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OSCILLATION OF FIXED POINTS OF SOLUTIONS TO COMPLEX LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the relationship between the derivatives of the solutions to the differential equation $f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_0 f = 0$ and entire functions of finite order.

1. Introduction and statement of results

Throughout this article, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory [13, 12]. In addition, we will use $\lambda(f)$ and $\lambda(1/f)$ to denote respectively the exponents of convergence of the zero-sequence and the pole-sequence of a meromorphic function f, $\rho(f)$ to denote the order of growth of f, $\overline{\lambda}(f)$ and $\overline{\lambda}(1/f)$ to denote respectively the exponents of convergence of the sequence of distinct zeros and distinct poles of f. A meromorphic function $\varphi(z)$ is called a small function of a meromorphic function $f(z)$ if $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$, where $T(r,f)$ is the Nevanlinna characteristic function of f. In order to express the rate of growth of meromorphic solutions of infinite order, we recall the following definitions.

Definition 1.1 ([11],[13]). Let f be a meromorphic function and let $z_1, z_2, \dots$ such that ($|z_j| = r_j$, $0 < r_1 \le r_2 \le \dots$) be the sequence of the fixed points of f, each point being repeated only once. The exponent of convergence of the sequence of distinct fixed points of f is defined by

$$\overline{\tau}(f) = \inf\Big\{\tau > 0 : \sum_{j=1}^{+\infty} |z_j|^{-\tau} < +\infty\Big\}.$$

Clearly,

$$\overline{\tau}(f) = \limsup_{r\to+\infty} \frac{\log \overline{N}(r, \frac{1}{f-z})}{\log r}, \tag{1.1}$$

where $\overline{N}(r, \frac{1}{f-z})$ is the counting function of distinct fixed points of $f(z)$ in $\{|z| < r\}$.

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Definition 1.2 ([4, 6, 12]). *Let f be a meromorphic function. Then the hyper-order $\rho_2(f)$ of $f(z)$ is defined by*

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r}. \tag{1.2}$$

Definition 1.3 ([4, 6]). Let f be a meromorphic function. Then the hyper exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_2(f) = \limsup_{r\to+\infty} \frac{\log\log \overline{N}(r,\frac{1}{f})}{\log r}, \tag{1.3}$$

where $\overline{N}(r,\frac{1}{f})$ is the counting function of distinct zeros of $f(z)$ in $\{|z|<r\}$.

For $k\geq 2$, we consider the linear differential equation

$$f^{(k)} + Af = 0 \tag{1.4}$$

where $A(z)$ is a transcendental meromorphic function of finite order $\rho(A)=\rho>0$. Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades (see [15]). However, there are a few studies on the fixed points of solutions of differential equations. In [13], Wang and Lü investigated the fixed points and hyper-order of solutions of second order linear differential equations with meromorphic coefficients and their derivatives, they obtained the following result.

Theorem 1.4 ([13]). *Suppose that $A(z)$ is a transcendental meromorphic function satisfying $\delta(\infty,A) = \liminf_{r\to+\infty}\frac{m(r,A)}{T(r,A)} = \delta > 0$, $\rho(A)=\rho<+\infty$. Then every meromorphic solution $f\not\equiv 0$ of the equation*

$$f'' + A(z)f = 0, \tag{1.5}$$

satisfies that f, f', f'' all have infinitely many fixed points and

$$\overline{\tau}(f) = \overline{\tau}(f') = \overline{\tau}(f'') = \rho(f) = +\infty, \tag{1.6}$$

$$\overline{\tau}_2(f) = \overline{\tau}_2(f') = \overline{\tau}_2(f'') = \rho_2(f) = \rho. \tag{1.7}$$

The above theorem has been generalized to higher order differential equations by Liu Ming-Sheng and Zhang Xiao-Mei as follows.

