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UNIQUENESS OF POSITIVE SOLUTIONS FOR AN ELLIPTIC SYSTEM ARISING IN A DIFFUSIVE PREDATOR-PREY MODEL

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Abstract. In this note, we study the uniqueness of positive solutions for an elliptic system which arises in a diffusive predator-prey model in the strong-predator case. The main result extends an earlier results by the same authors.

1. Introduction

In this note, we study the uniqueness of positive solutions for the system

$$\begin{gathered} -\Delta u=\lambda u-buv \quad \text{in } \Omega,\\ -\Delta v=\mu v\Big(1-\xi\frac{v}{u}\Big) \quad \text{in } \Omega,\\ \frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain, ν is the outward unit normal vector on $\partial\Omega$, λ, b, μ and ξ are positive constants, which arises in the diffusive predator-prey model in the strong-predator case ($\beta\to+\infty$):

$$\begin{gathered} -\Delta u=\lambda u-a(x)u^2-\beta uv \quad \text{in } \Omega,\\ -\Delta v=\mu v\Big(1-\frac{v}{u}\Big) \quad \text{in } \Omega,\\ \frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where β is a positive constant, and $a(x)$ is a continuous function satisfying $a(x)=0$ on $\overline{\Omega}_0$ and $a(x)>0$ in $\overline{\Omega}\setminus\overline{\Omega}_0$ for some smooth domain Ω_0 with $\overline{\Omega}_0\subset\Omega$. We refer the reader to [1, 2, 5] for some related studies on (1.2).

It is easy to see that $(u,v)=(\frac{\xi\lambda}{b},\frac{\lambda}{b})$ is a positive solution for problem (1.1). In [2, Remark 3.2], the authors pointed out that when $N=1$, the positive solution of (1.1) is unique for any $\mu>0$ by a simple variation of the arguments in [3]. When $N\geqslant 2$, it was proved in [6] that the uniqueness holds for all sufficiently large μ. In the present paper, we prove the uniqueness for $\mu\geqslant 2\lambda$. We point out that a key step of the proof is to establish a new a priori estimate on u for the solution (u,v) of problem (1.1), which is stated as follows.

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Theorem 1.1. *Let (u,v) be a positive solution of (1.1). If $\mu > \lambda$, then*

$$u \leqslant \frac{\xi\mu\lambda}{b(\mu-\lambda)} \quad \text{on } \overline{\Omega}. \tag{1.3}$$

Based on this estimate and the identity in [6, (2.13)], we have

Theorem 1.2. *Let $N \geqslant 2$. If $\mu \geqslant 2\lambda$, then there is a unique positive solution for (1.1).*

2. Proofs of main theorems

To prove Theorem 1.1, we need the following maximal principle due to Lou and Ni [4, Lemma 2.1].

Lemma 2.1. *Suppose that $g \in C^1(\overline{\Omega} \times \mathbb{R}^1)$, $b_j \in C(\overline{\Omega})$ for $j = 1, 2, \dots, N$.*
(i) If $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies

$$\begin{gathered}\Delta w(x) + \sum_{j=1}^{N} b_j(x) w_{x_j} + g(x, w(x)) \geqslant 0 \quad \text{in } \Omega,\\ \partial_\nu w \leqslant 0 \quad \text{on } \partial\Omega,\end{gathered}$$

and $w(x_0) = \max_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \geqslant 0$.
(ii) If $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies

$$\begin{gathered}\Delta w(x) + \sum_{j=1}^{N} b_j(x) w_{x_j} + g(x, w(x)) \leqslant 0 \quad \text{in } \Omega,\\ \partial_\nu w \geqslant 0 \quad \text{on } \partial\Omega,\end{gathered}$$

and $w(x_0) = \min_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \leqslant 0$.

