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POSITIVE SOLUTIONS FOR ANISOTROPIC DISCRETE BOUNDARY-VALUE PROBLEMS

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Abstract. Using mountain pass arguments and the Karsuh-Kuhn-Tucker Theorem, we prove the existence of at least two positive solution for anisotropic discrete Dirichlet boundary-value problems. Our results generalized and improve those in [16].

1. Introduction

In this note we consider an anisotropic difference equation with Dirichlet type boundary condition on the form

$$\begin{gathered}\Delta(|\Delta y(k-1)|^{p(k-1)-2}\Delta y(k-1)) + f(k,y(k)) = 0, \quad k\in[1,T],\\ y(0)=y(T+1)=0,\end{gathered} \tag{1.1}$$

where $T \geq 2$ is a integer, $f:[1,T]\times\mathbb{R}\to(0,+\infty)$ is a continuous function; $[1,T]$ is a discrete interval $\{1,2,\dots,T\}$, $\Delta y(k-1)=y(k)-y(k-1)$ is the forward difference operator; $y(k)\in\mathbb{R}$ for all $k\in[1,T]$; $p:[0,T+1]\to[2,+\infty)$. Let $p^-=\min_{k\in[0,T+1]}p(k)$; $p^+=\max_{k\in[0,T+1]}p(k)$.

About the nonlinear term, we assume the following condition

(C1) There exist a number $m>p^+$ and functions $\varphi_1,\varphi_2:[1,T]\to(0,\infty)$, $\psi_1,\psi_2:[1,T]\to(0,\infty)$ such that
$$\psi_1(k)+\varphi_1(k)|y|^{m-2}y \leq f(k,y) \leq \varphi_2(k)|y|^{m-2}y+\psi_2(k)$$
for all $y\geq 0$ and all $k\in[1,T]$.

Now, we show an example of a function that satisfies condition (C1).

Example 1.1. Let $f:[1,T]\times\mathbb{R}\to(0,\infty)$ be given by

$$f(k,y)=|y|^{m-2}y\frac{2+\arctan(y)}{T^2k}+\frac{\sin^2(k)e^{-|y|}+1}{T^3}$$

for $(k,y)\in[1,T]\times\mathbb{R}$; here $m>p^+$. We see that for $y\geq 0$ we have

$$\frac{1}{T^3}+\frac{2}{T^2k}|y|^{m-2}y \leq f(k,y) \leq \frac{4+\pi}{2T^2k}|y|^{m-2}y+\frac{2}{T^3}.$$

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Thus we may put

$$\varphi_1(k)=\frac{2}{T^2k};\quad \varphi_2(k)=\frac{4+\pi}{2T^2k};\quad \psi_1(k)=\frac{1}{T^3};\quad \psi_2(k)=\frac{2}{T^3}.$$

Solutions to (1.1) will be investigated in a space

$$Y=\{y:[0,T+1]\to\mathbb{R}:y(0)=y(T+1)=0\}$$

with a norm

$$\|y\|=\Big(\sum_{k=1}^{T+1}|\Delta y(k-1)|^2\Big)^{1/2}$$

with which Y becomes a Hilbert space. For $y\in Y$, let

$$y_+=\max\{y,0\},\quad y_-=\max\{-y,0\}.$$

Note that $y_+\geq 0$, $y_-\geq 0$, $y=y_+-y_-$, and $y_+\cdot y_-=0$.

In order to demonstrate that problem (1.1) has at least two positive solutions we assume additionally the condition

(C2)

$$T^{\frac{p^+-2}{2}}\Big(\frac{1}{\sqrt{T+1}}\Big)^{p^+}>\sum_{k=1}^{T}(\varphi_2(k)+\psi_2(k)).$$

Example 1.2. We show that the function defined in Example 1.1 satisfies condition (C2), by taking $p^+=18$ and $T=200$:

$$T^{\frac{p^+-2}{2}}\Big(\frac{1}{\sqrt{T+1}}\Big)^{p^+}=0.009>0.002=\sum_{k=1}^{T}(\varphi_2(k)+\psi_2(k)).$$

Theorem 1.3. *Suppose that assumptions* (C1), (C2) *hold. Then* (1.1) *has at least two positive solutions.*

Discrete boundary-value problems received some attention lately. Let us mention, far from being exhaustive, the following recent papers on discrete BVPs investigated via critical point theory, [1, 3, 4, 11, 14, 15, 18, 19, 20]. The tools employed cover the Morse theory, mountain pass methodology, linking arguments; i.e. methods usually applied in continuous problems.