Theorem 1.5 ([11]). *Suppose that $k\geq 2$ and $A(z)$ is a transcendental meromorphic function satisfying*

$$\delta(\infty,A) = \liminf_{r\to+\infty}\frac{m(r,A)}{T(r,A)} = \delta > 0,$$

and $\rho(A)=\rho<+\infty$. Then every meromorphic solution $f\not\equiv 0$ of (1.4) satisfies that f and $f', f'', \dots, f^{(k)}$ all have infinitely many fixed points and

$$\overline{\tau}(f) = \overline{\tau}(f') = \overline{\tau}(f'') = \dots = \overline{\tau}(f^{(k)}) = \rho(f) = +\infty, \tag{1.8}$$

$$\overline{\tau}_2(f) = \overline{\tau}_2(f^{'}) = \overline{\tau}_2(f'') = \dots = \overline{\tau}_2(f^{(k)}) = \rho_2(f) = \rho. \tag{1.9}$$

In [2], El Farissi and Belaidi extended the result of Theorem 1.5 and gave the following theorem.

Theorem 1.6. *Suppose that $k \geq 2$ and $A(z)$ is a transcendental meromorphic function satisfying $\delta(\infty, A) = \liminf_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$, and $0 < \rho(A) = \rho < +\infty$. If $\varphi \not\equiv 0$ is a meromorphic function with finite order $\rho(\varphi) < +\infty$, then every meromorphic solution $f \not\equiv 0$ of (1.4) satisfies*

$$\overline{\lambda}(f-\varphi) = \overline{\lambda}(f'-\varphi) = \cdots = \overline{\lambda}(f^{(k)}-\varphi) = \rho(f) = +\infty, \tag{1.10}$$

$$\overline{\lambda}_2(f-\varphi) = \overline{\lambda}_2(f'-\varphi) = \cdots = \overline{\lambda}_2(f^{(k)}-\varphi) = \rho_2(f) = \rho. \tag{1.11}$$

2. Our contribution

The main purpose of this article is to study the relationship between the derivatives of the solutions to the differential equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = 0, \quad k \geq 2, \tag{2.1}$$

and entire functions of finite order, where A_j are entire functions of finite order. We prove the following result.

Theorem 2.1. *Let $k \geq 2$ and A_j be entire functions of finite order such that $\max\{\rho(A_j), j = 1, \dots, k-1\} < \rho(A_0) < +\infty$. If $\varphi \not\equiv 0$ is an entire function with finite order, $\rho(\varphi) < +\infty$, then every solution $f \not\equiv 0$ of (2.1) satisfies*

$$\overline{\lambda}(f^{(i)}-\varphi) = \lambda(f^{(i)}-\varphi) = \rho(f) = +\infty, \quad i \in \mathbb{N} \tag{2.2}$$

and

$$\overline{\lambda}_2(f^{(i)}-\varphi) = \lambda_2(f^{(i)}-\varphi) = \rho_2(f) = \rho(A_0) = \rho,\ i \in \mathbb{N}. \tag{2.3}$$

For $\varphi(z) = z$ in Theorem 2.1, we obtain the following result.

Corollary 2.2. *Let $k \geq 2$ and A_j be entire functions of finite order such that $\max\{\rho(A_j), j = 1, \dots, k-1\} < \rho(A_0) < +\infty$. Then every solution $f \not\equiv 0$ of (2.1), its derivatives $f^{(i)}$ $(i \in \mathbb{N})$ have infinitely many fixed points and*

$$\overline{\tau}(f^{(i)}) = \tau(f^{(i)}) = \rho(f) = +\infty, \quad i \in \mathbb{N}, \tag{2.4}$$

$$\overline{\tau}_2(f^{(i)}) = \tau_2(f^{(i)}) = \rho_2(f) = \rho(A_0) = \rho, \quad i \in \mathbb{N}. \tag{2.5}$$

Corollary 2.3. *Suppose that $k \geq 2$ and $A(z)$ is a transcendental entire function such that $0 < \rho(A) = \rho < +\infty$. If $\varphi \not\equiv 0$ is an entire function with finite order, $\rho(\varphi) < +\infty$, then every solution $f \not\equiv 0$ of (1.4) satisfies (2.2) and (2.3).*

3. Auxiliary Lemmas

The following lemmas will be used in the proof of Theorem 2.1.