Proof of Theorem 1.1. Denote us denote

$$(U, V) = \Big(\frac{b}{\xi} u, bv\Big). \tag{2.1}$$

Then (U, V) satisfies

$$\begin{gathered}-\Delta U = U(\lambda - V) \quad \text{in } \Omega,\\ -\Delta V = \mu V\Big(1 - \frac{V}{U}\Big) \quad \text{in } \Omega,\\ \frac{\partial U}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{2.2}$$

Clearly, estimate (1.3) is equivalent to

$$U \leqslant \frac{\mu\lambda}{\mu - \lambda} \quad \text{on } \overline{\Omega}. \tag{2.3}$$

Let $\varphi = V/U$. Then $V = \varphi U$, and differentiating it twice yields

$$\Delta V = \varphi \Delta U + 2\nabla U \cdot \nabla \varphi + U \Delta \varphi \quad \text{in } \Omega;$$

therefore,

$$-\Delta\varphi - \frac{2}{U}\nabla U \cdot \nabla\varphi = -\frac{1}{U}\Delta V + \frac{\varphi}{U}\Delta U \quad \text{in } \Omega. \tag{2.4}$$

From (2.2), we obtain

$$\begin{gathered}
-\Delta U = U(\lambda - \varphi U) \quad \text{in } \Omega, \\
\frac{\partial U}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\
-\Delta V = \mu\varphi U(1-\varphi) \quad \text{in } \Omega, \\
\frac{\partial V}{\partial \nu} = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{2.5}$$

Substituting them into (2.4), we have

$$-\Delta\varphi - \frac{2}{U}\nabla U \cdot \nabla\varphi = \varphi(\mu - \lambda - \mu\varphi + \varphi U) \quad \text{in } \Omega, \tag{2.6}$$

and hence

$$-\Delta\varphi - \frac{2}{U}\nabla U \cdot \nabla\varphi \geqslant \varphi(\mu - \lambda - \mu\varphi) \quad \text{in } \Omega.$$

Using Lemma 2.1 (ii) and noticing that $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$, we obtain

$$\varphi \geqslant \frac{\mu - \lambda}{\mu} \quad \text{on } \overline{\Omega}.$$

From the estimate and the first equation of (2.5) it follows that

$$\begin{gathered}
-\Delta U \leqslant U\Big(\lambda - \frac{\mu-\lambda}{\mu} U\Big) \quad \text{in } \Omega, \\
\frac{\partial U}{\partial \nu} = 0 \quad \text{on } \partial\Omega.
\end{gathered}$$

By Lemma 2.1 (i), we obtain (2.3). The proof is complete. □

Proof of Theorem 1.2. It suffices to show that $(u,v) = (\frac{\xi\lambda}{b}, \frac{\lambda}{b})$ for any positive solution (u,v) of(1.1). Let (U,V) be the same as that in (2.1). Then (U,V) satisfies (2.2).

On the other hand, one can show the following identity (i.e. [6, (2.13)]):

$$\int_\Omega (U - 2\lambda)\frac{|\nabla U|^2}{U^3} dx - \frac{\lambda}{\mu}\int_\Omega \frac{|\nabla V|^2}{V^2} dx = \int_\Omega \frac{(\lambda - V)^2}{U} dx. \tag{2.7}$$

Indeed, multiplying the equations of U and V by $\frac{\lambda - U}{U^2}$ and $\frac{1}{\mu}\frac{\lambda - V}{V}$, respectively, we obtain

$$-2\lambda\int_\Omega \frac{|\nabla U|^2}{U^3} dx + \int_\Omega \frac{|\nabla U|^2}{U^2} dx = \int_\Omega \frac{(\lambda - U)(\lambda - V)}{U} dx,$$

and

$$\begin{aligned}
-\frac{\lambda}{\mu}\int_\Omega \frac{|\nabla V|^2}{V^2} dx &= \int_\Omega \frac{(U - V)(\lambda - V)}{U} dx \\
&= \int_\Omega \frac{(U - \lambda)(\lambda - V)}{U} dx + \int_\Omega \frac{(\lambda - V)^2}{U} dx.
\end{aligned}$$

Adding the two identities yields (2.7).

Noticing $\mu \geqslant 2\lambda$, we deduce from (2.3) that $U \leqslant 2\lambda$, so the first integral of left hand side of (2.7) is non-positive, hence

$$\int_\Omega \frac{(\lambda - V)^2}{U} dx \leqslant 0,$$

which implies that $V = \lambda$, so $U = \lambda$. Recalling (2.1), we complete the proof. □

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