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CUBIC SYSTEMS WITH INVARIANT AFFINE STRAIGHT LINES OF TOTAL PARALLEL MULTIPLICITY SEVEN

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Abstract. In this article, we study the planar cubic differential systems with invariant affine straight lines of total parallel multiplicity seven. We classify these system according to their geometric properties encoded in the configurations of invariant straight lines. We show that there are only 18 different topological phase portraits in the Poincaré disc associated to this family of cubic systems up to a reversal of the sense of their orbits, and we provide representatives of every class modulo an affine change of variables and rescaling of the time variable.

1. Introduction and statement of main results

We consider the real polynomial system of differential equations

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y), \quad \gcd(P,Q) = 1 \tag{1.1}$$

and the vector field $\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$ associated with system (1.1).

Denote $n = \max\{\deg(P), \deg(Q)\}$. If $n = 3$ then system (1.1) is called cubic.

A differentiable function $f : D \subset \mathbb{C}^2 \to \mathbb{C}$, f not constant is said to be an *elementary invariant* (or a *Darboux invariant*) for the vector field $\mathbb{X}$ if there exists a polynomial $K_f \in \mathbb{C}[x,y]$ with $\deg(K_f) \le n-1$ such that the identity

$$\mathbb{X}(f) \equiv f(x,y)K_f(x,y), \quad (x,y) \in D$$

holds. Denote by $I_{\mathbb{X}}$ the set of all elementary invariants of $\mathbb{X}$; $I_a = \{f \in \mathbb{C}[x,y] : f \in I_{\mathbb{X}}\}$, $I_e = \{\exp(\frac{g}{h}) : g,h \in \mathbb{C}[x,y], \gcd(g,h) = 1, \exp(\frac{g}{h}) \in I_{\mathbb{X}}\}$.

If $f \in I_a$ (respectively $f \in I_e$), then $f(x,y) = 0$; i.e., the set $\{(x,y) \in \mathbb{C} : f(x,y) = 0\}$, (respectively f) is called *an invariant algebraic curve* (respectively *an invariant exponential function*) for polynomial system (1.1). In the case $f \in I_a$, $\deg(f) = 1$; i.e., $f = ax + by + c$, $a,b,c \in \mathbb{C}$, $(a,b) \ne (0,0)$, we say that $f = 0$ (in brief f) is an *invariant straight line* for (1.1). Moreover, if m is the greatest positive integer such that f^m divides $X(f)$, then we will say that the invariant straight line f has the *parallel multiplicity* equal to m. If $f \in I_a$ has the parallel multiplicity equal to $m \ge 2$, then $exp(1/f), \dots, \exp(1/f^{m-1}) \in I_e$.

If the straight line $ax + by + c = 0$, $a,b,c \in \mathbb{C}$ passes through at least two distinct points with real coordinates, then the complex line $\{(x,y) \in \mathbb{C}^2 : ax + by + c = 0\}$

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contains a real line $\{(x,y) \in \mathbb{R}^2 : a'x + b'y + c' = 0\}$ with $a', b', c' \in \mathbb{R}$, which is the real line passing through these two real points. In this case the complex line could be written as $ax + by + c = \lambda(a'x + b'y + c') = 0$ with $\lambda \in \mathbb{C} \setminus \mathbb{R}$. We call *an essentially complex line*, a line which could not be written in this way. In what follows by complex line we shall mean essentially complex line.

System (1.1) is called *Darboux integrable* if there exists a non-constant function of the form $f = f_1^{\lambda_1} \dots f_s^{\lambda_s}$, where $f_j \in I_a \cup I_e$ and $\lambda_j \in \mathbb{C}$, $j = \overline{1,s}$, such that either f is a first integral or f is an integrating factor for (1.1) (about the theory of Darboux, presented in the context of planar polynomial differential systems on the affine plane, see [11]).

A great number of works are dedicated to the investigation of polynomial differential systems with invariant straight lines (see, for example [1]–[10], [12]–[18]). In particular we point out the following facts:

(1) The maximum number of invariant affine straight lines of cubic differential systems is 8 [1].

(2) The class of cubic systems possessing invariant straight lines of total multiplicity 9, including the line at infinity was completely investigated in [8].

In this article we proceed to the next step, namely to consider cubic systems with invariant affine straight lines of total parallel multiplicity 7. This is a continuation of the qualitative investigation started in [18]. Our main result is as follows:

Theorem 1.1. *Assume that a cubic system possesses invariant affine straight lines of total parallel multiplicity seven. Then all such systems are integrable and we give below their integrating factors as well as their first integrals. We give below normal forms modulo the action of affine transformations and time rescaling of such systems: normal forms $(I.1)-(I.17)$. Moreover in Fig. 1.1 - Fig. 1.17. we give the 18 topologically distinct phase portraits on the Poincaré disc of these systems. In the table below for each one of the systems (I.1)–(I.17) the first arrow points to the straight lines, the integrating factor and the first integral that corresponds to each system.*

$$\text{(I.1)} \quad \begin{cases} \dot{x} = x(x+1)(x-a),\ a > 0, \\ \dot{y} = y(y+1)(y-a),\ a \neq 1, \\ \text{configuration } (3r, 3r, 1r); \end{cases} \quad \to \quad (1.2) \quad \to \quad \text{Fig. 1.1;}$$

$$\text{(1.2)} \quad \begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = y^2(y+1), \\ \text{configuration } (3(2)r, 3(2)r, 1r); \end{cases} \quad \to \quad (1.3) \quad \to \quad \text{Fig. 1.2;}$$

$$\text{(I.3)} \quad \begin{cases} \dot{x} = x((x-a)^2+1), \\ \dot{y} = y((y-a)^2+1),\ a \neq 0, \\ \text{configuration } (1r + 2c_0, 1r + 2c_0, 1r); \end{cases} \quad \to \quad (1.4) \quad \to \quad \text{Fig. 1.3;}$$

$$\text{(I.4)} \quad \begin{cases} \dot{x} = x(-a + 2(a+1)y + x^2 - 3y^2), \\ \dot{y} = -ay - (a+1)(x^2 - y^2) + 3x^2 y - y^3, \\ \quad a \in (0;1),\ a \neq 1/2, \\ \text{configuration } (3c_1, 3c_1, 1r); \end{cases} \quad \to \quad (1.5) \quad \to \quad \text{Fig. 1.4;}$$

$$\text{(I.5)} \quad \begin{cases} \dot{x} = x(1 + 2ay - x^2 + 3y^2), \quad a > 0, \\ \dot{y} = a + y - ax^2 + ay^2 - 3x^2 y + y^3, \\ \text{configuration } (3c_1, 3c_1, 1r); \end{cases} \quad \to \quad (1.6) \quad \to \quad \text{Fig. 1.5;}$$

$$\text{(I.6)} \quad \begin{cases} \dot{x} = x(x^2 + 2y - 3y^2), \\ \dot{y} = -x^2 + y^2 + 3x^2y - y^3, \\ \text{configuration } (3(2)c_1, 3(2)c_1, 1r); \end{cases} \quad \rightarrow \quad (1.7) \quad \rightarrow \quad \text{Fig. 1.6;}$$

$$\text{(I.7)} \quad \begin{cases} \dot{x} = x(x+1)(x-a),\ a > 0,\ a \neq 1, \\ \dot{y} = y(y+1)((1-a)x + ay - a), \\ \text{configuration } (3r, 2r, 1r, 1r); \end{cases} \quad \rightarrow \quad (1.8) \quad \rightarrow \quad \text{Fig. 1.7;}$$