Lemma 3.1 ([6]). *Let f be a transcendental meromorphic function of finite order ρ, let $\Gamma = \{(k_1, j_1), (k_2, j_2), \dots, (k_m, j_m)\}$ denote a finite set of distinct pairs of integers that satisfy $k_i > j_i \geq 0$ for $i = 1, \dots, m$ and let $\varepsilon > 0$ be a given constant. Then the following estimations hold:*

(i) There exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi \in [0, 2\pi) - E_1$, then there is a constant $R_1 = R_1(\psi) > 1$ such that for all z satisfying $\arg z = \psi$ and $|z| \geqslant R_1$ and for all $(k,j) \in \Gamma$, we have

$$\Big|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Big| \leqslant |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{3.1}$$

(ii) There exists a set $E_2 \subset (1,\infty)$ that has finite logarithmic measure $lm(E_2) = \int_1^{+\infty} \frac{\chi_{E_2}(t)}{t}dt$, where χ_{E_2} is the characteristic function of E_2, such that for all z satisfying $|z| \notin E_2 \cup [0,1]$ and for all $(k,j) \in \Gamma$, we have

$$|\frac{f^{(k)}(z)}{f^{(j)}(z)}| \leqslant |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{3.2}$$

To avoid some problems caused by the exceptional set we recall the following Lemmas.

Lemma 3.2 ([1, p. 68]). *Let $g : [0,+\infty) \to \mathbb{R}$ and $h : [0,+\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite linear measure. Then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

Lemma 3.3 ([4]). *Let A_0, $A_1, \dots,$ A_{k-1}, $F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution with $\rho(f) = +\infty$ of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_1 f' + A_0 f = F, \tag{3.3}$$

then $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$.

Lemma 3.4 ([2]). *Let $A_0, A_1, \dots, A_{k-1}, F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution of the (3.3) with $\rho(f) = +\infty$ and $\rho_2(f) = \rho$, then f satisfies $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho$.*

Lemma 3.5 ([5]). *Let A_j be entire functions of finite order such that*

$$\max\{\rho(A_j), j = 1, \dots, k-1\} < \rho(A_0) = \rho < +\infty,$$

then every solution $f \not\equiv 0$, of (2.1), satisfies $\rho_2(f) = \rho$.

Let A_j $(j = 0, 1, \dots, k-1)$ be entire functions. We define a sequences of functions as follows:

$$\begin{gathered} A_j^0 = A_j \quad j = 0,1,\dots,k-1 \\ A_{k-1}^i = A_{k-1}^{i-1} - \frac{(A_0^{i-1})'}{A_0^{i-1}} \quad i \in \mathbb{N} \\ A_j^i = A_j^{i-1} + A_{j+1}^{i-1}\frac{(\Psi_{j+1}^{i-1})'}{\Psi_{j+1}^{i-1}} \quad j = 0,1,\dots,k-2;\ i \in \mathbb{N}, \end{gathered} \tag{3.4}$$

where $\Psi_{j+1}^{i-1} = \frac{A_{j+1}^{i-1}}{A_0^{i-1}}$.

Lemma 3.6. *Let A_j be entire functions of finite order such that*

$$\beta = \max\{\rho(A_j), j = 1, \dots, k-1\} < \rho(A_0) = \alpha < +\infty.$$

Then (1) There exists a set $E_i \subset (1,\infty)$ that has finite logarithmic measure such that for all z satisfying $|z| \notin E_i \cup [0,1]$, we have

$$|A_j^i| \leqslant M_i r^{\mu_i} \exp\{\gamma_i r^\beta\}, \quad \text{for } j = 1, \dots, k-1,$$

where M_i, μ_i, γ_i are positive real numbers.

