *is completely controllable.*

The proof of the apbove proposition is trivial as (1.2) and (3.1) are identical.

Let us consider the corresponding linear system of (3.1), which is given by
$$\dot\psi(t) = A_1(t)\psi(t) + B_1(t)\hat u(t) \tag{3.2}$$
Murty et al [6] established sufficient conditions for the complete controllability of the linear system (3.2).

Theorem 3.3. *([6, Murty et el.]) The system* (3.2) *is completely controllable if and only if the $n^2\times n^2$ symmetric controllability matrix*
$$W(t_0,t_1) = \int_{t_0}^{t_1}\Phi(t_0,s)(I_n\otimes F(s))(I_n\otimes F^T(s))\Phi^T(t_0,s)ds, \tag{3.3}$$
is nonsingular, where $\Phi(t,s) = \Phi_2(t,s)\otimes\Phi_1(t,s)$ is the transition matrix generated by $A_1(t)$ in which Φ_1 and Φ_2 are the transition matrices for systems $\dot X(t) = A(t)X(t)$ and $\dot X(t) = B^T(t)X(t)$, respectively. In this case the control
$$\hat u(t) = -(I_n\otimes F^T(t))\Phi^T(t_0,t)W^{-1}(t_0,t_1)[\psi_0 - \Phi(t_0,t_1)\psi_1], \tag{3.4}$$
transfers $\psi(t_0)=\psi_0$ to $\psi(t_1)=\psi_1$.

Remark 3.4. In the above theorem $W(t_0,t_1)$ can also be defined as follows:
$$W(t_0,t_1) = \int_{t_0}^{t_1}\Phi(t_1,s)(I_n\otimes F(s))(I_n\otimes F^T(s))\Phi^T(t_1,s)ds, \tag{3.5}$$
and in this case the control $\hat u$ is given by
$$\hat u(t) = (I_n\otimes F^T(t))\Phi^T(t_1,t)W^{-1}(t_0,t_1)[\psi_1 - \Phi(t_1,t_0)\psi_0]. \tag{3.6}$$

Remark 3.5. $W(t_0, t_1)$, as defined in (3.5), can also be written as CC^*, where $\mathrm{C} : L^2([t_0, t_1]; \mathbb{R}^{mn}) \to \mathbb{R}^{n^2}$ is defined as

$$\mathrm{C}u = \int_{t_0}^{t_1} \Phi(t_1, s)(I_n \otimes F(s))u(s)ds,$$

and $\mathrm{C}^* : \mathbb{R}^{n^2} \to L^2([t_0, t_1]; \mathbb{R}^{mn})$ is the adjoint of C and defined as follows

$$(\mathrm{C}^*\psi)(t) = (I_n \otimes F^T(t))\Phi^T(t_1, t)\psi.$$

Proposition 3.6. *Let $\Phi(t, s)$ be the same as in Theorem* (3.3)*. Then the solution of* (3.1) *with initial condition $\psi(t_0) = \psi_0$ is given by the following Volterra-type integral equation*

$$\psi(t) = \Phi(t, t_0)\psi_0 + \int_{t_0}^{t} \Phi(t, s)((I_n \otimes F(s))\hat{u}(s) + G_1(s, \psi(s)))ds. \tag{3.7}$$

The proof of the above proposition can be obtained by using the standard technique of the variation of parameter.

Note that we are interested in global controllability of (3.1); that is, the domain of controllability is $\mathbb{R}^{n^2}$. Furthermore, the controllability results for nonlinear system (3.1) will mainly depend on the controllability results of corresponding linear system (3.2). Therefore, we assume throughout this paper that the linear system (3.2) is globally completely controllable. We will now state some essential definitions from non-linear functional analysis.