Continuous versions of problems such as (1.1) are known to be mathematical models of various phenomena arising in the study of elastic mechanics (see [17]), electrorheological fluids (see [13]) or image restoration (see [5]). Variational continuous anisotropic problems have been started by Fan and Zhang in [7] and later considered by many methods and authors (see [9] for an extensive survey of such boundary value problems). The research concerning the discrete anisotropic problems of type (1.1) have only been started (see [10], [12] where known tools from the critical point theory are applied in order to get the existence of solutions).

When compared with [16] we see that our problem is more general since we consider variable exponent case instead of a constant one. While we do not include term depending on $\Phi_{p^-}(y)=|y|^{p^--2}y$ in the nonlinear part as is the case in [16], it is apparent that our results would also hold should we have made our nonlinearity more complicated. We note that term $\Phi_{p^-}(y)=|y|^{p^--2}y$ does not influence the growth of the nonlinearity.

2. Auxiliary results

We connect positive solutions to (1.1) with critical points of suitably chosen action functional. Let

$$F(k,y)=\int_0^y f(k,s)ds \quad \text{for } y\in\mathbb{R} \text{ and } k\in[1,T].$$

Let us define a functional $J:Y\to R$ by

$$J(y)=\sum_{k=1}^{T+1}\frac{1}{p(k-1)}|\Delta y(k-1)|^{p(k-1)}-\sum_{k=1}^{T}F(k,y_+(k)).$$

Functional J is slightly different from functionals applied in investigating the existence of positive solutions, compare with [15]. Thus we indicate its properties. The functional J is continuously Gâteaux differentiable and its derivative at y is

$$\begin{aligned}\langle J'(y),v\rangle=&\sum_{k=1}^{T+1}|\Delta y(k-1)|^{p(k-1)-2}\Delta y(k-1)\Delta v(k-1)\\&-\sum_{k=1}^{T}f(k,y_+(k))v(k)\end{aligned}\tag{2.1}$$

for all $v\in Y$. Suppose that y is a critical point to J; i.e., $\langle J'(y),v\rangle=0$ for all $v\in Y$. Summing by parts and taking boundary values into account, see [8], we observe that

$$0=-\sum_{k=1}^{T+1}\Delta(|\Delta y(k-1)|^{p(k-1)-2}\Delta y(k-1))v(k)-\sum_{k=1}^{T}f(k,y_+(k))v(k).$$

Since $v\in Y$ is arbitrary, we see that y satisfies (1.1).

Now, we recall some auxiliary material which we use later: For (A1)-(A3) see [12], for (A4)-(A5) see [8], for (A6) see [15].

(A1) For every $y\in Y$ with $\|y\|>1$, we have

$$\sum_{k=1}^{T+1}|\Delta y(k-1)|^{p(k-1)}\ge T^{\frac{2-p^-}{2}}\|y\|^{p^-}-T.$$

(A2) For every $y\in Y$ with $\|y\|\le 1$, we have

$$\sum_{k=1}^{T+1}|\Delta y(k-1)|^{p(k-1)}\ge T^{\frac{p^+-2}{2}}\|y\|^{p^+}.$$

(A3) For every $y\in Y$ and any $m\ge 2$, we have

$$(T+1)^{\frac{2-m}{2}}\|y\|^m\le\sum_{k=1}^{T+1}|\Delta y(k-1)|^m\le(T+1)\|y\|^m.$$