(2) There exists a set $E_i \subset (1,\infty)$ that has finite logarithmic measure such that for all z satisfying $|z| \notin E_i \cup [0,1]$, we have

$$|A_0^i - A_0| \leqslant M_i r^{\mu_i} \exp\{\gamma_i r^\beta\} \quad \text{for } i \in \mathbb{N},$$

where M_i, μ_i, γ_i are positive numbers
(3) For all $i \in \mathbb{N}$, $A_0^i \not\equiv 0$.

Proof. We use the induction on i: If $i = 1$, then by (3.4), we have

$$|A_j^1| = |A_j^0 + A_{j+1}^0 \frac{(\Psi_{j+1}^0)'}{\Psi_{j+1}^0}| \leqslant |A_j^0| + |A_{j+1}^0 \frac{(\Psi_{j+1}^0)'}{\Psi_{j+1}^0}|. \tag{3.5}$$

By Lemma 3.1 (ii), there exists a set $E_1 \subset (1, \infty)$ that has finite logarithmic measure such that for all z satisfying $|z| \notin E_1 \cup [0, 1]$, we have

$$|\frac{(\Psi_{j+1}^0)'}{\Psi_{j+1}^0}| \leqslant r^{\mu_1}, \tag{3.6}$$

$$|A_j^0| \leqslant \exp\{\gamma r^\beta\}, \quad j = 0, \dots, k-2; \tag{3.7}$$

$$|A_{j+1}^0| \leqslant \exp\{\gamma r^\beta\}, \quad j = 0, \dots, k-2. \tag{3.8}$$

Combining (3.5), (3.6), (3.7) and (3.8), yields

$$|A_j^1| \leqslant M_1 r^{\mu_1} \exp\{\gamma_1 r^\beta\}. \tag{3.9}$$

Then (1) is true for $i = 1$.

Now suppose that the assertion (1) is true for the values which are strictly smaller than a certain i. Let

$$|A_j^i| = \left|A_j^{i-1} + A_{j+1}^{i-1} \frac{(\Psi_{j+1}^{i-1})'}{\Psi_{j+1}^{i-1}}\right| \leqslant |A_j^{i-1}| + \left|A_{j+1}^{i-1} \frac{(\Psi_{j+1}^{i-1})'}{\Psi_{j+1}^{i-1}}\right|. \tag{3.10}$$

By the induction hypothesis, there exists a set $E_{i-1} \subset (1, \infty)$ that has finite logarithmic measure such that for all z satisfying $|z| \notin E_{i-1} \cup [0, 1]$ we have

$$|A_k^{i-1}| \leqslant M_{i-1} r^{\mu_{i-1}} \exp\{\gamma_{i-1} r^\beta\}, \quad (k = j, j+1) \tag{3.11}$$

and by Lemma 3.1 there exist a set $E'_{i-1} \subset (1, \infty)$ that has finite logarithmic measure such that for all z satisfying $|z| \notin E'_{i-1} \cup [0, 1]$ and we have

$$|\frac{(\Psi_{j+1}^{i-1})'}{\Psi_{j+1}^{i-1}}| \leqslant r^{\mu_{i-1}}. \tag{3.12}$$

Hence, thanks to (3.10),(3.11) and (3.12), there exist a set $E_i = E_{i-1} \cup E'_{i-1} \subset (1, \infty)$ that has finite logarithmic measure such that for all z satisfying $|z| \notin E_i \cup [0, 1]$ and we have

$$|A_j^i| \leqslant M_i r^{\mu_i} \exp\{\gamma_i r^\beta\}. \tag{3.13}$$

Where M_i, μ_i, γ_i are positive numbers; the proof of part (1) is complete.