Definition 3.7 ([4])**.** *Let X be a real Banach space. Let "Lip" be the set of all operators $N : X \to X$ which satisfy Lipschitz condition; that is, there exists a constant $\alpha > 0$ such that*

$$\|Nx_1 - Nx_2\| \le \alpha\|x_1 - x_2\|, \quad \textit{for all } x_1, x_2 \in X. \tag{3.8}$$

For $N \in$Lip we define

$$\|N\|_{\mathrm{Lip}} = \sup_{x_1, x_2 \in X,\, x_1 \neq x_2} \frac{\|Nx_1 - Nx_2\|}{\|x_1 - x_2\|}.$$

Definition 3.8 ([4])**.** Let H be a real Hilbert space. Let $\mathcal{M}$ be the set of all operators $N : H \to H$ such that $N \in \mathcal{M}$ if and only if

$$\langle Nx_1 - Nx_2, x_1 - x_2\rangle \ge \alpha\|x_1 - x_2\|^2,$$

for all x_1, $x_2 \in H$ and α is a constant in $\mathbb{R}$. For $N \in \mathcal{M}$ we define

$$\mu(N) = \inf_{\substack{x_1, x_2 \in H \\ x_1 \neq x_2}} \frac{< Nx_1 - Nx_2, x_1 - x_2 >}{\|x_1 - x_2\|^2}.$$

The operator N is called monotone (strongly monotone) if $\mu(N) \ge 0$ ($\mu(N) > 0$).

4. Reduction of controllability problem to a solvability problem

In this section we shall discuss the controllability of (3.1) in terms of the solvability of an equivalent feedback system of the form

$$\begin{aligned} e_1 &= u_1 - S_1 e_2, \\ e_2 &= u_2 + S_2 e_1 \end{aligned} \tag{4.1}$$

for some appropriate operator $S_1 : X_1 \to X_2$ and $S_2 : X_2 \to X_1$, where X_1 and X_2 are some suitable Banach spaces.

Suppose that the system (3.1) is completely controllable on $[t_0, t_1]$. That is, there exists a control u in $L^2([t_0,t_1];\mathbb{R}^{mn})$ which steers the initial state $\psi_0 \in \mathbb{R}^{n^2}$ of system (3.1) to the final state $\psi_1 \in \mathbb{R}^{n^2}$. Then according to Proposition 3.6 we have:

$$\psi_1 = \psi(t_1) = \Phi(t_1,t_0)\psi_0 + \int_{t_0}^{t_1} \Phi(t_1,\tau)B_1(\tau)u(\tau)d\tau + \int_{t_0}^{t_1} \Phi(t_1,\tau)G_1(\tau,\psi(\tau))d\tau.$$

That is,

$$\psi_1 - \Phi(t_1,t_0)\psi_0 - \int_{t_0}^{t_1} \Phi(t_1,\tau)G_1(\tau,\psi(\tau))d\tau = \int_{t_0}^{t_1} \Phi(t_1,\tau)B_1(\tau)u(\tau)d\tau.$$

Consider now the integral equation

$$\begin{aligned} \psi(t) &= \Phi(t,t_0)\psi_0 + \int_{t_0}^{t} \Phi(t,\tau)G_1(\tau,\psi(\tau))d\tau \\ &\quad + \int_{t_0}^{t} \Phi(t,\tau)B_1(\tau)\Big(\mathrm{C}^*(\mathrm{C}\mathrm{C}^*)^{-1}\big[\psi_1 - \Phi(t_1,t_0)\psi_0 \\ &\quad - \int_{t_0}^{t_1} \Phi(t_1,\tau)G_1(\tau,\psi(\tau))d\tau\big]\Big)(\tau)d\tau. \end{aligned} \tag{4.2}$$

Suppose that (4.2) is solvable for some ψ. Then it can be verified that $\psi(t_0) = \psi_0$ and $\psi(t_1) = \psi_1$. This implies that the system (3.1) is controllable with a control u given by

$$u(t) = (\mathrm{C}^*(\mathrm{C}\mathrm{C}^*)^{-1}[\psi_1 - \Phi(t_1,t_0)\psi_0 - \int_{t_0}^{t_1} \Phi(t_1,\tau)G_1(\tau,\psi(\tau))d\tau])(t)$$

Hence the controllability of nonlinear system (3.1) is equivalent to the solvability of coupled equations:

$$\begin{aligned} \psi(t) &= \Phi(t,t_0)\psi_0 + \int_{t_0}^{t} \Phi(t,\tau)G_1(\tau,\psi(\tau))d\tau + \int_{t_0}^{t} \Phi(t,\tau)B_1(\tau)u(\tau)d\tau, \\ u(t) &= (\mathrm{C}^*(\mathrm{C}\mathrm{C}^*)^{-1}[\psi_1 - \Phi(t_1,t_0)\psi_0 - \int_{t_0}^{t_1} \Phi(t_1,\tau)G_1(\tau,\psi(\tau))d\tau])(t). \end{aligned} \tag{4.3}$$

