

**UNIQUENESS AND ASYMPTOTIC BEHAVIOR OF POSITIVE
SOLUTIONS FOR A FRACTIONAL-ORDER INTEGRAL
BOUNDARY-VALUE PROBLEM**

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ABSTRACT. In this note, we extend the results by Jia et al [3] to a more general case. By refining the conditions imposed on f and finding more suitable upper and lower solution, we remove some key conditions used in [3], and still establish their results.

1. INTRODUCTION

In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see [1, 2, 3, 4, 5, 10, 11, 12, 13]. Yuan [1] studied the $(n - 1, 1)$ -type conjugate boundary-value problem

$$\begin{aligned} \mathcal{D}_t^\alpha u(t) + f(t, u(t)) &= 0, \quad 0 < t < 1, n - 1 < \alpha \leq n, n \geq 3, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) &= 0, \quad u(1) = 0, \end{aligned} \quad (1.1)$$

where f is continuous and semipositone, \mathcal{D}_t^α is the standard Riemann-Liouville derivative. By giving properties of Green's function and using the Guo-Krasnosel'skii fixed point theorem on cones, the existence of multiple positive solutions of (1.1) were obtained. Zhang [2] considered the existence and uniqueness of higher-order fractional differential equation

$$\begin{aligned} \mathcal{D}_t^\alpha x(t) + q(t)f(x, x', \dots, x^{(n-2)}) &= 0, \quad 0 < t < 1, n - 1 < \alpha \leq n, \\ x(0) = x'(0) = \dots = x^{(n-2)}(0) &= x^{(n-2)}(1) = 0, \end{aligned} \quad (1.2)$$

where \mathcal{D}_t^α is the standard Riemann-Liouville fractional derivative of order α , q may be singular at $t = 0$ and f may be singular at $x = 0, x' = 0, \dots, x^{(n-2)} = 0$. By using fixed point theorem of the mixed monotone operator, the author established the existence and uniqueness result of positive solution for the above problem (1.2). Recently, Jia et al [3], considered the existence, uniqueness and asymptotic behavior of positive solutions for the following higher nonlocal fractional differential equation

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with Riemann-Stieltjes integral condition

$$\begin{aligned} -\mathcal{D}_t^\alpha x(t) &= f(t, x(t), x'(t), x''(t), \dots, x^{(n-2)}(t)), \quad 0 < t < 1, \\ x(0) = x'(0) = \dots = x^{(n-2)}(0) &= 0, \quad x^{(n-2)}(1) = \int_0^1 x^{(n-2)}(s) dA(s), \end{aligned} \quad (1.3)$$

where $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$ and $n \geq 2$, \mathcal{D}_t^α is the standard Riemann-Liouville derivative, $\int_0^1 x^{(n-2)}(s) dA(s)$ is linear functionals given by Riemann-Stieltjes integrals, A is a function of bounded variation and dA can be a changing-sign measure, and $f : (0, 1) \times (0, +\infty)^{n-1} \rightarrow [0, +\infty)$ is continuous, f may be singular at $x_i = 0$ and $t = 0, 1$. By using upper and lower solution method and Schauder's fixed point theorem, the existence, uniqueness and asymptotic behavior of positive solutions of (1.3) are obtained provided that f satisfies suitable growth condition and integral conditions.

Motivated by the results mentioned above, in this paper, we study the existence, uniqueness and asymptotic behavior of positive solutions for the fractional differential equation with Riemann-Stieltjes integral condition

$$\begin{aligned} -\mathcal{D}_t^\mu x(t) &= f(t, x(t), \mathcal{D}_t^{\mu_1} x(t), \mathcal{D}_t^{\mu_2} x(t), \dots, \mathcal{D}_t^{\mu_{n-2}} x(t)), \quad 0 < t < 1, \\ x(0) = \mathcal{D}_t^{\mu_1} x(0) = \dots = \mathcal{D}_t^{\mu_{n-2}} x(0) &= 0, \quad \mathcal{D}_t^{\mu_{n-2}} x(1) = \int_0^1 \mathcal{D}_t^{\mu_{n-2}} x(s) dA(s), \end{aligned} \quad (1.4)$$

where $n - 1 < \mu \leq n$, $n \in \mathbb{N}$ and $n \geq 2$ with $0 < \mu_1 < \mu_2 < \dots < \mu_{n-2}$ and $n - 2 < \mu_{n-2} < \mu - 1$, \mathcal{D}_t^μ is the standard Riemann-Liouville derivative, and $f : (0, 1) \times (0, +\infty)^{n-1} \rightarrow [0, +\infty)$ is continuous, f may be singular at $x_i = 0$ and $t = 0, 1$. By refining the conditions imposed on f and finding more suitable upper and lower solution, we remove some key conditions which are required in the works of Jia et al [3], but a similar result is still established for the more general form (1.4).