$$\text{(I.8)} \quad \begin{cases} \dot{x} = x(x+1)(x-a),\ a > 0,\ a \neq 1, \\ \dot{y} = y(y+1)(-a + (2+a)x - (1+a)y), \\ \text{configuration } (3r, 2r, 1r, 1r); \end{cases} \quad \rightarrow \quad (1.9) \quad \rightarrow \quad \text{Fig. 1.8;}$$

$$\text{(I.9)} \quad \begin{cases} \dot{x} = x^3, \\ \dot{y} = y^2(ax + y - ay), \\ a \in \mathbb{R} \setminus \{0; 1; 3/2; 2; 3\}, \\ \text{configuration } (3(3)r, 2(2)r, 1r, 1r); \end{cases} \quad \rightarrow \quad (1.10) \quad \rightarrow \quad \text{Fig. 1.9a, 1.9b;}$$

$$\text{(I.10)} \quad \begin{cases} \dot{x} = x^3, \\ \dot{y} = y^2(2ax - y),\ a \in (-1, 0) \cup (0, 1), \\ \text{configuration } (3(3)r, 2(2)r, 1c_1, 1c_1); \end{cases} \quad \rightarrow \quad (1.11) \quad \rightarrow \quad \text{Fig. 1.10;}$$

$$\text{(I.11)} \quad \begin{cases} \dot{x} = (x-a)(x^2+1), \quad a > 0, \\ \dot{y} = y(1+y)(2ax - (a^2+1)y - a^2 + 1), \\ \text{configuration } (1r + 2c_0, 2r, 1c_1, 1c_1); \end{cases} \quad \rightarrow \quad (1.12) \quad \rightarrow \quad \text{Fig. 1.11;}$$

$$\text{(I.12)} \quad \begin{cases} \dot{x} = x(1+x)(-1 + ax - (2+a)y), \\ \dot{y} = y(1+y)(-a - (1+2a)x + y), \\ a > 0,\ a \neq 1, \\ \text{configuration } (2r, 2r, 2r, 1r); \end{cases} \quad \rightarrow \quad (1.13) \quad \rightarrow \quad \text{Fig. 1.12;}$$

$$\text{(I.13)} \quad \begin{cases} \dot{x} = x^2(ax + y), \\ \dot{y} = y^2((2+3a)x - (1+2a)y), \\ \quad a(a+1)(3a+2)(3a+1)(2a+1) \neq 0, \\ \text{configuration } (2(2)r, 2(2)r, 2(2)r, 1r); \end{cases} \quad \rightarrow \quad (1.14) \quad \rightarrow \quad \text{Fig. 1.13;}$$

$$\text{(I.14)} \quad \begin{cases} \dot{x} = x(x+1)(1 + a^2 + 2x - 2ay), \\ \dot{y} = (1+a^2)y + (3+a^2)xy - 2ay^2 \\ \quad +ax^3 + 3x^2y - axy^2 + y^3, \quad a \neq 0, \\ \text{configuration } (2r, 2c_1, 2c_1, 1r); \end{cases} \quad \rightarrow \quad (1.15) \quad \rightarrow \quad \text{Fig. 1.14;}$$

$$\text{(I.15)} \quad \begin{cases} \dot{x} = 2x^2(x + ay), \quad a > 0, \\ \dot{y} = -ax^3 + 3x^2y + axy^2 + y^3, \\ \text{configuration } (2(2)r, 2(2)c_1, 2(2)c_1, 1r); \end{cases} \quad \rightarrow \quad (1.16) \quad \rightarrow \quad \text{Fig. 1.15;}$$

$$\text{(I.16)} \quad \begin{cases} \dot{x} = (x^2+1)(ax - 2y + ay), \\ \dot{y} = (y^2+1)(-x + 2ax - y), \\ a(2a-1)(a-1)(a-2) \neq 0, \\ \text{configuration } (2c_0, 2c_0, 2c_0, 1r); \end{cases} \quad \rightarrow \quad (1.17) \quad \rightarrow \quad \text{Fig. 1.16;}$$

$$\text{(I.17)} \quad \begin{cases} \dot{x} = x(1 - (1+a^2)x^2 + 4axy - 3y^2), \\ \dot{y} = 2(ax - y)(1 + y^2), \quad a > 0, \\ \text{configuration } (2c_0, 2c_1, 2c_1, 1r). \end{cases} \quad \rightarrow \quad (1.18) \quad \rightarrow \quad \text{Fig. 1.17.}$$

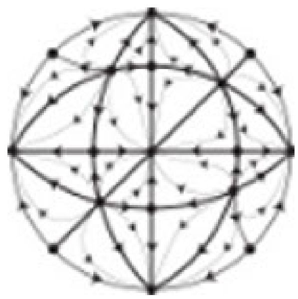
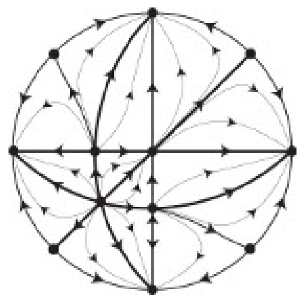


Figure 1.1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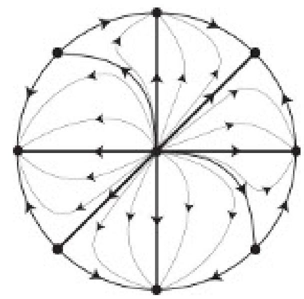
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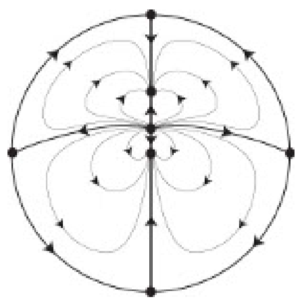
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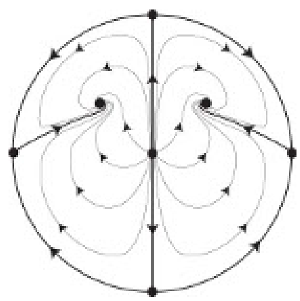
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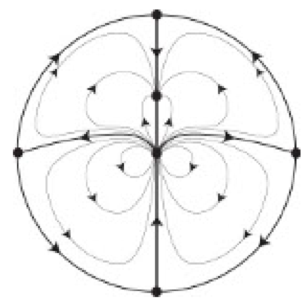
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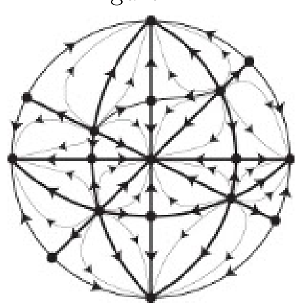
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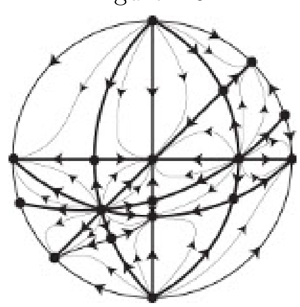
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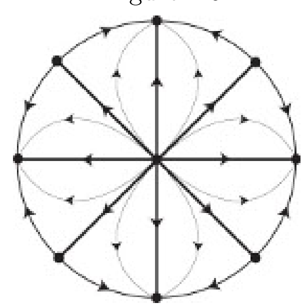
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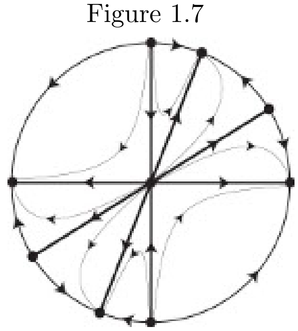
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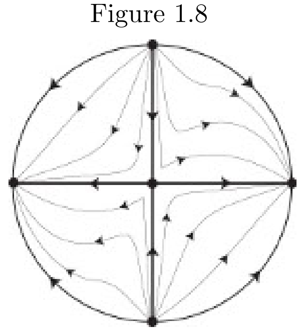
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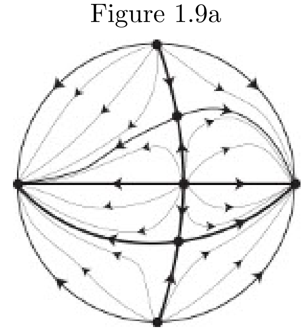
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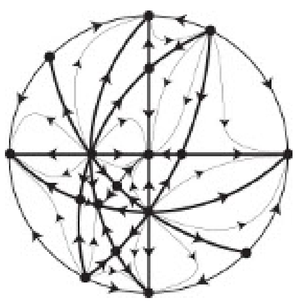