Let $X_1 = L^2([t_0,t_1];\mathbb{R}^{mn})$, $X_2 = L^2([t_0,t_1];\mathbb{R}^{n^2})$. Define operators $K, N : X_2 \to X_2$, $H : X_1 \to X_2$ and $R : X_2 \to X_1$ as follows:

$$(K\psi)(t) = \int_{t_0}^{t} \Phi(t,\tau)\psi(\tau)d\tau, \quad (N\psi)(t) = G_1(t,\psi(t)),$$

$$(Hu)(t) = \int_{t_0}^{t} \Phi(t,\tau)B_1(\tau)u(\tau)d\tau,$$

$$(R\psi)(t) = (\mathrm{C}^*(\mathrm{C}\mathrm{C}^*)^{-1} \int_{t_0}^{t_1} \Phi(t_1,\tau)\psi(\tau)d\tau)(t).$$

With this notation, equations (4.3) can be written as a pair of operator equations

$$\begin{gathered} \psi = u_0 + KN\psi + Hu, \\ u = u_1 - RN\psi \end{gathered} \tag{4.4}$$

where $u_0(t) = \Phi(t,t_0)\psi_0$ and $u_1(t) = (\mathrm{C}^*(\mathrm{C}\mathrm{C}^*)^{-1}[\psi_1 - \Phi(t_1,t_0)\psi_0])(t)$. Without loss of generality ψ_0 can be taken as 0 as indicated in the following theorem.

Theorem 4.1. *The system* (3.1) *is globally controllable if and only if for $x_1 \in \mathbb{R}^{n^2}$ there is a control $u \in L^2([t_0, t_1]; \mathbb{R}^{mn})$ which steers 0 to x_1.*

The proof of the above theorem follows by the same argument as in [8, Proposition 2.2]. Now using the above theorem the coupled system (4.4) can be written as follows:

$$\begin{gathered} \psi = KN\psi + Hu, \\ u = u_1 - RN\psi, \end{gathered} \tag{4.5}$$

where $u_1 = \mathrm{C}^*(\mathrm{CC}^*)^{-1}\psi_1$. Thus the nonlinear system (3.1) is controllable if and only if the above pair of operator equations (4.5) is solvable. We now introduce operators $\mathcal{M}_1 : X_1 \to X_2$ and $\mathcal{M}_2 : X_2 \to X_1$ as follows:

$$\mathcal{M}_1 = (I - KN)^{-1}H, \quad \mathcal{M}_2 = RN.$$

Now the following lemma is immediate.

Lemma 4.2. *If the operator $(I - KN)$ is invertible then the controllability of the system* (3.1) *is equivalent to the solvability of the feed-back system*

$$\begin{gathered} \psi = \mathcal{M}_1 u, \\ u = u_1 - \mathcal{M}_2\psi. \end{gathered} \tag{4.6}$$

4.1. **Controllability results with Lipschitzian nonlinearity.** Now we make the following assumptions.

(A1) Let $b = \sup_{t_0 \le t \le t_1} \|B_1(t)\|$ and the transition matrix $\Phi(t,s)$ is such that $\|\Phi(t,s)\| \le h(t,s)$, where $h(\cdot,\cdot) : [t_0,t_1] \times [t_0,t_1] \to \mathbb{R}^+$ is a function satisfying

$$\Big[\int_{t_0}^{t_1}\int_{t_0}^{t} h^2(t,s)\,ds\,dt\Big]^{\frac{1}{2}} = k < \infty.$$

(A2) The function $G : [t_0,t_1] \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ satisfies the Caratheodory conditions; that is, $G(\cdot, x)$ is measurable with respect to t for all $x \in \mathbb{R}^{n\times n}$ and $G(t,\cdot)$ is continuous with respect to x for almost all $t \in [t_0,t_1]$. Further G satisfies Lipschitz condition with Lipschitz constant α. That is,

$$\|G(t,x) - G(t,y)\| \le \alpha\|x - y\|.$$

Lemma 4.3. *Under Assumptions* (A1)–(A2)*, the bounds for $\|K\|$, $\|H\|$ and $\|R\|$ are estimated as $\|K\| \le k$, $\|H\| \le bk \triangleq h$ and $\|R\| \le bk_1^2 c \triangleq \gamma$ where $c = \|(CC^*)^{-1}\|$ and $k_1 = [\int_{t_0}^{t_1} h^2(t_1,s)ds]^{\frac{1}{2}}$.*