(A4) If $p^+\ge 2$, there exists $C_{p^+}>0$ such that for every $y\in Y$,

$$\sum_{k=1}^{T+1}|\Delta y(k-1)|^{p(k-1)}\le 2^{p^+}(T+1)(C_{p^+}\|y\|^{p^+}+1).$$

(A5) For every $y \in Y$ and any $m \geq 2$, we have
$$\sum_{k=1}^{T+1} |\Delta y(k-1)|^m \leq 2^m \sum_{k=1}^{T} |y(k)|^m.$$
(A6) For every $y \in Y$ and any $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have
$$\|y\|_C = \max_{k\in[1,T]} |y(k)| \leq (T+1)^{\frac{1}{q}} (\sum_{k=1}^{T+1} |\Delta y(k-1)|^p)^{1/p}.$$

Let E be a real Banach space. We say that a functional $J : E \to \mathbb{R}$ satisfies Palais-Smale condition if every sequence (y_n) such that $\{J(y_n)\}$ is bounded and $J'(y_n) \to 0$, has a convergent subsequence.

Lemma 2.1 ([6])**.** *Let E be a Banach space and $J \in C^1(E, \mathbb{R})$ satisfy Palais-Smale condition. Assume that there exist $x_0, x_1 \in E$ and a bounded open neighborhood Ω of x_0 such that $x_1 \notin \overline{\Omega}$ and*
$$\max\{J(x_0), J(x_1)\} < \inf_{x\in\partial\Omega} J(x).$$
Let
$$\Gamma = \{h \in C([0,1], E) : h(0) = x_0, h(1) = x_1\},$$
$$c = \inf_{h\in\Gamma} \max_{s\in[0,1]} J(h(s)).$$
Then c is a critical value of J; that is, there exists $x^\star \in E$ such that $J'(x^\star) = 0$ and $J(x^\star) = c$, where $c > \max\{J(x_0), J(x_1)\}$.

Finally we recall the Karush-Kuhn-Tucker theorem with Slater qualification conditions (for one constraint), see [2].

Theorem 2.2. *Let X be a finite-dimensional Euclidean space, $\eta, \mu : X \to \mathbb{R}$ be differentiable functions, with μ convex and $\inf_X \mu < 0$, and $S = \{x \in X : \mu(x) \leq 0\}$. Moreover, let $\overline{x} \in S$ be such that $\eta(\overline{x}) = \inf_S \eta$. Then, there exists $\sigma \geq 0$ such that*
$$\eta'(\overline{x}) + \sigma\mu'(\overline{x}) = 0 \quad \text{and} \quad \sigma\mu(\overline{x}) = 0.$$

We will provide now some results which are used in the proof of the Main Theorem. The following lemma may be viewed as a kind of a discrete maximum principle.

Lemma 2.3. *Assume that $y \in Y$ is a solution of the equation*
$$\begin{gathered}\Delta(|\Delta y(k-1)|^{p(k-1)-2}\Delta y(k-1)) + f(k, y_+(k)) = 0, k \in [1,T],\\ y(0) = y(T+1) = 0,\end{gathered} \tag{2.2}$$
then $y(k) > 0$ for all $k \in [1,T]$ and moreover y is a solution of (1.1).

Proof. We will show that
$$\Delta y(k-1)\Delta y_-(k-1) \leq 0 \quad \text{for every } k \in [1, T+1].$$
Indeed,
$$\begin{aligned}&\Delta y(k-1)\Delta y_-(k-1)\\ &= (y(k) - y(k-1))(y_-(k) - y_-(k-1))\\ &= [(y_+(k) - y_+(k-1)) - (y_-(k) - y_-(k-1))](y_-(k) - y_-(k-1))\\ &= (y_+(k) - y_+(k-1))(y_-(k) - y_-(k-1)) - (y_-(k) - y_-(k-1))^2\end{aligned}$$

$$\begin{aligned}
&= y_+(k)y_-(k) - y_+(k)y_-(k-1) - y_+(k-1)y_-(k) \\
&\quad + y_+(k-1)y_-(k-1) - (y_-(k) - y_-(k-1))^2 \\
&= -[y_+(k)y_-(k-1) + y_+(k-1)y_-(k) + (y_-(k) - y_-(k-1))^2] \le 0.
\end{aligned}$$