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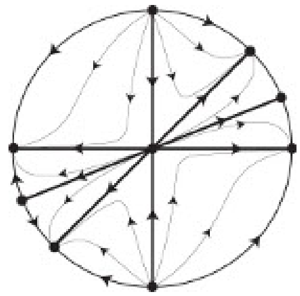

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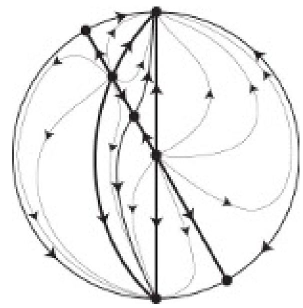

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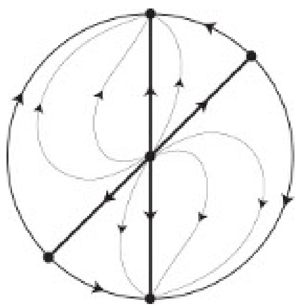

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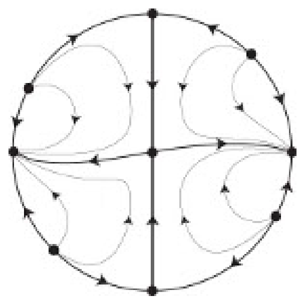

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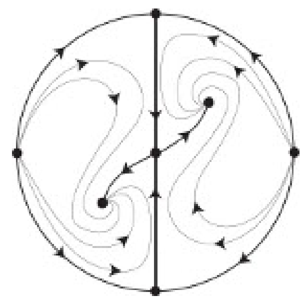

Figure 1.17

The systems (I.1)-(I.17) have the following straight lines, Darboux integrating factor μ and elementary first integral $\mathcal{F}$, respectively, (see [11])

$$\begin{gathered} l_1 = x, \quad l_2 = x+1, \quad l_3 = x-a, \quad l_4 = y, \quad l_5 = y+1, \quad l_6 = y-a, \\ l_7 = y-x; \quad \mu = 1/(l_1l_2l_3l_4l_5l_6), \quad \mathcal{F} \equiv (\frac{x}{y})^{a+1}(\frac{y+1}{x+1})^a\frac{y-a}{x-a}; \end{gathered} \tag{1.2}$$

$$\begin{gathered} l_{1,2} = x, \quad l_3 = x+1, \quad l_{4,5} = y, \quad l_6 = y+1, \quad l_7 = y-x; \\ \mu = 1/(l_1^2l_3l_4^2l_6), \quad \mathcal{F} \equiv x^{-1}e^{-1/x}(x+1)ye^{1/y}(y+1)^{-1} = \text{const}; \end{gathered} \tag{1.3}$$

$$\begin{gathered} l_1 = x, \, l_{2,3} = x-a \mp i, \quad l_4 = y, \quad l_{5,6} = y-a \mp i, \quad l_7 = y-x; \\ \mu = \frac{1}{l_1l_2l_3l_4l_5l_6}, \quad \mathcal{F} = \frac{l_2l_3l_4^2}{l_1^2l_5l_6}\exp(-2a\arctan\frac{l_7}{-1-a^2+ax+ay-xy}); \end{gathered} \tag{1.4}$$

$$\begin{gathered} l_1 = y-ix, \quad l_2 = y-ix-1, \quad l_3 = y-ix-a, \quad l_4 = y+ix, \quad l_5 = y+ix-1, \\ l_6 = y+ix-a, \quad l_7 = x; \quad \mu = 1/(l_1l_2l_3l_4l_5l_6), \\ \mathcal{F} = \arctan(ax/(x^2-ay+y^2)) - a\arctan(x/(x^2-y+y^2)); \end{gathered} \tag{1.5}$$

$$\begin{gathered} l_1 = y-ix+i, \quad l_2 = y+ix-i, \quad l_3 = y-ix-i, \quad l_4 = y+ix+i, \\ l_5 = y-ix+a, \quad l_6 = y+ix+a, \quad l_7 = x; \quad \mu = 1/(l_1l_2l_3l_4l_5l_6), \\ \mathcal{F} = (\frac{l_3l_4}{l_1l_2})^a\exp(4\arctan\frac{x}{a+y} - 2\arctan\frac{2xy}{1-x^2+y^2}); \end{gathered} \tag{1.6}$$

$$\begin{gathered} l_1 = l_2 = y-ix, \quad l_3 = y-ix-1, \quad l_4 = l_5 = y+ix, \\ l_6 = y+ix-1, \quad l_7 = x; \quad \mu = 1/(l_1^2l_3l_4^2l_6), \\ \mathcal{F} = ((l_1l_4-l_7-y)(l_1l_4+l_7-y)\cos\frac{2l_7}{l_1l_4} + 2l_7(l_1l_4-y)\sin\frac{2l_7}{l_1l_4})/(l_1l_3l_4l_6); \end{gathered} \tag{1.7}$$

$$\begin{gathered} l_1 = x, \quad l_2 = x+1, \quad l_3 = x-a, \quad l_4 = y, \quad l_5 = y+1, \quad l_6 = y-x, \\ l_7 = x+ay; \quad \mu = l_1/(l_2 l_3 l_4 l_6 l_7), \quad \mathcal{F} = l_2 l_3^a l_4^{a+1} l_6^{-1} l_7^{-a}; \end{gathered} \tag{1.8}$$

$$\begin{gathered} l_1 = x, \quad l_2 = x+1, \quad l_3 = x-a, \quad l_4 = y, \quad l_5 = y+1, \quad l_6 = y-x, \\ l_7 = x-(a+1)y-a; \quad \mu = l_2/(l_1 l_3 l_5 l_6 l_7), \quad \mathcal{F} = l_1 l_3^{-a-1} l_5^{-a} l_6^{-1} l_7^{a+1}; \end{gathered} \tag{1.9}$$

$$\begin{gathered} l_{1,2,3} = x, \quad l_{4,5} = y, \quad l_6 = y-x, \quad l_7 = x+y-ay; \\ \mu = 1/(l_1 l_4 l_6 l_7), \quad \mathcal{F} = (l_1 l_4)^{a-2} l_6 l_7^{1-a}; \end{gathered} \tag{1.10}$$

$$\begin{gathered} l_{1,2,3} = x, \quad l_{4,5} = y, \quad l_{6,7} = y-(a \pm i\sqrt{1-a^2})x; \quad \mu = 1/(l_1 l_4 l_6 l_7), \\ \mathcal{F} = (xy)^{2\sqrt{1-a^2}}(((1-a^2)x^2 + (y-ax)^2)^{-\sqrt{1-a^2}}) \exp(-2a \arctan \frac{\sqrt{1-a^2}x}{y-ax}); \end{gathered} \tag{1.11}$$