Proof. We will show that $\|K\| \le k$.

$$\begin{aligned} \|Kx\|_{X_2}^2 &= \int_{t_0}^{t_1} \|(Kx)(t)\|^2 dt \\ &= \int_{t_0}^{t_1} \|\int_{t_0}^{t} \Phi(t,\tau)x(\tau)d\tau\|^2 dt \\ &\le \int_{t_0}^{t_1} \Big(\int_{t_0}^{t} \|\Phi(t,\tau)x(\tau)\|d\tau\Big)^2 dt. \end{aligned}$$

By using Holder's inequality on the last expression, we have

$$\|Kx\|_{X_2}^2 \le \int_{t_0}^{t_1} \Big(\int_{t_0}^{t} \|\Phi(t,\tau)\|^2 d\tau\Big)\Big(\int_{t_0}^{t} \|x(\tau)\|^2 d\tau\Big) dt.$$

$$\|Kx\|_{X_2}^2 \le \Big(\int_{t_0}^{t_1}\int_{t_0}^{t} h^2(t,\tau)d\tau dt\Big)\|x\|_{X_2}^2.$$

Now $\|K\| \le k$ follows from the last inequality. Bounds for $\|H\|$ and $\|R\|$ can be obtained by using almost similar arguments, therefore the details are skipped. □

Lemma 4.4. *Under Assumption* (A1)–(A2) *the nonlinear operator N is Lipschitz continuous and bounded from X_2 into itself with Lipschitz constant $\beta = \sqrt{n}\alpha$.*

Proof. We shall prove this lemma by using the following well known matrix norm inequality (see [5, p. 64])

$$\|A\| \le \|A\|_F \le \sqrt{n}\|A\| \tag{4.7}$$

for any matrix $A \in \mathbb{R}^{n\times n}$. Given any $x_1,\ x_2 \in \mathbb{R}^{n\times n}$, let $\psi_1 = \mathrm{Vec}(x_1)$, $\psi_2 = \mathrm{Vec}(x_2)$. By using (4.7) the following relation holds for all $t \in [t_0,t_1]$

$$\|G(t,x_1) - G(t,x_2)\| \le \|G(t,x_1) - G(t,x_2)\|_F \le \sqrt{n}\|G(t,x_1) - G(t,x_2)\|. \tag{4.8}$$

Furthermore, we have

$$\|G_1(t,\psi_1) - G_1(t,\psi_2)\| = \|G(t,x_1) - G(t,x_2)\|_F. \tag{4.9}$$

By using (4.9) in (4.8) we have

$$\|G(t,x_1) - G(t,x_2)\| \le \|G_1(t,\psi_1) - G_1(t,\psi_2)\| \le \sqrt{n}\|G(t,x_1) - G(t,x_2)\|. \tag{4.10}$$

Let $\psi_1(\cdot), \psi_2(\cdot) \in X_2$ be arbitrary. Using relation (4.10) and Assumption (A1)–(A2) we can show that

$$\|(N\psi_1)(t) - (N\psi_2)(t)\| \le \sqrt{n}\alpha\|\psi_1(t) - \psi_2(t)\|. \tag{4.11}$$

The last inequality will in turn implies that N is Lipschitz continuous with Lipschitz constant $\sqrt{n}\alpha$. □

Theorem 4.5. *Let β be the Lipschitz constant for nonlinear operator N. Then the operator $I - KN$ is invertible if Assumptions* (A1)–(A2) *hold along with the condition $k\beta < 1$. Furthermore, $(I-KN)^{-1}$ is Lipschitz continuous with Lipschitz constant $\frac{1}{1-k\beta}$.*

Proof. First we show that under the assumptions of the theorem the operator $KN : X_2 \to X_2$ is a contraction. Since,

$$\begin{aligned}\|KN(x_1) - KN(x_2)\|_{X_2} &\le k\|N(x_1) - N(x_2)\|_{X_2}\\ &\le k\beta\|x_1 - x_2\|_{X_2}\\ &< \|x_1 - x_2\|_{X_2}.\end{aligned}$$