Assume that $y \in Y$ is a solution of (2.2). Taking $v = y_-$ in (2.1) we obtain

$$\sum_{k=1}^{T+1} |\Delta y(k-1)|^{p(k-1)-2}\Delta y(k-1)\Delta y_-(k-1) = \sum_{k=1}^{T} f(k, y_+(k))y_-(k).$$

Since the term on the left is non-positive and the one on the right is non-negative, so this equation holds true if the both terms are equal zero, which leads to $y_-(k) = 0$ for all $k \in [1, T]$. Then $y = y_+$. Therefore, y is a positive solution of (1.1). Arguing by contradiction, assume that there exists $k \in [1, T]$ such that $y(k) = 0$, while we can assume $y(k-1) > 0$. Then, by (2.2) we have

$$|y(k+1)|^{p(k)-2}y(k+1) = -y(k-1)^{p(k-1)-1} - f(k, 0) < 0,$$

which implies $y(k+1) < 0$, a contradiction. So $y(k) > 0$ for all $k \in [1, T]$. $\square$

Finally we prove that J satisfies Palais-Smale condition.

Lemma 2.4. *Assume that **(C1)** holds. Then the functional J satisfies Palais-Smale condition.*

Proof. Assume that $\{y_n\}$ is such that $\{J(y_n)\}$ is bounded and $J'(y_n) \to 0$. Since Y is finitely dimensional, it is sufficient to show that $\{y_n\}$ is bounded. Note that

$$\Delta y_+(k)\Delta y_-(k) \le 0 \quad \text{for every } k \in [0, T].$$

Using the above inequality we obtain

$$\begin{aligned}
&-\sum_{k=1}^{T+1} |\Delta y(k-1)|^{p(k-1)-2}\Delta y(k-1)\Delta y_-(k-1) \\
&= -\sum_{k=1}^{T+1} |\Delta y(k-1)|^{p(k-1)-2}\Delta(y_+(k-1) - y_-(k-1))\Delta y_-(k-1) \\
&= -\sum_{k=1}^{T+1} |\Delta y(k-1)|^{p(k-1)-2}\Delta y_+(k-1)\Delta y_-(k-1) \\
&\quad + \sum_{k=1}^{T+1} |\Delta y(k-1)|^{p(k-1)-2}\Delta y_-(k-1)\Delta y_-(k-1) \\
&\ge \sum_{k=1}^{T+1} |\Delta y(k-1)|^{p(k-1)-2}(\Delta y_-(k-1))^2 \\
&\ge \sum_{k=1}^{T+1} |\Delta y_-(k-1)|^{p(k-1)}.
\end{aligned} \tag{2.3}$$

Since $y_n = (y_n)_+ - (y_n)_-$, we will show that $\{(y_n)_-\}$ and $\{(y_n)_+\}$ are bounded. Suppose that $\{(y_n)_-\}$ is unbounded. Then we may assume that there exists $N_0 > 0$ such that for $n \ge N_0$ we have $\|(y_n)_-\| \ge T \ge 2$. Using (2.3) we obtain

$$\langle J'(y_n), (y_n)_-\rangle$$

$$
\begin{aligned}
&= \sum_{k=1}^{T+1} |\Delta y_n(k-1)|^{p(k-1)-2} \Delta y_n(k-1) \Delta (y_n)_-(k-1) \\
&\quad - \sum_{k=1}^{T} f(k,(y_n)_+(k))(y_n)_-(k) \\
&\le - \sum_{k=1}^{T+1} |\Delta (y_n)_-(k-1)|^{p(k-1)}.
\end{aligned}
$$

So by (A1) we obtain

$$
\begin{aligned}
T^{\frac{2-p^-}{2}} \|(y_n)_-\|^{p^-} - T &\le \sum_{k=1}^{T+1} |\Delta (y_n)_-(k-1)|^{p(k-1)} \\
&\le \langle J'(y_n), -(y_n)_- \rangle \le \|J'(y_n)\| \, \|(y_n)_-\|.
\end{aligned}
$$