$$\begin{gathered} l_1 = x-a, \quad l_{2,3} = x \pm i, \quad l_4 = y, \quad l_5 = y+1, \\ l_{6,7} = x-(a \pm i)y - a; \quad \mu = l_1/(l_2 l_3 l_4 l_6 l_7), \\ \mathcal{F} = \frac{y^2(x^2+1)}{y^2+(x-a-ay)^2} \exp(2a(\arctan \frac{1}{x} + \arctan \frac{y}{x-a-ay})); \end{gathered} \tag{1.12}$$

$$\begin{gathered} l_1 = x, \quad l_2 = x+1, \quad l_3 = y, \quad l_4 = y+1, \quad l_5 = ax-y+a, \quad l_6 = ax-y-1, \\ l_7 = x+y+1; \quad \mu = l_7/(l_1 l_2 l_3 l_4 l_5 l_6), \quad \mathcal{F} = (l_1/l_2)^a (l_4/l_3)(l_5/l_6)^{a+1}; \end{gathered} \tag{1.13}$$

$$\begin{gathered} l_{1,2} = x, \quad l_{3,4} = y, \quad l_{5,6} = x-y, \quad l_7 = ax-y-2ay; \\ \mu = (l_1 l_3 l_5)/l_7^5, \quad \mathcal{F} = (l_1 l_3 l_5)/(l_7^2); \end{gathered} \tag{1.14}$$

$$\begin{gathered} l_1 = x, \quad l_2 = x+1, \quad l_{3,5} = y \mp ix, \quad l_{4,6} = y \mp i(x+1) - a, \quad l_7 = y+ax; \\ \mu = l_7/(l_1 l_2 l_3 l_4 l_5 l_6), \quad \mathcal{F} = \frac{l_2^2 l_3 l_5}{l_1^2 l_4 l_6} \exp(2a(\arctan \frac{l_2}{y-a} - \arctan \frac{x}{y})); \end{gathered} \tag{1.15}$$

$$\begin{gathered} l_{1,2} = x, \quad l_{3,4} = y-ix, \quad l_{5,6} = y+ix, \quad l_7 = y-ax; \\ \mu = l_1 l_3 l_5/l_7^5, \quad \mathcal{F} = l_1 l_3 l_5/l_7^2; \end{gathered} \tag{1.16}$$

$$\begin{gathered} l_{1,2} = x \mp i, \, l_{3,4} = y \mp i, \, l_{5,6} = y - a(x \pm i) \pm i, \, l_7 = y-x; \\ \mu = l_7/(l_1 l_2 l_3 l_4 l_5 l_6); \quad \mathcal{F} = a \arctan \frac{l_7}{a l_1 l_2 - 1 - xy} + \arctan \frac{a l_7}{l_3 l_4 - a(1+xy)}; \end{gathered} \tag{1.17}$$

$$\begin{gathered} l_{1,2} = y \mp i, \quad l_{3,4} = y-(a+i)x \mp i, \quad l_{5,6} = y-(a-i)x \pm i, \\ l_7 = x; \quad \mu = l_7/(l_1 l_2 l_3 l_4 l_5 l_6), \\ \mathcal{F} = (\frac{l_4 l_6}{l_3 l_5})^a \exp(2 \arctan \frac{2y-2ax}{x^2-1+(y-a*x)^2} - 4 \arctan \frac{1}{y}). \end{gathered} \tag{1.18}$$

2. Properties of the cubic systems with invariant straight lines

We consider the real cubic differential systems

$$\begin{aligned} \frac{dx}{dt} &= \sum_{r=0}^{3} P_r(x,y) \equiv P(x,y), \\ \frac{dy}{dt} &= \sum_{r=0}^{3} Q_r(x,y) \equiv Q(x,y), \\ &\gcd(P,Q) = 1, \end{aligned} \tag{2.1}$$

where $P_r(x,y)$ and $Q_r(x,y)$ are homogeneous polynomials of degree r and $|P_3(x,y)| + |Q_3(x,y)| \not\equiv 0$.

By a *straight lines parallel configuration of invariant straight lines* of a cubic system we understand the set of all its invariant affine straight lines, each endowed with its own parallel multiplicity.

The goal of this section is to enumerate such properties for invariant straight lines which will allow the construction of configurations of straight lines realizable for (2.1). Some of these properties are obvious or easy to prove and others were proved in [18].

2.1. **Points and straight lines.**

(II.1) In the finite part of the phase plane each system (2.1) has at most nine singular points.

(II.2) In the finite part of the phase plane, on any straight line there are located at most three singular points of the system (2.1).

(II.3) The system (2.1) has no more than eight invariant affine straight lines ([1]).

(II.4) At infinity the system (2.1) has at most four distinct singular points (in the Poincaré compactification [12]) if $yP_3(x,y) - xQ_3(x,y) \not\equiv 0$. In the case $yP_3(x,y) - xQ_3(x,y) \equiv 0$ the infinity is degenerate, i.e. consists only of singular points.

(II.5) If $yP_3(x,y) - xQ_3(x,y) \not\equiv 0$, then the infinity represents for (2.1) a non-singular invariant straight line, i.e. a line that is not filled up with singularities.

(II.6) Through one point cannot pass more than four distinct invariant straight lines of the system (2.1).

We say that the straight lines $l_j \equiv \alpha_j x + \beta_j y + \gamma_j \in \mathbb{C}[x,y]$, $(\alpha_j, \beta_j) \neq (0,0)$, $j = 1,2$, are *parallel* if $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$. Otherwise the straight lines are called *concurrent*. If an invariant affine straight line l has the parallel multiplicity equal to m, then we will consider that we have m parallel invariant straight lines identical with l.

(II.7) The intersection point (x_0, y_0) of two concurrent invariant straight lines l_1 and l_2 of system (2.1) is a singular point for this system. If $l_1, l_2 \in \mathbb{R}[x,y]$ or $l_2 \equiv \bar{l_1}$, i.e. the straight lines l_1 and l_2 are complex conjugate, then $x_0, y_0 \in \mathbb{R}$.

(II.8) A complex straight line l which passes through a point M_0 with real coordinates, could be described by an equation of the form: $y = \alpha x + \beta$, $Im\,\alpha \neq 0$, and M_0 is the intersection point of the straight lines l and $\bar{l}$.

Definition 2.1. A complex straight line whose equation is verified by a point with real coordinates will be called *relatively complex straight line*.

Unlike the complex straight lines, a straight line $ax + by + c = 0$, $a,b,c \in \mathbb{R}$, $a^2 + b^2 \neq 0$, passes through an infinite number of real points and through an

infinite number of points with at least one complex coordinate. Indeed, if $x_0, y_0 \in \mathbb{R}$ and $ax_0 + by_0 + c = 0$, then this straight line passes through complex points $(x_0 + \alpha b, y_0 - \alpha a)$, $\alpha \in \mathbb{C} \setminus \mathbb{R}$.
(II.9) To a straight line $L: \ ax + by + c = 0$, $a, b, c \in \mathbb{C}$ such that L passes through two distinct real points or through two complex conjugate points we can associate a straight line $L: \ a'x + b'y + c' = 0$ with $a', b', c' \in \mathbb{R}$ such that

$$\{(x, y) \in \mathbb{R}^2 : a'x + b'y + c' = 0\} \subset \{(x, y) \in \mathbb{C}^2 : ax + by + c = 0\}.$$

(II.10) The complex conjugate straight lines l and $\bar{l}$ can be invariant lines for system (2.1) only together.
(II.11) The complex conjugate invariant straight lines l and $\bar{l}$ have the same parallel multiplicity.
(II.12) The number of complex singular points of a system (2.1) on an invariant straight line $\{(x, y) \in \mathbb{C}^2 : ax + by + c = 0\}$ where $a, b, c \in \mathbb{R}$ is even and is at most two. In the last case the singular points are complex conjugate.
(II.13) An invariant straight line with real coefficients either intersects none of the complex invariant straight lines of the system (2.1) in complex points, or it intersects exactly two complex conjugate invariant straight lines in complex points.
(II.14) A cubic system with at least seven invariant affine straight lines counted with parallel multiplicity has non-degenerate infinity and, therefore, there exist at most four directions (slopes) for these lines.