Hence KN is a contraction. Now by using Banach contraction principle it can be shown that for each fixed $y \in X_2$ the equation $(I - KN)x = y$ has the unique solution; say x_y. Indeed x_y is the unique limit of the iterates

$$x_{n+1} = KNx_n + y.$$

Now the correspondence $(I - KN)^{-1} : X_2 \to X_2$ given by

$$(I - KN)^{-1}y = x_y \tag{4.12}$$

is well defined. Hence $(I - KN)^{-1}$ is invertible. Furthermore,

$$\begin{aligned}\|(I - KN)^{-1}(y_1) - (I - KN)^{-1}(y_2)\| &= \|x_{y_1} - x_{y_2}\|\\ &= \|KNx_{y_1} + y_1 - KNx_{y_2} - y_2\|\end{aligned}$$

$$\leq \|K\|\|Nx_{y_1} - Nx_{y_2}\| + \|y_1 - y_2\|$$
$$\leq k\beta\|x_{y_1} - x_{y_1}\| + \|y_1 - y_2\|.$$

Hence,

$$(1 - k\beta)\|x_{y_1} - x_{y_2}\| \leq \|y_1 - y_2\|. \tag{4.13}$$

That is,

$$\|(I - KN)^{-1}(y_1) - (I - KN)^{-1}(y_2)\| \leq \frac{1}{(1 - k\beta)}\|(y_1 - y_2)\|. \tag{4.14}$$

Equation (4.14) shows that $(I - KN)^{-1}$ is Lipschitz continuous with Lipschitz constant $\frac{1}{1-k\beta}$. □

Now we describe briefly the solvability of the feedback system (4.1).

Definition 4.6. Let X_1 and X_2 be real Banach spaces. The feedback system (4.1) is said to be globally solvable if for every $(u_1, u_2) \in X_1 \times X_2$, there exists a solution $(e_1, e_2) \in (X_1 \times X_2)$ of (4.1). If this solution is unique then it is said to be uniquely globally solvable.

The following Lemma describes the conditions when the system (4.1) is uniquely globally solvable.

Lemma 4.7. *Let $S_1 : X_1 \to X_2$ and $S_2 : X_2 \to X_1$ belong to the class* Lip. *If $\|S_1\|_{\mathrm{Lip}}\|S_2\|_{\mathrm{Lip}} < 1$, then the system* (4.1) *is uniquely globally solvable. Moreover the iterates $[e_1^n]$, $[e_2^n]$ defined by*

$$e_1^{(n+1)} = u_1 - S_2 e_2^{(n)}$$
$$e_2^{(n)} = u_2 + S_1 e_1^{(n)},$$

converge to the unique solution of $(e_1, e_2) \in (X_1 \times X_2)$ starting from arbitrary $e_1^{(0)} \in X_1$.

Theorem 4.8. *Let β be the Lipschitz constant for the nonlinear operator N. Suppose that the linear system* (3.2) *is controllable and Assumptions* (A1)–(A2) *are satisfied with $k\beta < 1$ and $(\frac{\beta}{1-k\beta})\gamma h < 1$ then*

(1) *system* (3.1) *is controllable.*
(2) *the control vector $u(t)$ can be approximated by iterates $u^n(t)$ defined by*

$$u^{(n+1)}(t) = (\mathrm{C}^*(\mathrm{CC}^*)^{-1}[\psi_1 - \int_{t_0}^{t_1} \Phi(t_1, s)G_1(s, \psi^n(s))ds])(t). \tag{4.15}$$

The state vector approximation at n^{th} stage is given by the iterates

$$\psi_{j+1}^{(n)}(t) = \int_{t_0}^{t} \Phi(t, s)G_1(s, \psi_j^{(n)}(s))ds + \int_{t_0}^{t} \Phi(t, s)B_1(s)u^{(n)}(s)ds. \tag{4.16}$$

Proof. Since $k\beta < 1$, therefore by using Lemma 4.5 the operator $I - KN$ is invertible. Furthermore by Lemma 4.2 the controllability of semilinear system (3.1) is equivalent to the solvability of system (4.6). Furthermore,

$$\|\mathcal{M}_1\|_{\mathrm{Lip}} = \|(I - KN)^{-1}H\|_{\mathrm{Lip}} < \frac{h}{1 - k\beta}, \quad \|\mathcal{M}_2\|_{\mathrm{Lip}} = \|RN\|_{\mathrm{Lip}} < \gamma\beta. \tag{4.17}$$

Therefore, by the assumption $(\frac{\beta}{1-k\beta})\gamma h < 1$, we have $\|\mathcal{M}_1\|_{\mathrm{Lip}}\|\mathcal{M}_2\|_{\mathrm{Lip}} < 1$. Since (4.6) is special form of (4.1). Also system (4.6) satisfies $\|\mathcal{M}_1\|_{\mathrm{Lip}}\|\mathcal{M}_2\|_{\mathrm{Lip}} < 1$.