2. PRELIMINARIES

In this section, we present the necessary definitions from fractional calculus theory.

Definition 2.1 ([6, 7]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $x : (0, +\infty) \rightarrow \mathbb{R}$ is given by

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2 ([6, 7]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $x : (0, +\infty) \rightarrow \mathbb{R}$ is given by

$$\mathcal{D}_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} x(s) ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α , provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Proposition 2.3 ([6, 7]). (1) If $x \in L^1(0, 1)$, $\nu > \sigma > 0$, then

$$I^\nu I^\sigma x(t) = I^{\nu+\sigma} x(t), \quad \mathcal{D}_t^\sigma I^\nu x(t) = I^{\nu-\sigma} x(t), \quad \mathcal{D}_t^\sigma I^\sigma x(t) = x(t).$$

(2) If $\alpha > 0$, $\sigma > 0$, then

$$\mathcal{D}_t^\alpha t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)} t^{\sigma-\alpha-1}.$$

Proposition 2.4 ([6, 7]). Let $\alpha > 0$, and $f(x)$ be integrable, then

$$I^\alpha \mathcal{D}_t^\alpha f(x) = f(x) + c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + \cdots + c_n x^{\alpha-n},$$

where $c_i \in \mathbb{R}$ ($i = 1, 2, \dots, n$), n is the smallest integer greater than or equal to α .

Let

$$x(t) = I^{\mu_{n-2}} y(t), \quad y(t) \in C[0, 1],$$

by Propositions 2.3-2.4 and a discussion similar to [3], we easily reduce the order of (1.4) to the equivalent problem

$$\begin{aligned} -\mathcal{D}_t^{\mu-\mu_{n-2}} y(t) &= f(t, I^{\mu_{n-2}} y(t), I^{\mu_{n-2}-\mu_1} y(t), \dots, I^{\mu_{n-2}-\mu_{n-3}} y(t), y(t)), \\ y(0) &= 0, \quad y(1) = \int_0^1 y(s) dA(s). \end{aligned} \quad (2.1)$$

Lemma 2.5 ([13]). Given $h \in L^1(0, 1)$, then the problem

$$\begin{aligned} \mathcal{D}_t^{\mu-\mu_{n-2}} y(t) + h(t) &= 0, \quad 0 < t < 1, \\ y(0) &= 0, \quad y(1) = 0, \end{aligned} \quad (2.2)$$

has the unique solution

$$y(t) = \int_0^1 G(t, s) h(s) ds,$$

where $G(t, s)$ is the Green function of (2.2), given by

$$G(t, s) = \begin{cases} \frac{t^{\mu-\mu_{n-2}-1} (1-s)^{\mu-\mu_{n-2}-1} - (t-s)^{\mu-\mu_{n-2}-1}}{\Gamma(\mu-\mu_{n-2})}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\mu-\mu_{n-2}-1} (1-s)^{\mu-\mu_{n-2}-1}}{\Gamma(\mu-\mu_{n-2})}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (2.3)$$

By Proposition 2.4, the unique solution of the problem

$$\begin{aligned} \mathcal{D}_t^{\mu-\mu_{n-2}} y(t) &= 0, \quad 0 < t < 1, \\ y(0) &= 0, \quad y(1) = 1, \end{aligned} \quad (2.4)$$

is $t^{\mu-\mu_{n-2}-1}$. Let

$$\mathcal{C} = \int_0^1 t^{\mu-\mu_{n-2}-1} dA(t), \quad (2.5)$$

and define

$$\mathcal{G}_A(s) = \int_0^1 G(t, s) dA(t).$$

Then the Green function for the nonlocal BVP (2.1) is (for details see [8] or [9])

$$K(t, s) = \frac{t^{\mu-\mu_{n-2}-1}}{1-\mathcal{C}} \mathcal{G}_A(s) + G(t, s). \quad (2.6)$$

In this article we use the following assumption

(H0) A is a function of bounded variation such that $\mathcal{G}_A(s) \geq 0$ for $s \in [0, 1]$ and $0 \leq \mathcal{C} < 1$, where \mathcal{C} is defined by (2.5).