2.2. **Parallel invariant straight lines.**

Definition 2.2. An affine straight line not passing through any real finite point will be called *absolutely complex straight line.*

(II.15) A complex invariant straight line ($l \in \mathbb{C}[x, y] \setminus \mathbb{R}[x, y]$) *of the system* (2.1) *is absolutely complex if and only if it is parallel with its conjugate line.*
(II.16) Through a complex point of any complex straight line can pass at most one straight line with real coefficients.
(II.17) Via a non-degenerate linear transformation of the phase plane any absolutely complex straight line can be made parallel to one of the axes of the coordinate system, i.e. it is described by one of the equations $x = \gamma$ or $y = \gamma, \quad \gamma \in \mathbb{C} \setminus \mathbb{R}$. Moreover, if we have two such straight lines l_1 and l_2, $l_1 \nparallel l_2$, $l_1 \parallel \overline{l_1}$, $l_2 \parallel \overline{l_2}$, then by a suitable transformation we can at the same time make the straight line l_1 to be parallel to the coordinate axis Ox, and the straight line l_2 to be parallel to Oy axis.
(II.18) Let l be a relatively complex line. Then neither an absolutely complex line nor a straight line with real coefficients could be parallel to l.
(II.19) If l_1 and l_2 are two distinct parallel invariant affine straight lines of the system (1.1), then either

- (a) $l_1, l_2 \in \mathbb{R}[x, y]$, or
- (b) $l_1 \in \mathbb{R}[x, y]$ and l_2 is absolutely complex, or
- (c) l_1 and l_2 are absolutely complex and $l_2 = \overline{l_1}$, or
- (d) l_1 and l_2 are relatively complex straight lines and $l_2 \neq \overline{l_1}$.

(II.20) The system (2.1) cannot have invariant affine parallel straight lines of total parallel multiplicity greater than 3.

2.3. Multiple invariant straight lines.

Definition 2.3. By a triplet of parallel invariant affine straight lines we shall mean a set of parallel invariant affine straight lines of total parallel multiplicity 3.

(II.21) If the cubic system (2.1) has a triplet of parallel invariant affine straight lines, then all its finite singular points lie on these straight lines.
(II.22) The cubic system (2.1) cannot have more than two triplets of parallel invariant affine straight lines.
(II.23) If l_1, l_2, l_3 form a triplet of parallel invariant affine straight lines of a cubic system (2.1), then either

(a) $l_1, l_2, l_3 \in \mathbb{R}[x, y]$, or
(b) l_1, l_2, l_3 are relatively complex, or
(c) $l_1 \in \mathbb{R}[x, y]$ and $l_{2,3}$ are absolutely complex.

(II.24) The parallel multiplicity of an invariant affine straight line of the cubic system (2.1) is at most three.
(II.25) The parallel multiplicity of any absolutely complex invariant straight line of the cubic system (2.1) is equal to one.
(II.26) If the cubic system (2.1) has two concurrent invariant affine straight lines l_1, l_2 and l_1 has the parallel multiplicity equal to m, $1 \leq m \leq 3$, then this system cannot have more than $3 - m$ singular points on $l_2 \setminus l_1$.

We say that three affine straight lines are in generic position if no pair of the lines could be parallel and no more that two lines could pass through the same point.
(II.27) For the cubic system (2.1) the total parallel multiplicity of three invariant affine straight lines in generic position is at most four.

3. Proof of Theorem 1.1

The classes of cubic systems (2.1) with invariant affine straight lines of total multiplicity seven, where six of them form two triplets of parallel straight lines, i.e. the systems (I.1)–(I.6) of Theorem 1.1, were studied in [18]. In the present paper we will investigate the cubic system with invariant affine straight lines of total multiplicity seven when the system: (A) has exactly one triplet of parallel straight lines and (B) has not triplets of parallel straight lines.

3.1. A. Cases of one triplet of parallel invariant affine straight lines.

We write down the type of a configuration in italic (respectively, bold face; normal form) if this configuration is a subconfiguration (a part) of a configuration with eight invariant straight lines (respectively, unrealizable; realizable). We denote by c_0 (respectively c_1) an absolutely (respectively relatively) complex invariant straight line.

We denote by $(3r, 2r, 2r)$ (see (A1) below) the configuration which consists of seven distinct straight lines with real coefficients $l_1, \dots l_7 \in \mathbb{R}[x; y]$, among which l_1, l_2, l_3 form a triplet of parallel straight lines, i.e. $l_1 \parallel l_2 \parallel l_3$. Moreover the lines $l_{4,5}$ and $l_{6,7}$ form two pairs of parallel straight lines and $l_j \nparallel l_k$, $(j, k) = (1, 4), (1, 6), (4, 6)$.

In the case of configuration $(3(2)r, 2c_1, 2c_1)$ (see (A20) below) we have $l_1 \equiv l_2 \parallel l_3$, $l_1, l_3 \in \mathbb{R}[x, y]$, $l_1 \neq l_3$, the straight lines l_4 and l_5 are relatively complex, $l_4 \parallel l_5$, $l_6 = \overline{l_4}$, $l_7 = \overline{l_5}$ and the slopes of the straight lines l_1, l_4, l_6 are distinct.

The configuration $(1r+2c_0, 2c_0, 1c_1, 1c_1)$ (see below (A54)) consists of a straight line l_1 with real coefficients and distinct complex straight lines $l_2, \dots, l_7$, $l_1 \parallel l_2 \parallel l_3, l_4 \parallel l_5, l_7 = \overline{l_6}$, $l_j \nparallel l_k, (j,k) = (1,4),(1,6),(1,7),(4,6),(4,7)$, the straight lines l_2, l_3, l_4, l_5 are absolutely complex and l_6, l_7 are relatively complex.

In $(3(2)r, 2r, 2r)$ (see below (A2)) the straight line l_1 with real coefficients has the parallel multiplicity equal to two ($l_1 \equiv l_2 \parallel l_3, l_1 \neq l_3$). In $(3(3)r, 2(2)c_1, 2(2)c_1)$ (see below (A24)) the straight line l_1 with real coefficients has the parallel multiplicity equal to three ($l_1 \equiv l_2 \equiv l_3$), the relatively complex straight line l_4 has the parallel multiplicity equal to two ($l_4 \equiv l_5, l_6 \equiv l_7, l_4 \neq l_6, l_6 = \overline{l_4}$) and so on.

According to property (II.14), if the cubic system has seven invariant affine straight lines, then there exist at most four direction (slopes) for these lines.