Hence Lemma 4.7 implies that the feedback system (4.6) is uniquely globally solvable, which in turn implies that the nonlinear system (3.1) is completely controllable. Furthermore the convergence of iterates (4.15) and (4.16) and uniqueness of control u and state ψ follow directly from the Lemma 4.7. □

The following theorem gives sufficient conditions for the complete controllability of nonlinear system (3.1) without assuming KN to be a contraction. However we will impose a slightly stringent condition on Assumption (A1).

Theorem 4.9. *Suppose that the linear system* (3.2) *is controllable and Assumptions* (A1)–(A2) *hold along with $h(s,t) = M$ (where M being a positive constant). Furthermore $e^{M\beta(t_1-t_0)}\gamma h\beta < 1$, where β is the Lipschitz constant for G_1. Then the conclusions of Theorem 4.8 hold.*

Proof. We will first show that the operator $(I - KN)^{-1}$ is Lipschitz continuous with $\|(I - KN)^{-1}\|_{\rm Lip} \leq e^{M\beta(t_1-t_0)}$. Let $y \in X_2$ be arbitrary. We will start by showing that $(I - KN)^{-1}(y)$ is well defined. Consider the Volterra type integral equation

$$x(t) = \int_{t_0}^{t} \Phi(t,\tau)G_1(\tau, x(\tau))d\tau + y(t). \tag{4.18}$$

Define the following iterates

$$x_0(t) = y(t) \quad \forall t \in [t_0, t_1]. \tag{4.19}$$

$$x_{n+1}(t) = y(t) + \int_{t_0}^{t} \Phi(t,\tau)G_1(\tau, x_n(\tau))d\tau, \quad n = 0, 1, 2 \dots \tag{4.20}$$

By using Lipschitz continuity of $G_1(t,x)$ and the boundedness of $\Phi(t,\tau)$ in $[t_0,t_1]$, it can shown that the iterates $\{x_n\}$ converges to the solution of $(*)$. Furthermore, by applying Gronwall's inequality [10, pp.92] uniqueness of the solution of $(*)$ can be easily proved. This in turn shows that the operator $(I - KN)^{-1}$ is well defined. Furthermore, given any $y_1, y_2 \in X_2$, let $(I-KN)^{-1}(y_1) = x_1$ and $(I-KN)^{-1}(y_2) = x_2$. Then we have

$$\begin{aligned}
&\|(I - KN)^{-1}(y_1)(t) - (I - KN)^{-1}(y_2)(t)\| \\
&= \|x_1(t) - x_2(t)\| \\
&= \|y_1(t) - y_2(t)\| + \Big\| \int_{t_0}^{t} \Phi(t,\tau)[G_1(\tau, x_1(\tau)) - G_1(\tau, x_2(\tau))]d\tau \Big\| \\
&\leq \|y_1(t) - y_2(t)\| + \int_{t_0}^{t} M\beta \|x_1(\tau) - x_2(\tau)\| d\tau.
\end{aligned}$$

Now again by applying Gronwall's inequality we have

$$\|x_1(t) - x_2(t)\| \leq e^{M\beta(t_1-t_0)}\|y_1(t) - y_2(t)\|.$$

Hence we have,

$$\|(I - KN)^{-1}(y_1) - (I - KN)^{-1}(y_2)\|_{X_2} \leq e^{M\beta(t_1-t_0)}\|y_1 - y_2\|_{X_2}.$$

Thus we have shown that $\|(I - KN)^{-1}\|_{\rm Lip} \leq e^{M\beta(t_1-t_0)}$. Now it follows that

$$\|\mathcal{M}_1\|_{\rm Lip} = \|(I - KN)^{-1}H\|_{\rm Lip} < e^{M\beta(t_1-t_0)}h, \quad \|\mathcal{M}_2\|_{\rm Lip} = \|RN\|_{\rm Lip} < \gamma\beta.$$