The following Lemma follows from (2.3) and (2.6).

Lemma 2.6. *Suppose (H0) holds. Then the Green function defined by (2.6) satisfies:*

(1) $K(t, s) > 0$, for all $t, s \in (0, 1)$.

(2)

$$\frac{t^{\mu-\mu_{n-2}-1}}{1-\mathcal{C}} \mathcal{G}_A(s) \leq K(t, s) \leq \mathcal{H}(s)t^{\mu-\mu_{n-2}-1}, \quad (2.7)$$

where

$$\mathcal{H}(s) = \frac{(1-s)^{\mu-\mu_{n-2}-1}}{\Gamma(\mu-\mu_{n-2})} + \frac{\mathcal{G}_A(s)}{1-\mathcal{C}}.$$

Definition 2.7. *A continuous function $\Psi(t)$ is called a lower solution of (2.1), if it satisfies*

$$\begin{aligned} -\mathcal{D}_t^{\mu-\mu_{n-2}}\Psi(t)(t) &\leq f(t, I^{\mu_{n-2}}\Psi(t), I^{\mu_{n-2}-\mu_1}\Psi(t), \dots, I^{\mu_{n-2}-\mu_{n-3}}\Psi(t), \Psi(t)), \\ \Psi(0) &\geq 0, \quad \Psi(1) \geq \int_0^1 \Psi(s)dA(s). \end{aligned}$$

Definition 2.8. *A continuous function $\Phi(t)$ is called an upper solution of (2.1), if it satisfies*

$$\begin{aligned} -\mathcal{D}_t^{\mu-\mu_{n-2}}\Phi(t)(t) &\geq f(t, I^{\mu_{n-2}}\Phi(t), I^{\mu_{n-2}-\mu_1}\Phi(t), \dots, I^{\mu_{n-2}-\mu_{n-3}}\Phi(t), \Phi(t)), \\ \Phi(0) &\leq 0, \quad \Phi(1) \leq \int_0^1 \Phi(s)dA(s). \end{aligned}$$

3. MAIN RESULTS

Let $E = C[0, 1]$. Define the following continuous functions on E :

$$\begin{aligned} \kappa_0(t) &= I^{\mu_{n-2}}s^{\mu-\mu_{n-2}-1} = \int_0^t \frac{(t-s)^{\mu_{n-2}-1}s^{\mu-\mu_{n-2}-1}}{\Gamma(\mu_{n-2})} ds = \frac{\Gamma(\mu-\mu_{n-2})}{\Gamma(\mu)}s^{\mu-1}, \\ \kappa_1(t) &= I^{\mu_{n-2}-\mu_1}s^{\mu-\mu_{n-2}-1} = \int_0^t \frac{(t-s)^{\mu_{n-2}-\mu_1-1}s^{\mu-\mu_{n-2}-1}}{\Gamma(\mu_{n-2}-\mu_1)} ds \\ &= \frac{\Gamma(\mu-\mu_{n-2})}{\Gamma(\mu-\mu_1)}t^{\mu-1-\mu_1}, \\ &\dots \\ \kappa_{n-3}(t) &= I^{\mu_{n-2}-\mu_{n-3}}s^{\mu-\mu_{n-2}-1} = \int_0^t \frac{(t-s)^{\mu_{n-2}-\mu_{n-3}-1}s^{\mu-\mu_{n-2}-1}}{\Gamma(\mu_{n-2}-\mu_{n-3})} ds \\ &= \frac{\Gamma(\mu-\mu_{n-2})}{\Gamma(\mu-\mu_{n-3})}t^{\mu-1-\mu_{n-3}}, \\ \kappa_{n-2}(t) &= t^{\mu-1-\mu_{n-2}}. \end{aligned}$$

Set

$$\begin{aligned} P = \left\{ y \in E : \text{there exist positive numbers } 0 < l_y < 1, L_y > 1 \text{ such that} \right. \\ \left. l_y \kappa_{n-2}(t) \leq y(t) \leq L_y \kappa_{n-2}(t), t \in [0, 1] \right\}. \end{aligned} \quad (3.1)$$