By properties (II.19), (II.23), (II.24) and (II.25), if the system (2.1) has one triplet of parallel invariant affine straight lines, one of the following 54 configurations is possible:

(A1) $(3r, 2r, 2r)$;
(A2) $\mathbf{(3(2)r, 2r, 2r)}$;
(A3) $\mathbf{(3(3)r, 2r, 2r)}$;
(A4) $\mathbf{(3r, 2(2)r, 2r)}$;
(A5) $\mathbf{(3(2)r, 2(2)r, 2r)}$;
(A6) $\mathbf{(3(3)r, 2(2)r, 2r)}$;
(A7) $\mathbf{(3r, 2(2)r, 2(2)r)}$;
(A8) $\mathbf{(3(2)r, 2(2)r, 2(2)r)}$;
(A9) $(3(3)r, 2(2)r, 2(2)r)$;
(A10) $\mathbf{(3r, 2r, 2c_0)}$;
(A11) $\mathbf{(3(2)r, 2r, 2c_0)}$;
(A12) $\mathbf{(3(3)r, 2r, 2c_0)}$;
(A13) $\mathbf{(3r, 2(2)r, 2c_0)}$;
(A14) $\mathbf{(3(2)r, 2(2)r, 2c_0)}$;
(A15) $\mathbf{(3(3)r, 2(2)r, 2c_0)}$;
(A16) $\mathbf{(3r, 2c_0, 2c_0)}$;
(A17) $\mathbf{(3(2)r, 2c_0, 2c_0)}$;
(A18) $\mathbf{(3(3)r, 2c_0, 2c_0)}$;
(A19) $(3r, 2c_1, 2c_1)$
(A20) $\mathbf{(3(2)r, 2c_1, 2c_1)}$;
(A21) $\mathbf{(3(3)r, 2c_1, 2c_1)}$;
(A22) $\mathbf{(3r, 2(2)c_1, 2(2)c_1)}$;
(A23) $\mathbf{(3(2)r, 2(2)c_1, 2(2)c_1)}$;
(A24) $(3(3)r, 2(2)c_1, 2(2)c_1)$;
(A25) $\mathbf{(1r+2c_0, 2r, 2r)}$;
(A26) $\mathbf{(1r+2c_0, 2(2)r, 2r)}$;
(A27) $\mathbf{(1r+2c_0, 2(2)r, 2(2)r)}$;
(A28) $(1r+2c_0, 2c_0, 2c_0)$;
(A29) $(1r+2c_0, 2c_1, 2c_1)$;
(A30) $\mathbf{(1r+2c_0, 2(2)c_1, 2(2)c_1)}$;
(A31) $(3r, 2r, 1r, 1r)$;
(A32) $\mathbf{(3(2)r, 2r, 1r, 1r)}$;
(A33) $\mathbf{(3(3)r, 2r, 1r, 1r)}$;
(A34) $\mathbf{(3r, 2(2)r, 1r, 1r)}$;
(A35) $\mathbf{(3(2)r, 2(2)r, 1r, 1r)}$;
(A36) $(3(3)r, 2(2)r, 1r, 1r)$;
(A37) $\mathbf{(3r, 2c_0, 1r, 1r)}$;
(A38) $\mathbf{(3(2)r, 2c_0, 1r, 1r)}$;
(A39) $\mathbf{(3(3)r, 2c_0, 1r, 1r)}$;
(A40) $\mathbf{(3r, 2r, 1c_1, 1c_1)}$;
(A41) $\mathbf{(3(2)r, 2r, 1c_1, 1c_1)}$;
(A42) $\mathbf{(3(3)r, 2r, 1c_1, 1c_1)}$;
(A43) $\mathbf{(3r, 2(2)r, 1c_1, 1c_1)}$;
(A44) $\mathbf{(3(2)r, 2(2)r, 1c_1, 1c_1)}$;
(A45) $(3(3)r, 2(2)r, 1c_1, 1c_1)$;
(A46) $(3r, 2c_0, 1c_1, 1c_1)$;
(A47) $\mathbf{(3(2)r, 2c_0, 1c_1, 1c_1)}$;
(A48) $\mathbf{(3(3)r, 2c_0, 1c_1, 1c_1)}$;
(A49) $\mathbf{(1r+2c_0, 2r, 1r, 1r)}$;
(A50) $\mathbf{(1r+2c_0, 2(2)r, 1r, 1r)}$;
(A51) $(1r+2c_0, 2r, 1c_1, 1c_1)$;
(A52) $\mathbf{(1r+2c_0, 2(2)r, 1c_1, 1c_1)}$;
(A53) $(1r+2c_0, 2c_0, 1r, 1r)$;
(A54) $\mathbf{(1r+2c_0, 2c_0, 1c_1, 1c_1)}$.

Next, we will examine the configurations (A1)–(A54) and their realization in the class of cubic systems.

3.1.1. Unrealizable configurations. Property (II.27) does not allow the realization of configurations (A6), (A7), (A8), (A22), (A23), (A27), (A30) and (A44); Properties (II.7), (II.26) do not allow the realization of configurations (A17), (A18), (A32), (A34), (A50), (A52); (II.7),(II.21) → (A11), (A12), (A15), (A20), (A21), (A39), (A41), (A42), (A47), (A48); (II.7), (II.12), (II.21) $\to$ (A16); (II.2), (II.7), (II.8), (II.16) $\to$ (A26), (A40), (A49); (II.2), (II.7), (II.21) $\to$ (A2), (A3), (A33); (II.2), (II.7), (II.26) $\to$ (A4); (II.7), (II.16), (II.21) $\to$ (A10), (A13), (A14), (A37), (A38); (II.2), (II.7), (II.8), (II.26), (II.27) $\to$ (A5); (II.2), (II.7), (II.16), (II.21) $\to$ (A25); (II.7), (II.26), (II.27) $\to$ (A35); (II.7), (II.21), (II.26) $\to$ (A43); (II.2), (II.7), (II.9), (II.21) $\to$ (A54).

3.1.2. Subconfigurations of configurations with eight straight lines. We denote by $O_{j,k}$ the point of intersection of concurrent straight lines l_j and l_k.

Configuration (A1): $(3r, 2r, 2r)$. Via affine transformations of coordinates we can make that $l_1 = x$, $l_2 = x + 1$, $l_3 = x - a$, $a > 0$, $l_4 = y$, $l_5 = y + 1$. Properties (II.2), (II.7) and (II.21) impose the straight lines l_6 and l_7 to pass, respectively, through the points: (a) $O_{2,5}(-1, -1)$, $O_{1,4}(0, 0)$ and $O_{1,5}(0, -1)$, $O_{3,4}(a, 0)$ or (b) $O_{1,5}(0, -1)$, $O_{2,4}(-1, 0)$ and $O_{1,4}(0, 0)$, $O_{3,5}(a, -1)$ (Fig. 3.1). Taking into account that $l_6 \parallel l_7$, in the case (a) we have $l_6 = y - x$, $l_7 = y - x + 1$, and in the case (b): $l_6 = y + x - 1$, $l_7 = y + x$. In both cases $a = 1$. We observe that the configuration of the straight lines $l_1, \dots l_7$ in the case (a) is symmetrical with respect to the coordinate axis Oy to the configuration of the same lines in the case (b). Therefore, it is enough to consider the case when $l_1 = x$, $l_2 = x + 1$, $l_3 = x - 1$, $l_4 = y$, $l_5 = y + 1$, $l_6 = y - x$, $l_7 = y - x + 1$. The cubic system (2.1) for which these straight lines are invariant look as:

$$\dot{x} = x(x^2 - 1), \quad \dot{y} = y(y + 1)(3x - 2y - 1). \tag{3.1}$$

It is easy to show that (3.1), besides the invariant straight lines $l_1, \dots, l_7$, has one more invariant affine straight line $l_8 = x - 2y - 1$.