By the given condition $e^{M\beta(t_1-t_0)}\gamma h\beta < 1$, it follows that $\|\mathcal{M}_1\|_{\rm Lip}\|\mathcal{M}_2\|_{\rm Lip} < 1$. Now the remaining part of the proof is obvious and is same as in Theorem 4.8. □

4.2. **Controllability results with non-Lipschitzian nonlinearity.** In this section we establish the controllability results for the non-linear matrix Lyapunov system (1.2) with non-Lipschitzian nonlinearity. In particular we require monotonicity type of condition on the non-linear term G. Such assumption are quite reasonable because practically we have situations where the derivatives of the nonlinearities are bounded below by a constant.

We will use the following Lemma which guarantees the solvability of the feedback system (4.1).

Lemma 4.10. *Let X_1 and X_2 be Hilbert spaces. Let $S_1 : X_1 \to X_2$, $S_2 : X_2 \to X_1$ be the operators satisfying the following conditions:*

(i) *S_1 is compact, continuous and satisfy the growth condition of the type*
$$S_1 e_1 \leq \overline{s_1} + s_1 e_1, \quad \forall e_1 \in X_1,\ and s_1, \overline{s_1} > 0.$$
(ii) *S_2 is continuous and satisfy the growth condition of the type*
$$S_2 e_2 \leq \overline{s_2} + s_2 e_2, \quad \forall e_2 \in X_2,\ s_2, \overline{s_2} > 0.$$

If $(1 - s_1 s_2) > 0$ then the feedback system (4.1) is solvable.

Let us now assume that the system (1.2) satisfies the following assumptions.

(B1) There exists a positive constant μ such that the matrix $A_1(t)$ satisfies
$$< -A_1(t)\psi, \psi >\geq \mu\|\psi\|^2.$$
(B2) The nonlinear function $-G$ is monotone. In fact $-G$ should satisfy a weaker condition then monotonicity as given below. Given any $x_1,\ x_2 \in \mathbb{R}^{n\times n}$
$$< (G(t, x_1) - G(t, x_2))e_j, (x_1 - x_2)e_j >\leq 0, 1 \leq j \leq n,$$
where $[e_j]$ denotes the canonical basis in $\mathbb{R}^n$.

(B3) G also satisfies a growth condition of the form
$$\|G(t, x)\| \leq d(t) + w\|x\|,$$
for all $(t, x) \in [t_0, t_1] \times \mathbb{R}^{n\times n}$, $d(\cdot) \in L^2([t_0, t_1]; \mathbb{R})$ and $w > 0$.

Theorem 4.11. *Under the assumptions (B1)–(B3), the operator $(I-KN)^{-1}$ exists and continuous. Furthermore it satisfies a growth condition of the type*
$$\|(I - KN)^{-1}y\| \leq \frac{d\sqrt{n}}{\mu} + \big(\frac{w\sqrt{n}}{\mu} + 1\big)\|y\|, \tag{4.21}$$
where $d = \|d(\cdot)\|_{L^2[t_0,t_1]}$.

Proof. Assumption (B2) implies that $< (G_1(t, \psi_1) - G_1(t, \psi_2)), (\psi_1 - \psi_2) >\leq 0$ for every ψ_1 and $\psi_2 \in \mathbb{R}^{n^2}$. Furthermore, inequality (4.10) together with assumption $[B3]$ implies that
$$\|G_1(t, \psi)\| \leq \sqrt{n}(d(t) + w\|\psi\|),$$
for all $(t, \psi) \in [t_0, t_1] \times \mathbb{R}^{n^2}$. Now all the requirements of [4, Theorem 5.1] are satisfied. A careful trace of the [4, Theorem 5.1] will prove the theorem. □

Theorem 4.12. *Suppose that the linear system (3.2) is controllable and the assumptions (A1) and (B1)–(B3) are satisfied. If $[1 - (\frac{w\sqrt{n}}{\mu} + 1)\sqrt{n}w\gamma h] > 0$, then the nonlinear system (3.1) is controllable.*