Clearly, P is nonempty since $\kappa_{n-2}(t) \in P$. For any $y \in P$, define an operator T by

$$(Ty)(t) = \int_0^1 K(t,s)f(s, I^{\mu_{n-2}}y(s), I^{\mu_{n-2}-\mu_1}y(s), \dots, I^{\mu_{n-2}-\mu_{n-3}}y(s), y(s))ds. \quad (3.2)$$

In this note, we will use the following conditions:

(H1) $f \in C((0,1) \times (0, \infty)^{n-1}, [0, +\infty))$, and $f(t, x_0, x_1, x_2, \dots, x_{n-2})$ is nonincreasing in $x_i > 0$ for $i = 0, 1, 2, \dots, n-2$;

(H2) For any $\lambda_i > 0$,

$$0 < \int_0^1 \mathcal{H}(s)f(s, \lambda_0\kappa_0(s), \lambda_1\kappa_1(s), \lambda_2\kappa_2(s), \dots, \lambda_{n-2}\kappa_{n-2}(s))ds < +\infty.$$

Lemma 3.1. *Suppose (H0)–(H2) hold. Then T is well defined, $T(P) \subset P$, and T is nonincreasing relative to y .*

Proof. For any $y \in P$, by the definition of P , there exist two positive numbers $0 < l_y < 1$, $L_y > 1$ such that

$$l_y\kappa_{n-2}(s) \leq y(t) \leq L_y\kappa_{n-2}(s) \quad (3.3)$$

for any $s \in [0, 1]$. It follows from (2.7) and (H1)–(H2) that

$$\begin{aligned} (Ty)(t) &= \int_0^1 K(t,s)f(s, I^{\mu_{n-2}}y(s), I^{\mu_{n-2}-\mu_1}y(s), \dots, I^{\mu_{n-2}-\mu_{n-3}}y(s), y(s))ds \\ &\leq \kappa_{n-2}(s) \int_0^1 \mathcal{H}(s)f(s, l_y\kappa_0(s), l_y\kappa_1(s), \dots, l_y\kappa_{n-3}(s), l_y\kappa_{n-2}(s))ds \\ &< +\infty. \end{aligned} \quad (3.4)$$

By (2.7), (3.3) and (3.4), we have

$$\begin{aligned} (Ty)(t) &= \int_0^1 K(t,s)f(s, I^{\mu_{n-2}}y(s), I^{\mu_{n-2}-\mu_1}y(s), \dots, I^{\mu_{n-2}-\mu_{n-3}}y(s), y(s))ds \\ &\geq \frac{t^{\mu-\mu_{n-2}-1}}{1-\mathcal{C}} \int_0^1 \mathcal{G}_A(s)f(s, L_y\kappa_0(s), L_y\kappa_1(s), \dots, L_y\kappa_{n-3}(s), L_y\kappa_{n-2}(s))ds. \end{aligned} \quad (3.5)$$

Take

$$\begin{aligned} l'_y &= \min \left\{ 1, \frac{1}{1-\mathcal{C}} \int_0^1 \mathcal{G}_A(s)f(s, L_y\kappa_0(s), L_y\kappa_1(s), \dots, L_y\kappa_{n-3}(s), L_y\kappa_{n-2}(s))ds \right\}, \\ L'_y &= \max \left\{ 1, \int_0^1 \mathcal{H}(s)f(s, l_y\kappa_0(s), l_y\kappa_1(s), \dots, l_y\kappa_{n-2}(s))ds \right\}. \end{aligned} \quad (3.6)$$

It follows from (3.3)–(3.6) that T is well defined and $T(P) \subset P$. Moreover, by (H1), T is nonincreasing relative to y . \square

Theorem 3.2 (Existence). *Suppose $rm(H0)$ –(H2) hold. Then (1.4) has at least one positive solution $x(t)$.*

Proof. From (3.2) and simple computation, we have

$$\begin{aligned} -\mathcal{D}_t^{\mu-\mu_{n-2}}(Ty)(t) &= f(t, I^{\mu_{n-2}}y(t), I^{\mu_{n-2}-\mu_1}y(t), \dots, I^{\mu_{n-2}-\mu_{n-3}}y(t), y(t)), \\ (Ty)(0) &= 0, \quad (Ty)(1) = \int_0^1 (Ty)(s)dA(s). \end{aligned} \quad (3.7)$$