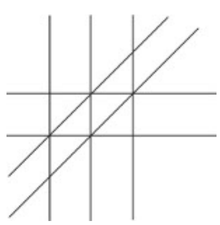
Figure 3.1a

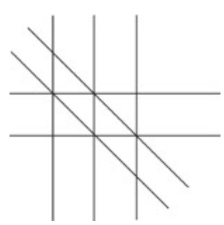
Figure 3.1b

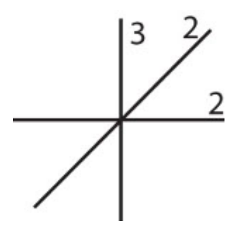

Figure 3.2

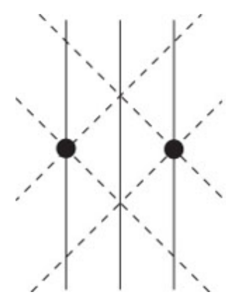
Figure 3.3

Configuration (A9): $(3(3)r, 2(2)r, 2(2)r)$. Assume that $l_1 = l_2 = l_3$, $l_4 = l_5$, $l_6 = l_7$, $l_j \nparallel l_k$, $(j, k) \neq (1, 4), (1, 6), (4, 6)$. We can consider $l_{1,2,3} = x$, $l_{4,5} = y$, $l_{6,7} = x - y$ (see Fig. 3.2). There is only one cubic system for which these straight lines are invariant (l_1 with parallel multiplicity equal to three, l_4 and l_6 both with parallel multiplicity equal to two):

$$\dot{x} = x^3, \quad \dot{y} = y^2(3x - 2y).$$

It is easy to verify that this system, together with the straight lines $l_1, \dots, l_7$, has also the invariant affine straight line $l_8 = x - 2y$.

Configuration (A19): $(3r, 2c_1, 2c_1)$. Properties (II.7), (II.12) and (II.21) allow only the configuration given in Fig. 3.3. By an affine transformation we can make $l_1 = x$, $l_2 = x - a$, $a \in (0, 1)$, $l_3 = x - 1$, $l_{4,6} = y \mp ix$, $l_{5,7} = y \mp i(x - 1) - \alpha$, $\alpha \in \mathbb{R}$. The cubic systems for which the straight lines $l_1, \dots, l_4$ and l_6 are invariant look as:

$$\begin{aligned}\dot{x} &= x(x-1)(x-a),\\ \dot{y} &= ay + b_{20}x^2 - (a+1)xy + b_{20}y^2 + b_{30}x^3 + b_{21}x^2y\\ &\quad + b_{30}xy^2 + (b_{21}-1)y^3.\end{aligned} \tag{3.2}$$

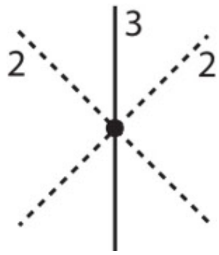

Figure 3.4

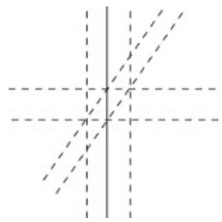
Figure 3.5

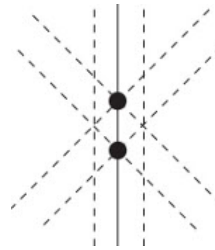
Figure 3.6

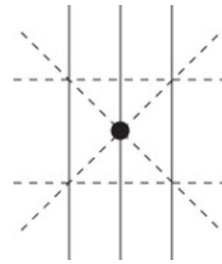
Figure 3.7

If the straight lines $l_{5,7} = y \mp i(x-1) - \alpha$ are invariant for system (3.2) then it has the form

$$\dot{x} = x(x-1)(2x-1), \quad \dot{y} = y(1-3x+3x^2+y^2). \tag{3.3}$$

Totally the system (3.3) has the following invariant affine straight lines: $l_1 = x$, $l_2 = x - 1/2$, $l_3 = x - 1$, $l_{4,6} = y \mp ix$, $l_{5,7} = y \mp i(x-1)$, $l_8 = y$.
Configuration (A24): (*3(3)r,2(2)c*$_1$*,2(2)c*$_1$) (Fig. 3.4). Without loss of generality, we consider $l_1 = l_2 = l_3 = x$ and $l_{5,7} = \overline{l_{4,6}} = y \pm ix$. There is only one cubic system for which these straight lines are invariant and this is the system

$$\dot{x} = 2x^3, \quad \dot{y} = y(3x^2+y^2). \tag{3.4}$$

Clearly, for cubic system (3.4) and the straight line $l_8 = y$ is also invariant.
Configuration (A28): (*1r+2c*$_0$*,2c*$_0$*,2c*$_0$) (Fig. 3.5). We can take $l_1 = x - a$, $a \in \mathbb{R}$, $l_2 = x - i$, $l_3 = x + i$, $l_4 = y - i$, $l_5 = y + i$. Therefore, we have the following cubic system possessing these lines:

$$\dot{x} = (x-a)(x^2+1), \quad \dot{y} = (y^2+1)(bx+cy+d). \tag{3.5}$$

We may assume that the straight line l_6 passes through the singular points $O_{3,5}(-i,-i)$, $O_{1,4}(a,i)$, otherwise we could apply the substitution $x \to -x$ or/and $y \to -y$ which preserves the form of the system (3.5). Then the line l_6 is described by the equation $2x - (1+ia)y - a + i = 0$. Hence, $l_7 = 2x - (1+ia)y - a - i = 0$. The fact that the straight lines l_6 and l_7 are parallel implies $a = 0$, and therefore, $l_{6,7} = 2x - y \pm i$. If the straight lines $l_{6,7}$ are invariant for system (3.5) it becomes

$$\dot{x} = x(x^2+1), \quad \dot{y} = (3x-y)(y^2+1)/2.$$

It is easy to see that besides the invariant straight lines $l_1, \dots l_7$ defined above, the obtained system has also the invariant affine straight line $l_8 = x - y$.
Configuration (A29): (*1r+2c*$_0$*,2c*$_1$*,2c*$_1$) (Fig. 3.6). We can consider $l_1 = x$, $l_4 = y - ix$, $l_5 = y - ix - 2$, $l_6 = y + ix$, $l_7 = y + ix - 2$. The absolutely complex straight line l_2 (respectively l_3) pass through the point $O_{4,7}(-i,1)$ (respectively $O_{5,6}(i,1)$), i.e. it is described by the equation $x + i = 0$ (respectively $x - i = 0$). The cubic system for which these straight lines are invariant look as:

$$\dot{x} = 2x(x^2+1), \quad \dot{y} = (y-1)(-2y+3x^2+y^2).$$

Evidently, the straight line $l_8 = y - 1$ is also invariant for the obtained system. Therefore, it has eight invariant affine straight line.

Configuration (A46): ($3r,2c_0,1c_1,1c_1$) (Fig. 3.7). We start with the system

$$\dot{x} = x(x+1)(x-a),\ a > 0, \quad \dot{y} = (y^2+1)(bx+cy+d) \tag{3.6}$$

for which the straight lines $l_1 = x$, $l_2 = x+1$, $l_3 = x - a$, $l_4 = y - i$, $l_5 = y + i$ are invariant. The straight line l_6 passes through the points $O_{2,5}(-1,-i)$, $O_{3,4}(a,i)$ and therefore it is described by the equation $y = \frac{2i}{a+1}x + \frac{1-a}{a+1}i$. We put $l_6 = y - \frac{2i}{a+1}x - \frac{1-a}{a+1}i$, $l_7 = \overline{l_6}$. The straight lines $l_{6,7}$ are invariant for system (3.6) if and only if this system has the form

$$\dot{x} = x(x+1)(x-1), \quad \dot{y} = -y(y^2+1). \tag{3.7}$$

It is easy to check that the straight lines $l_1 = x$, $l_{2,3} = x \pm 1$, $l_{4,5} = y \mp i$, $l_{6,7} = y \mp ix$, $l_8 = y$ are invariant for (3.7).