Next, we see that

$$
\begin{aligned}
T^{\frac{2-p^-}{2}} \|(y_n)_-\|^{p^-} &\le \|J'(y_n)\| \, \|(y_n)_-\| + T \\
&\le \|J'(y_n)\| \, \|(y_n)_-\| + \|(y_n)_-\| \\
&\le (\|J'(y_n)\| + 1) \|(y_n)_-\|
\end{aligned}
$$

and

$$
T^{\frac{2-p^-}{2}} \|(y_n)_-\|^{p^- - 1} \le (\|J'(y_n)\| + 1).
$$

Since, for a fixed $\varepsilon > 0$, there exists some $N_1 \ge N_0$ such that $\|J'(y_n)\| < \varepsilon$ for every $n \ge N_1$, we obtain

$$
\|(y_n)_-\|^{p^- - 1} \le \frac{(\varepsilon + 1)}{T^{\frac{2-p^-}{2}}}.
$$

This means that $\{(y_n)_-\}$ is bounded.

Now, we will show that $\{(y_n)_+\}$ is bounded. Suppose that $\{(y_n)_+\}$ is unbounded. We may assume that $\|(y_n)_+\| \to \infty$. Since

$$
f(k,y) \ge \varphi_1(k)|y|^{m-2}y + \psi_1(k) \quad \text{for all } k \in [1,T],
$$

it follows that

$$
F(k,y) \ge \frac{\varphi_1(k)}{m}|y|^m + \psi_1(k)y.
$$

Thus by (A3) and (A5), we obtain

$$
\sum_{k=1}^{T} F(k,(y_n)_+(k)) \ge \frac{\varphi_1^-}{m} \sum_{k=1}^{T} |(y_n)_+(k)|^m \ge \frac{\varphi_1^-}{m} 2^{-m} (T+1)^{\frac{2-m}{2}} \|(y_n)_+\|^m,
$$

where $\varphi_1^- = \min_{k \in [1,T]} \varphi_1(k)$. Therefore by (A4), we have

$$
\begin{aligned}
J(y_n) &= \sum_{k=1}^{T+1} [\frac{1}{p(k-1)} |\Delta y_n(k-1)|^{p(k-1)} - F(k,(y_n)_+(k))] \\
&\le 2^{p^+}(T+1)(C_{p^+}\|(y_n)_+ - (y_n)_-\|^{p^+} + 1) - \frac{\varphi_1^-}{m} 2^{-m}(T+1)^{\frac{2-m}{2}} \|(y_n)_+\|^m \\
&\le 2^{p^+}(T+1)(C_{p^+}2^{p^+-1}(\|(y_n)_+\|^{p^+} + \|(y_n)_-\|^{p^+}) + 1) \\
&\quad - \frac{\varphi_1^-}{m} 2^{-m}(T+1)^{\frac{2-m}{2}} \|(y_n)_+\|^m.
\end{aligned}
$$

Since $p^+ < m$ and $\{(y_n)_+\}$ is unbounded and $\{(y_n)_-\}$ is bounded, so $J(y_n) \to -\infty$. Thus we obtain a contradiction with the assumption $\{J(y_n)\}$ is bounded, so $\{(y_n)_+\}$ is bounded. It follows that $\{y_n\}$ is bounded. □

3. Proof of the main result

In this section we present the proof of Theorem 1.3.