Let

$$\alpha(t) = \min\{\kappa_{n-2}(t), (T\kappa_{n-2})(t)\}, \quad \beta(t) = \max\{\kappa_{n-2}(t), (T\kappa_{n-2})(t)\}, \quad (3.8)$$

then, if $\kappa_{n-2}(t) = (T\kappa_{n-2})(t)$, the conclusion of Theorem 3.2 holds. If $\kappa_{n-2}(t) \neq (T\kappa_{n-2})(t)$, clearly, $\alpha(t), \beta(t) \in P$ and

$$\alpha(t) \leq \kappa_{n-2}(t) \leq \beta(t). \quad (3.9)$$

Set

$$\Phi(t) = (T\beta)(t), \Psi(t) = (T\alpha)(t),$$

then by (3.8)-(3.9) and Lemma 3.1, one has

$$\begin{aligned} \Phi(t) &= (T\beta)(t) \leq (T\kappa_{n-2})(t) \leq T(\alpha)(t) = \Psi(t), \\ \Phi(t) &\leq (T\kappa_{n-2})(t) \leq \beta(t), \quad \Psi(t) \geq (T\kappa_{n-2})(t) \geq \alpha(t), \end{aligned} \quad (3.10)$$

and $\Phi(t), \Psi(t) \in P$.

On the other hand, by (3.7), (3.10) and Lemma 3.1, we have

$$\begin{aligned} &\mathcal{D}_t^{\mu-\mu_{n-2}}\Phi(t) + f(t, I^{\mu_{n-2}}\Phi(t), I^{\mu_{n-2}-\mu_1}\Phi(t), \dots, I^{\mu_{n-2}-\mu_{n-3}}\Phi(t), \Phi(t)) \\ &\geq \mathcal{D}_t^{\mu-\mu_{n-2}}(T\beta)(t) + f(t, I^{\mu_{n-2}}\beta(t), I^{\mu_{n-2}-\mu_1}\beta(t), \dots, I^{\mu_{n-2}-\mu_{n-3}}\beta(t), \beta(t)) \\ &= -f(t, I^{\mu_{n-2}}\beta(t), I^{\mu_{n-2}-\mu_1}\beta(t), \dots, I^{\mu_{n-2}-\mu_{n-3}}\beta(t), \beta(t)) \\ &\quad + f(t, I^{\mu_{n-2}}\beta(t), I^{\mu_{n-2}-\mu_1}\beta(t), \dots, I^{\mu_{n-2}-\mu_{n-3}}\beta(t), \beta(t)) = 0, \\ (T\Phi)(0) &= 0, \quad (T\Phi)(1) = \int_0^1 (T\Phi)(s)dA(s). \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} &\mathcal{D}_t^{\mu-\mu_{n-2}}\Psi(t) + f(t, I^{\mu_{n-2}}\Psi(t), I^{\mu_{n-2}-\mu_1}\Psi(t), \dots, I^{\mu_{n-2}-\mu_{n-3}}\Psi(t), \Psi(t)) \\ &\leq \mathcal{D}_t^{\mu-\mu_{n-2}}(T\alpha)(t) + f(t, I^{\mu_{n-2}}\alpha(t), I^{\mu_{n-2}-\mu_1}\alpha(t), \dots, I^{\mu_{n-2}-\mu_{n-3}}\alpha(t), \alpha(t)) \\ &= -f(t, I^{\mu_{n-2}}\alpha(t), I^{\mu_{n-2}-\mu_1}\alpha(t), \dots, I^{\mu_{n-2}-\mu_{n-3}}\alpha(t)) \\ &\quad + f(t, I^{\mu_{n-2}}\alpha(t), I^{\mu_{n-2}-\mu_1}\alpha(t), \dots, I^{\mu_{n-2}-\mu_{n-3}}\alpha(t)) = 0, \\ (T\Psi)(0) &= 0, \quad (T\Psi)(1) = \int_0^1 (T\Psi)(s)dA(s). \end{aligned} \quad (3.12)$$

Inequalities (3.10)-(3.12) imply that $\Phi(t), \Psi(t)$ are lower and upper solution of (2.1), respectively.