Proof. Let $X_1 = L^2([t_0, t_1]; \mathbb{R}^{mn})$, $X_2 = L^2([t_0, t_1]; \mathbb{R}^{n^2})$. By Lemma 4.2 the controllability of system (3.1) is equivalent to the solvability of the coupled system

$$\begin{aligned} \psi &= \mathcal{M}_1 u, \\ u &= u_1 - \mathcal{M}_2 \psi, \end{aligned}$$

where $\mathcal{M}_1 = (I - KN)^{-1}H : X_1 \to X_2$ and $\mathcal{M}_2 = RN : X_2 \to X_1$. By Theorem 4.11 the operator $(I - KN)^{-1}$ is continuous and satisfies the growth condition (4.21). Since the operator H is compact it follows that operator $\mathcal{M}_1$ is also compact and satisfies the following growth condition

$$\|\mathcal{M}_1 u\| \le \frac{hd\sqrt{n}}{\mu} + (\frac{w\sqrt{n}}{\mu} + 1)h\|u\|.$$

Similarly it can be shown $\mathcal{M}_2$ is continuous with growth condition

$$\begin{aligned} \|\mathcal{M}_2 \psi\| &= \|R(N\psi)\| \\ &\le \gamma \|G_1(\cdot, \psi(\cdot))\| \\ &\le \gamma d\sqrt{n} + \gamma w \sqrt{n}\|\psi\| \end{aligned}$$

Thus the operator $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfy all the conditions of Lemma 4.10, which implies the solvability of system (4.6) that in turn implies the controllability of the non-linear system (3.1). □

We will finally give one example to illustrate our results.

Example 4.13. Consider the matrix Laypunov non-linear differential equation

$$\begin{aligned} \begin{bmatrix} \dot{x}_{11}(t) & \dot{x}_{12}(t) \\ \dot{x}_{21}(t) & \dot{x}_{22}(t) \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{bmatrix} + \begin{bmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \\ &\quad + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix} + c \begin{bmatrix} \sin(x_{11}(t)) & \cos(x_{12}(t)) \\ \cos(x_{21}(t)) & \sin(x_{22}(t)) \end{bmatrix}, \end{aligned} \tag{4.22}$$

By applying the Vec operator to above equation, we have the following equation of the form (3.1)

$$\begin{bmatrix} \dot{x}_{11}(t) \\ \dot{x}_{21}(t) \\ \dot{x}_{12}(t) \\ \dot{x}_{22}(t) \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 & 0 \\ 3 & 3 & 0 & 2 \\ 1 & 0 & 2 & 2 \\ 0 & 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \\ x_{12}(t) \\ x_{22}(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + c \begin{bmatrix} \sin(x_{11}(t)) \\ \cos(x_{21}(t)) \\ \cos(x_{12}(t)) \\ \sin(x_{22}(t)) \end{bmatrix}.$$

In this example,

$$A_1 = \begin{bmatrix} 2 & 2 & 2 & 0 \\ 3 & 3 & 0 & 2 \\ 1 & 0 & 2 & 2 \\ 0 & 1 & 3 & 3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix},$$

and the nonlinear operator G_1 is given by

$$G_1(t, x(t)) = [\sin(x_{11}(t)), \cos(x_{21}(t)), \cos(x_{12}(t)), \sin(x_{22}(t))]^T,$$

where $x(t) = [x_{11}(t), x_{21}(t), x_{12}(t), x_{22}(t)]^T$.

We have taken $h(t, s) = (\sum e^{A_1 t})(\sum e^{-A_1 s})$. The time interval $[t_0, t_1] = [0, \cdot 1]$. Using above definition of $h(t, s)$, the bounds for the norm of operators K, H and R are computed as 2.0708(k), 2.9286(h), and $1.0186 \times 10^4(\gamma)$, respectively. Let $\beta \triangleq \rho c$, for some $\rho > 0$ be the Lipschitz constant for the non-linear operator N. It can be easily shown that $k\beta < 1$ and $(\frac{\beta}{1-k\beta}\gamma h) < 1$ for sufficiently small value of

c. Thus all the conditions of Theorem 4.8 are satisfied. Hence the system (4.22) is completely controllable during time interval $[0, \cdot 1]$.

Remark 4.14. Note that sharper bounds for $\|K\|$, $\|H\|$ and $\|R\|$ can be obtained by suitably choosing the function $h(t, s)$. Thus a higher value of c can be obtained.

Remark 4.15. The norm $\|R\|$ is proportional to the norm of the inverse of controllability Grammian $W^{-1}(t_0, t_1)$. Therefore value of c can be increased by decreasing the value of $\|W^{-1}(t_0, t_1)\|$.

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