(2) We use the same arguments as before. For $i = 1$ by (3.4) we have

$$|A_0^1 - A_0| = \left|A_0^0 + A_1^0 \frac{(\Psi_1^0)'}{\Psi_1^0} - A_0\right| = |A_1^0 \frac{(\Psi_1^0)'}{\Psi_1^0}|$$

Using Lemma 3.1 (ii), we can state that there exists a set $E_1 \subset (1, \infty)$ that has finite logarithmic measure such that for all z satisfying $z| \notin E_1 \cup [0, 1]$ and we have

$$|\frac{(\Psi_1^0)'}{\Psi_1^0}| \leqslant r^{\mu_1}, \tag{3.14}$$

$$|A_1^0| \leqslant \exp\{\gamma_1 r^\beta\}. \tag{3.15}$$

From these two inequalities, we find that

$$|A_0^1 - A_0| \leqslant M_1 r^{\mu_1} \exp\{\gamma_1 r^\beta\}. \tag{3.16}$$

Consequently (2) is true for $i = 1$. Now suppose that the assertion is true for the values which are strictly smaller than a certain i, then using (3.4) we obtain

$$|A_0^i - A_0| = \big|A_0^{i-1} + A_1^{i-1}\frac{(\Psi_1^{i-1})'}{\Psi_1^{i-1}} - A_0\big| \leqslant |A_0^{i-1} - A_0| + \big|A_1^{i-1}\frac{(\Psi_1^{i-1})'}{\Psi_1^{i-1}}\big| \tag{3.17}$$

By induction hypothesis, there exists a set $E_{i-1} \subset (1,\infty)$ that has finite logarithmic measure such that for all z satisfying $|z| \notin E_{i-1} \cup [0,1]$ and we have

$$|A_0^{i-1} - A_0| \leqslant M_{i-1} r^{\mu_{i-1}} \exp\{\gamma_{i-1} r^\beta\}. \tag{3.18}$$

Using Lemma 3.1, we deduce that there exist a set $E'_{i-1} \subset (1,\infty)$ that has finite logarithmic measure such that for all z satisfying $|z| \notin E'_{i-1} \cup [0,1]$ and we can write

$$\big|\frac{(\Psi_1^{i-1})'}{\Psi_1^{i-1}}\big| \leqslant r^{\mu_{i-1}}. \tag{3.19}$$

Using assertion (1), we obtain

$$|A_k^{i-1}| \leqslant M_{i-1} r^{\mu_{i-1}} \exp\{\gamma_{i-1} r^\beta\}. \tag{3.20}$$

By (3.17), (3.18), (3.19) and (3.20) there exists a set $E_i = E_{i-1} \cup E'_{i-1} \subset (1,\infty)$ that has finite logarithmic measure such that for all z satisfying $|z| \notin E_i \cup [0,1]$ we have

$$|A_0^i - A_0| \leqslant M_i r^{\mu_i} \exp\{\gamma_i r^\beta\}, \tag{3.21}$$

where M_i, μ_i, γ_i are positive real numbers. The proof of part (2) is complete.

Now we prove part (3) Suppose that there exists $i_0 \in \mathbb{N}$, such that. $A_0^{i_0} \equiv 0$, this implies that $-A_0 = A_0^{i_0} - A_0$. By (2), there exists a set $E_{i_0} \subset (1,\infty)$ that has finite logarithmic measure such that for all z satisfying $|z| \notin E_{i_0} \cup [0,1]$, we have

$$|A_0| = |A_0^{i_0} - A_0| \leqslant M_{i_0} r^{\mu_{i_0}} \exp\{\gamma_{i_0} r^\beta\},$$

which contradicts $\rho(A_0) > \beta$. □

Lemma 3.7. *Let A_j be entire functions of finite order such that $\max\{\rho(A_j), j = 1,\dots,k-1\} = \beta < \rho(A_0) = \alpha < +\infty$. Then every non trivial meromorphic solution of the equation*

$$g^{(k)} + A_{k-1}^i g^{(k-1)} + \dots + A_0^i g = 0, \quad k \geq 2 \tag{3.22}$$

has infinite order, where A_j^i, $j = 0,1,\dots,k-1$ are defined as in (3.4).