Define the function F and the operator A in E by

$$F(t, y) = \begin{cases} f(t, I^{\mu_{n-2}}\Phi(t), I^{\mu_{n-2}-\mu_1}\Phi(t), \dots, I^{\mu_{n-2}-\mu_{n-3}}\Phi(t), \Phi(t)), & y < \Phi(t), \\ f(t, I^{\mu_{n-2}}y(t), I^{\mu_{n-2}-\mu_1}y(t), \dots, I^{\mu_{n-2}-\mu_{n-3}}y(t), y(t)), & \Phi(t) \leq y \leq \Psi(t), \\ f(t, I^{\mu_{n-2}}\Psi(t), I^{\mu_{n-2}-\mu_1}\Psi(t), \dots, I^{\mu_{n-2}-\mu_{n-3}}\Psi(t), \Psi(t)), & y > \Psi(t), \end{cases} \quad (3.13)$$

and

$$(\mathfrak{A}y)(t) = \int_0^1 K(t,s)F(s,y(s))ds, \quad \forall y \in E.$$

Clearly, $F : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous by (3.13). Consider the following boundary value problem

$$\begin{aligned} -\mathcal{D}_t^{\mu-\mu_{n-2}}y(t) &= F(t,y), \quad 0 < t < 1, \\ y(0) &= 0, \quad y(1) = \int_0^1 y(s)dA(s). \end{aligned} \quad (3.14)$$

Obviously, a fixed point of the operator \mathfrak{A} is a solution of (3.14). As in [3], \mathfrak{A} has at least a fixed point y such that $y = \mathfrak{A}y$.

In the end, we claim

$$\Phi(t) \leq y(t) \leq \Psi(t), \quad t \in [0, 1].$$

In fact, since y is fixed point of \mathfrak{A} and (3.12), we obtain

$$y(0) = 0, \quad y(1) = \int_0^1 y(s)dA(s), \quad \Psi(0) = 0, \quad \Psi(1) = \int_0^1 \Psi(s)dA(s). \quad (3.15)$$

We firstly claim $y(t) \leq \Psi(t)$. Otherwise, suppose $x(t) > \Psi(t)$. According to the definition of F , we have

$$\begin{aligned} -\mathcal{D}_t^{\mu-\mu_{n-2}}y(t) &= F(t,y(t)) \\ &= f(t, I^{\mu_{n-2}}\Psi(t), I^{\mu_{n-2}-\mu_1}\Psi(t), \dots, I^{\mu_{n-2}-\mu_{n-3}}\Psi(t), \Psi(t)). \end{aligned} \quad (3.16)$$

On the other hand, it follows from ψ is an upper solution to (2.1) that

$$-\mathcal{D}_t^{\mu-\mu_{n-2}}\Psi(t) \geq f(t, I^{\mu_{n-2}}\Psi(t), I^{\mu_{n-2}-\mu_1}\Psi(t), \dots, I^{\mu_{n-2}-\mu_{n-3}}\Psi(t), \Psi(t)). \quad (3.17)$$

Let $z(t) = \Psi(t) - y(t)$, (3.15)-(3.17) imply that

$$\mathcal{D}_t^{\mu-\mu_{n-2}}z(t) = \mathcal{D}_t^{\mu-\mu_{n-2}}\Psi(t) - \mathcal{D}_t^{\mu-\mu_{n-2}}y(t) \leq 0,$$

and

$$z(0) = 0, \quad z(1) = \int_0^1 z(s)dA(s).$$

It follows from (2.6) that

$$z(t) \geq 0,$$

i.e., $y(t) \leq \Psi(t)$ on $[0, 1]$, which contradicts $y(t) > \Psi(t)$. Hence, $y(t) > \Psi(t)$ is impossible.

By the same way, we also have $y(t) \geq \Phi(t)$ on $[0, 1]$. So

$$\Phi(t) \leq y(t) \leq \Psi(t), \quad t \in [0, 1]. \quad (3.18)$$

Consequently, $F(t, y(t)) = f(t, I^{n-2}y(t), I^{n-3}y(t), \dots, I^1y(t), y(t))$, $t \in [0, 1]$. Then $y(t)$ is a positive solution of the problem (2.1). It follows from (2.1) that $x(t) = I^{\mu_{n-2}}y(t)$ is positive solution of (1.4). \square

Remark 3.3. In this work, we not only extend the main result of [3] to more general form with fractional derivatives in nonlinearity and boundary condition, but also by finding more suitable upper and lower solution, we omit the following key conditions of [3]:

- (i) For any $\lambda_i > 0$, $f(t, \lambda_0 t^{n-2}, \lambda_1 t^{n-3}, \dots, \lambda_{n-3} t, \lambda_{n-2}) \neq 0$, $t \in (0, 1)$.