Proof. Assume that $y_0 \in Y$ is a local minimizer of J in

$$B := \{y \in Y : \mu(y) \le 0\},$$

where $\mu(y) = \frac{\|y\|^2}{2} - \frac{1}{2(T+1)}$. Note that for $y \in B$ by (A6) it follows that for all $k \in [1, T]$,

$$|y(k)| \le \max_{s\in[1,T]} |y(s)| \le \sqrt{T+1}\|y\| \le \frac{1}{\sqrt{T+1}}\sqrt{T+1} = 1.$$

We prove that $y_0 \in IntB$, by contradiction. Thus suppose otherwise; i.e., we suppose that $y_0 \in \partial B$. Then by Theorem 2.2 there exists $\sigma \ge 0$ such that for all $v \in Y$

$$\langle J'(y_0), v\rangle + \sigma\langle y_0, v\rangle = 0.$$

Hence

$$\begin{aligned} &\sum_{k=1}^{T+1} |\Delta y_0(k-1)|^{p(k-1)-2}\Delta y_0(k-1)\Delta v(k-1) \\ &- \sum_{k=1}^{T} f(k, (y_0)_+(k))v(k) + \sigma \sum_{k=1}^{T}\langle y_0(k), v(k)\rangle = 0. \end{aligned}$$

Taking $v = y_0$, we see that

$$\sum_{k=1}^{T+1} |\Delta y_0(k-1)|^{p(k-1)} + \sigma\|y_0\|^2 = \sum_{k=1}^{T} f(k, (y_0)_+(k))y_0(k).$$

Since $y_0 \in \partial B$, we see that $\|y_0\| = \frac{1}{\sqrt{T+1}}$. Thus by (A2), we have

$$\sum_{k=1}^{T+1} |\Delta y_0(k-1)|^{p(k-1)} + \sigma\|y_0\|^2 \ge \sum_{k=1}^{T+1} |\Delta y_0(k-1)|^{p(k-1)} \ge T^{\frac{p^+-2}{2}}\Big(\frac{1}{\sqrt{T+1}}\Big)^{p^+}.$$

On the other hand

$$\begin{aligned} &\sum_{k=1}^{T} f(k, (y_0)_+(k))y_0(k) \\ &= \sum_{k=1}^{T} f(k, (y_0)_+(k))(y_0)_+(k) - \sum_{k=1}^{T} f(k, (y_0)_+(k))(y_0)_-(k) \\ &\le \sum_{k=1}^{T} \varphi_2(k)|(y_0)_+(k)|^m + \sum_{k=1}^{T} \psi_2(k)|(y_0)_+(k)| \\ &\le \sum_{k=1}^{T} \varphi_2(k) + \sum_{k=1}^{T} \psi_2(k). \end{aligned}$$

Thus,

$$T^{\frac{p^+-2}{2}}(\frac{1}{\sqrt{T+1}})^{p^+} \leq \sum_{k=1}^{T}(\varphi_2(k)+\psi_2(k)).$$

A contradiction with (C2). Hence $y_0 \in IntB$ and y_0 is a local minimizer of J. Thus $J(y_0) < \min_{y\in\partial B} J(y)$. We will show that there exists y_1 such that $y_1 \in Y \setminus B$ and $J(y_1) < \min_{y\in\partial B} J(y)$. Let $y_\lambda \in Y$ be define as follows: $y_\lambda(k) = \lambda$ for $k = 1,\dots,T$ and $y_\lambda(0) = y_\lambda(T+1) = 0$. Then for $\lambda > 1$ we have

$$J(y_\lambda) \leq \frac{\lambda^{p(0)}}{p(0)} + \frac{\lambda^{p(T)}}{p(T)} - \sum_{k=1}^{T}\frac{\varphi_1(k)\lambda^m}{m} \leq \frac{\lambda^{p^+}}{p(0)} + \frac{\lambda^{p^+}}{p(T)} - \frac{\varphi_1^-\lambda^m}{m}T - \psi_1^-\lambda T.$$

Since $m > p^+$, then $\lim_{\lambda\to\infty} J(y_\lambda) = -\infty$. Thus there exists λ_0 with $J(y_{\lambda_0}) < \min_{y\in\partial B} J(y)$. By Lemma 2.1 and Lemma 2.4 we obtain a critical value of the functional J for some $y^\star \in Y \setminus \partial B$. Then y_0 and $y^\star$ are two different critical points of J and therefore by Lemma 2.3 these are positive solutions of problem (1.1). □

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