Proof. Assume that (3.22) has a meromorphic solution g with $\rho(g) < \infty$. We rewrite (3.22) as

$$\frac{g^{(k)}}{g} + A_{k-1}^i \frac{g^{(k-1)}}{g} + \dots + A_0^i - A_0 = -A_0, \quad k \geq 2. \tag{3.23}$$

By Lemma 3.1 (ii), there exists a set $E \subset (1,\infty)$ of finite logarithmic measure such that for all z, $|z| \notin E \cup [0,1]$, we have

$$\big|\frac{g^{(j)}}{g}\big| \leqslant r^\alpha, \quad j = k, k-1, \dots, 1. \tag{3.24}$$

On the other hand, Lemma 3.6 (1) implies that there exists a set $E_i \subset (1,\infty)$ of finite logarithmic measure such that for all z, $|z| \notin E_i \cup [0,1]$, we have

$$|A_j^i| \leqslant M_i r^{\mu_i} \exp\{\gamma_i r^\beta\}. \tag{3.25}$$

And by 3.6 (2) there exists a set $E_i' \subset (1,\infty)$ of finite logarithmic measure such that for all z, $|z| \notin E_i' \cup [0,1]$, we have

$$|A_0^i - A_0| \leqslant M_i r^{\mu_i} \exp\{\gamma_i r^\beta\}. \tag{3.26}$$

By (3.23), (3.24), (3.25), and (3.26), we can find a set $E' := E \cup E_i' \cup E_i'$ of finite logarithmic measure such that for all z, $|z| \notin E_i' \cup [0,1]$, we have

$$\begin{aligned}|A_0| &\leqslant |\frac{g^{(k)}}{g}| + |A_{k-1}^i||\frac{g^{(k-1)}}{g}| + \dots + |A_0^i - A_0|\\ &\leqslant r^\alpha + M_i r^{\mu_i} \exp\{\gamma_i r^\beta\} r^\alpha + M_i r^{\mu_i} \exp\{\gamma_i r^\beta\} r^\alpha\\ &\leqslant M r^\mu \exp\{\gamma r^\beta\} r^\alpha,\end{aligned}$$

where M, μ, γ are positive real numbers. This leads to a contradiction with $\beta < \rho(A_0)$, hence $\rho(g) = \infty$. □

Lemma 3.8. *Let A_j $(j = 0, 1, \dots, k-1)$ be entire functions. If f is a solution of an equation of the form (2.1) then $g_i = f^{(i)}$ $(i \in \mathbb{N})$ is an entire solution of the equation*

$$g_i^{(k)} + A_{k-1}^i g_i^{(k-1)} + \dots + A_0^i g_i = 0, \tag{3.27}$$

where A_j^i, $j = 0, 1, \dots, k-1$ are defined in (3.4).

Proof. Assume that f is a solution of (2.1) and let $g_i := f^{(i)}$ $(i \in \mathbb{N})$. We shall prove that g_i is an entire solution of (3.27). To do this, we use induction. For $i = 1$, differentiating both sides of (2.1), we write

$$f^{(k+1)} + A_{k-1} f^{(k)} + (A_{k-1}' + A_{k-2}) f^{(k-1)} + \dots + (A_1' + A_0) f' + A_0' f = 0.$$

Taking

$$f = -\frac{(f^{(k)} + A_{k-1} f^{(k-1)} + \dots + A_1 f')}{A_0},$$

we obtain

$$\begin{aligned}&f^{(k+1)} + (A_{k-1} - \frac{A_0'}{A_0}) f^{(k)} + (A_{k-1}' + A_{k-2} - A_{k-1}\frac{A_0'}{A_0}) f^{(k-1)}\\ &+ \dots + (A_1' + A_0 - A_1 \frac{A_0'}{A_0}) f' = 0;\end{aligned}$$

that is,

$$g_1^{(k)} + A_{k-1}^1 g_1^{(k-1)} + A_{k-2}^1 g_1^{(k-2)} \dots + A_0^1 g_1 = 0.$$

Hence (3.27) is true for $i = 1$.