(ii)

$$\int_0^1 \mathcal{G}_A(s) f\left(s, \frac{L}{l} \kappa_0(s), \frac{L}{l} \kappa_1(s), \dots, \frac{L}{l} \kappa_{n-3}(s), \frac{L}{l} \kappa_{n-2}(s)\right) ds \geq 1 - \mathcal{C}.$$

This implies our result essentially improves those of [3].

Theorem 3.4 (Asymptotic Behavior). *Suppose (H0)–(H2) hold. Then there exist two constants $\mathcal{B}_1, \mathcal{B}_2$ such that the positive solution $x(t)$ of (1.4) satisfies*

$$\mathcal{B}_1 \kappa_{n-2}(t) \leq x(t) \leq \mathcal{B}_2 \kappa_{n-2}(t). \quad (3.19)$$

Proof. By (3.18), and $\Phi, \Psi \in P$, we know that there exist two positive constants $0 < l_\Phi < 1$, $L_\Psi > 1$ such that

$$l_\Phi \kappa_{n-2}(t) \leq \Phi(t) \leq y(t) \leq \Psi(t) \leq L_\Psi \kappa_{n-2}(t).$$

Notice that $x(t) = I^{\mu-2}y(t)$, we have

$$\begin{aligned} \frac{l_\Phi \Gamma(\mu - \mu_{n-2})}{\Gamma(\mu)} t^{\mu-1} &= l_\Phi I^{\mu-2} \kappa_{n-2}(s) \leq x(t) \\ &\leq L_\Psi I^{\mu-2} \kappa_{n-2}(s) = \frac{L_\Psi \Gamma(\mu - \mu_{n-2})}{\Gamma(\mu)} t^{\mu-1}. \end{aligned}$$

Let

$$\mathcal{B}_1 = \frac{l_\Phi \Gamma(\mu - \mu_{n-2})}{\Gamma(\mu)}, \quad \mathcal{B}_2 = \frac{L_\Psi \Gamma(\mu - \mu_{n-2})}{\Gamma(\mu)},$$

then (3.19) holds. \square

If $\mu > 1$ is a integer, then we also have the following uniqueness result similar to [3].

Theorem 3.5 (Uniqueness). *Suppose (H0)–(H2) hold, and $\mu = n > 1$. Then the positive solution $x(t)$ of (1.4) is unique.*

Proof. Notice that

$$\begin{aligned} &f(t, I^{\mu-2}w_2(t), I^{\mu-2-\mu_1}w_2(t), \dots, I^{\mu-2-\mu_{n-3}}w_2(t), w_2(t)) \\ &\leq f(t, I^{\mu-2}w_1(t), I^{\mu-2-\mu_1}w_1(t), \dots, I^{\mu-2-\mu_{n-3}}w_1(t), w_2(t)), \end{aligned}$$

for

$$w_2(t) \geq w_1(t), \quad t \in [a, b].$$

Thus similar to [3], the proof is completed. \square

Example 3.6. Consider the existence of positive solutions for the nonlinear fractional differential equation

$$\begin{aligned} -\mathcal{D}_t^{11/3} x(t) &= t^{2/3} [x^{-1/2} + (\mathcal{D}_t^{2/3} x)^{-1/3} + (\mathcal{D}_t^{7/3} x)^{-2/3}], \quad 0 < t < 1, \\ x(0) &= \mathcal{D}_t^{2/3} x(0) = \mathcal{D}_t^{7/3} x(0) = 0, \quad \mathcal{D}_t^{7/3} x(1) = \int_0^1 \mathcal{D}_t^{7/3} x(s) dA(s), \end{aligned} \quad (3.20)$$

where

$$A(t) = \begin{cases} 0, & t \in [0, 1/2), \\ 3/2, & t \in [1/2, 3/4), \\ 1, & t \in [3/4, 1]. \end{cases}$$

Therefore, (3.20) has at least a positive solution.

Proof. Clearly, (3.20) is equivalent to the following 4-point BVP with coefficients of both signs

$$\begin{aligned}
 -\mathcal{D}_t^{11/3}x(t) &= t^{2/3}[x^{-1/2} + (\mathcal{D}_t^{2/3}x)^{-1/3} + (\mathcal{D}_t^{7/3}x)^{-2/3}], \quad 0 < t < 1, \\
 x(0) = \mathcal{D}_t^{2/3}x(0) = \mathcal{D}_t^{7/3}x(0) &= 0, \quad \mathcal{D}_t^{7/3}(1) = \frac{3}{2}\mathcal{D}_t^{7/3}\left(\frac{1}{2}\right) - \frac{1}{2}\mathcal{D}_t^{7/3}\left(\frac{3}{4}\right),
 \end{aligned}$$

Thus $f(t, x_0, x_1, x_2) = t^{2/3}[x_0^{-1/2} + x_1^{-1/3} + x_2^{-2/3}]$, $\kappa_2(t) = t^{1/3}$, and

$$0 \leq C = \int_0^1 t^{1/3}dA(t) = 1 - \left[\int_{1/2}^{3/4} \frac{3}{2}dt^{1/3} + \int_{3/4}^1 dt^{1/3} \right] \approx 0.7363 < 1,$$

and

$$G(t, s) = \begin{cases} G_1(t, s) = \frac{t^{1/3}(1-s)^{1/3}}{\Gamma(4/3)}, & 0 \leq t \leq s \leq 1, \\ G_2(t, s) = \frac{t^{1/3}(1-s)^{1/3} - (t-s)^{1/3}}{\Gamma(4/3)}, & 0 \leq s \leq t \leq 1. \end{cases}$$

Thus,

$$\begin{aligned}
 \mathcal{G}_A(s) &= \begin{cases} \frac{3}{2}G_2(1/2, s) - \frac{1}{2}G_2(3/4, s), & 0 \leq s < \frac{1}{2}, \\ \frac{3}{2}G_1(1/2, s) - \frac{1}{2}G_2(3/4, s), & \frac{1}{2} \leq s < \frac{3}{4}, \\ \frac{3}{2}G_1(1/2, s) - \frac{1}{2}G_1(3/4, s), & \frac{3}{4} \leq s \leq 1, \end{cases} \\
 &= \begin{cases} \frac{(\frac{3}{2} \times (\frac{1}{2})^{1/3} - \frac{1}{2} \times (\frac{3}{4})^{1/3})(1-s)^{1/3} + \frac{1}{2}(\frac{3}{4}-s)^{1/3} - \frac{3}{2}(\frac{1}{2}-s)^{1/3}}{\Gamma(4/3)}, & 0 \leq s < \frac{1}{2}, \\ \frac{(\frac{3}{2} \times (\frac{1}{2})^{1/3} - \frac{1}{2} \times (\frac{3}{4})^{1/3})(1-s)^{1/3} + \frac{1}{2}(\frac{3}{4}-s)^{1/3}}{\Gamma(4/3)}, & \frac{1}{2} \leq s < \frac{3}{4}, \\ \frac{(\frac{3}{2} \times (\frac{1}{2})^{1/3} - \frac{1}{2} \times (\frac{3}{4})^{1/3})(1-s)^{1/3}}{\Gamma(4/3)}, & \frac{3}{4} \leq s \leq 1, \end{cases}
 \end{aligned}$$

and

$$\mathcal{H}(s) = \frac{(1-s)^{1/3}}{\Gamma(4/3)} + \frac{\mathcal{G}_A(s)}{0.2637}.$$

Clearly, (H0) and (H1) hold.

On the other hand, since

$$\kappa_0(t) = \frac{\Gamma(4/3)}{\Gamma(11/3)}t^{\frac{8}{3}}, \quad \kappa_1(t) = \frac{\Gamma(4/3)}{\Gamma(3)}t^2, \quad \kappa_2(t) = t^{1/3},$$

for any $\lambda_i > 0$, $i = 0, 1, 2$, we have

$$\begin{aligned}
 0 &< \int_0^1 \mathcal{H}(s)f(s, \lambda_0\kappa_0(s), \lambda_1\kappa_1(s), \lambda_2\kappa_2(s))ds \\
 &= \int_0^1 \left[\frac{(1-s)^{1/3}}{\Gamma(4/3)} + \frac{\mathcal{G}_A(s)}{0.2637} \right] s^{2/3} \left[\left(\frac{\Gamma(4/3)}{\Gamma(11/3)}\lambda_0 \right)^{-1/2} s^{-\frac{4}{3}} \right. \\
 &\quad \left. + \left(\frac{\Gamma(4/3)}{\Gamma(3)}\lambda_1 \right)^{-1/3} s^{-2/3} + \lambda_2^{-2/3} s^{-\frac{2}{9}} \right] ds \\
 &< +\infty.
 \end{aligned}$$

Thus (H2) is satisfied. Then by Theorem 3.2, the BVP (3.20) has at least a positive solution. □

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