Now suppose that (3.27) is true for the values which are strictly smaller than a certain i. If g_{i-1} is a solution of the equation

$$g_{i-1}^{(k)} + A_{k-1}^{i-1} g_{i-1}^{(k-1)} + A_{k-2}^{i-1} g_{i-1}^{(k-2)} \dots + A_0^{i-1} g_{i-1} = 0, \tag{3.28}$$

then by differentiation both sides of (3.28), we obtain

$$\begin{aligned}&g_{i-1}^{(k+1)} + A_{k-1}^{i-1} g_{i-1}^{(k)} + ((A_{k-1}^{i-1})' + A_{k-2}^{i-1}) g_{i-1}^{(k-1)} + \dots\\ &+ ((A_1^{i-1})' + A_0^{i-1}) g_{i-1}' + A_0' g_{i-1} = 0.\end{aligned}$$

Taking

$$g_{i-1} = -\frac{(g_{i-1}^{(k)} + A_{k-1}^{i-1} g_{i-1}^{(k-1)} + A_{k-2}^{i-1} g_{i-1}^{(k-2)} \cdots + A(g_{i-1})')}{A_0^{i-1}},$$

we obtain

$$\begin{aligned} &g_{i-1}^{(k+1)} + \Big(A_{k-1}^{i-1} - \frac{(A_0^{i-1})'}{A_0^{i-1}}\Big) g_{i-1}^{(k)} + \Big((A_{k-1}^{i-1})' + A_{k-2}^{i-1} - A_{k-1}^{i-1} \frac{(A_0^{i-1})'}{A_0^{i-1}}\Big) g_{i-1}^{(k-1)} \\ &+ \cdots + \Big((A_1^{i-1})' + A_0^{i-1} - A_1^{i-1} \frac{(A_0^{i-1})'}{A_0^{i-1}}\Big) g_{i-1}' = 0; \end{aligned}$$

that is,

$$g_i^{(k)} + A_{k-1}^{i-1} g_i^{(k-1)} + A_{k-2}^{i-1} g_i^{(k-2)} \cdots + A_0^{i-1} g_i = 0.$$

Lemma 3.8 is thus proved. □

3.1. **Proof of Theorem 2.1.** Assume that f is a solution of (2.1). By Lemma 3.5, we have $\rho_2(f) = \rho(A_0)$. Using Lemma 3.8, we can state that $g_i := f^{(i)}$ $(i \in \mathbb{N})$ is a solution of (3.27). Let $w(z) := g_i(z) - \varphi(z)$; φ is an entire finite order function. Then $\rho(w) = \rho(g_i) = \rho(f) = \infty$ and $\rho_2(w) = \rho_2(g_i) = \rho_2(f) = \rho(A_0)$.

To prove $\overline{\lambda}(g_i - \varphi) = \lambda(g_i - \varphi) = \infty$ and $\overline{\lambda}_2(g_i - \varphi) = \lambda_2(g_i - \varphi) = \rho(A_0)$, we need to prove only that $\overline{\lambda}(w) = \infty$ and $\overline{\lambda}_2(w) = \rho(A_0)$. Using the fact that $g_i = w + \varphi$, and by Lemma 3.8, we can write

$$w^{(k)} + A_{k-1}^i w^{(k-1)} + \cdots + A_0^i w = -\Big(\varphi^{(k)} + A_{k-1}^i \varphi^{(k-1)} + \cdots + A_0^i \varphi\Big) = F. \quad (3.29)$$

By $\rho(\varphi) < \infty$ and Lemma 3.7, we obtain $F \not\equiv 0$ and $\rho(F) < \infty$. By Lemma 3.4 $\overline{\lambda}(w) = \lambda(w) = \rho(w) = \infty$ and $\overline{\lambda}_2(w) = \lambda_2(w) = \rho_2(w) = \rho(A_0)$. The proof of Theorem 2.1 is complete.

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