Configuration (A53): ($1r+2c_0,2c_0,1r,1r$) (Fig. 3.8). We consider the system (3.5) which has the following invariant straight lines: $l_1 = x - a$, $a \in \mathbb{R}$, $l_2 = x - i$, $l_3 = x + i$, $l_4 = y - i$, $l_5 = y + i$. The straight lines l_6 and l_7 with real coefficients pass through the complex conjugate points $O_{3,5}(-i,-i)$, $O_{2,4}(i,i)$ and $O_{2,5}(i,-i)$, $O_{3,4}(-i,i)$, respectively. Therefore, $l_6 = y - x$ and $l_7 = y + x$. The straight lines $l_1, \dots, l_7$ are invariant for system (3.5) if and only if the system looks as:

$$\dot{x} = x(x^2+1), \quad \dot{y} = y(y^2+1). \tag{3.8}$$

Evidently, and the straight line $l_8 = y$ is also invariant for (3.8).

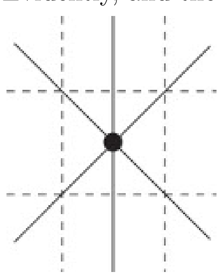

Figure 3.8

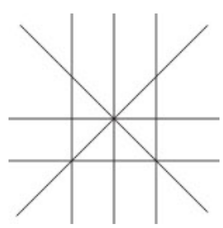

Figure 3.9a

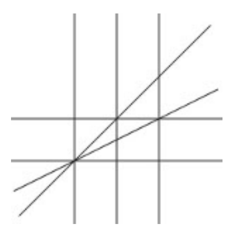

Figure 3.9b

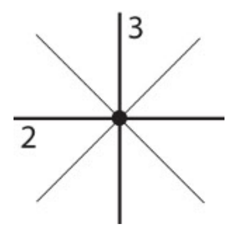

Figure 3.10

3.1.3. Realizable configurations. Configuration (A31): $(3r, 2r, 1r, 1r)$. Via affine transformations of the phase plane we can make the straight lines $l_1, \dots, l_6$ to be described by equations: $x = 0$, $x + 1 = 0$, $x - a = 0$, $a > 0$, $y = 0$, $y + 1 = 0$ and $x - y = 0$. Properties (II.7) and (II.21) allow only configurations from Fig. 3.9. In the case of Fig. 3.9a) (Fig. 3.9b)) we can write $l_7 = x + ay$ ($l_7 = x - (a+1)y - a$).

System (I.7) (respectively (I.8)) from Theorem 1.1 is the unique cubic system possessing the invariant affine straight lines: $l_1 = x$, $l_2 = x+1$, $l_3 = x - a$, $l_4 = y$, $l_5 = y+1$, $l_6 = y - x$ and $l_7 = x + ay$ (respectively $l_7 = x - (a+1)y - a$). Moreover this system could not have other invariant affine straight line if $a \neq 1$. If $a = 1$ then (I.7) (respectively (I.8)) has an additional invariant affine straight line $l_8 = y - 1$ (respectively $l_8 = x - y - 1$).

Configuration (A36): $(3(3)r, 2(2)r, 1r, 1r)$. Using properties (II.7) and (II.21), we obtain the configuration Fig. 3.10. We can consider $l_1 = l_2 = l_3 = x$, $l_4 = l_5 = y$ and $l_6 = y - x$. The cubic system with these invariant straight lines coincides with the system (I.9) from Theorem 1.1 and this system possess also the invariant straight line $l_7 = x + y - ay$ (see (1.10)). If $a = 0$ (respectively $a = 3/2$; $a = 3$), then the straight line l_4 (respectively l_7; l_6) has parallel multiplicity equal to three (two). In the case $a = 1$, we have $\gcd(P, Q) = x$, and in the case $a = 2$ the straight lines l_6 and l_7 (see (1.10)) coincide, have parallel multiplicity equal to one, and the system (I.9) does not have other invariant affine straight lines, except $l_1, \dots, l_5$. Therefore if $a = 2$ the system (I.9) has exactly six invariant affine straight lines (counting also their parallel multiplicity).

Configuration (A45): $(3(3)r, 2(2)r, 1c_1, 1c_1)$ (Fig. 3.10). We take $l_1 = l_2 = l_3 = x$, $l_4 = l_5 = y$ and the system

$$\dot{x} = x^3, \quad \dot{y} = y^2(b + cx + dy), \tag{3.9}$$

for which these straight lines are invariant.

By property (II.27), the conjugate and relative complex straight lines $l_{6,7}$ pass through origin of coordinates, so they can be described by the equations $y - (\alpha \pm \beta i)x = 0$, where $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$. Rescaling the coordinate axes, we can make $\beta = 1$. The conditions imposed to systems (3.9) to have the invariant straight lines $l_{6,7} = y - (\alpha \pm i)x$ lead to the system

$$\dot{x} = (1 + \alpha^2)x^3, \quad \dot{y} = y^2(2\alpha x - y), \ \alpha \neq 0. \tag{3.10}$$

Applying the substitutions $x \to x/\sqrt{1 + \alpha^2}$, $y \to y$, $a = \alpha/\sqrt{1 + \alpha^2}$, we obtain the system (I.10) from Theorem 1.1.

Configuration (A51): $(1r + 2c_0, 2r, 1c_1, 1c_1)$ (Fig. 3.11). We consider $l_1 = x - a$, $a \in [0, +\infty)$, $l_2 = x + i$, $l_3 = x - i$, $l_4 = y$, $l_5 = y + 1$. In the case given by Fig. 3.11a (respectively Fig. 3.11b) the straight line l_6 passes through the points $O_{2,5}(-i, -1)$ and $O_{1,4}(a, 0)$ (respectively $O_{2,5}(-i, -1)$ and $O_{3,4}(i, 0)$). Therefore, it is described by the equation $x - (a + i)y - a = 0$ $(2y + ix + 1 = 0)$. In the first case (given by Fig. 3.11a) assuming that $l_7 = \bar{l}_6$, we obtain the straight lines from (1.12) and the system (I.11), for which these straight lines are invariant (see Theorem 1.1). If $a = 0$, then the system (I.11) has the invariant affine straight lines $l_1 = x$, $l_{2,3} = x \pm i$, $l_4 = y$, $l_5 = y + 1$, $l_{6,7} = x \mp yi$, $l_8 = y - 1$.

In the case Fig. 3.11b we have $l_{6,7} = 2y \pm ix + 1$. The intersection point $O(0, -1/2)$ of the straight lines l_6 and l_7 lies on the straight line $l_1 = x - a$, so $a = 0$. There exists only one cubic system: $\dot{x} = x(x^2 + 1)$, $\dot{y} = -2y(1 + y)(1 + 2y)$, with invariant affine straight lines $l_1 = x$, $l_{2,3} = x \pm i$, $l_4 = y$, $l_5 = y + 1$, $l_{6,7} = 2y \pm ix + 1$. This system has an additional invariant affine straight line $l_8 = 1 + 2y$.

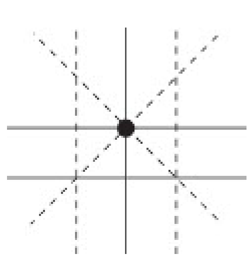

Figure 3.11a

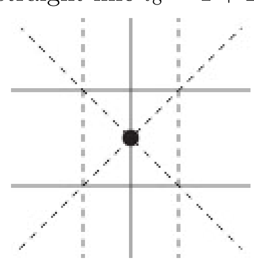

Figure 3.11b

3.1.4. Qualitative study of systems (I.7)–(I.11)

In this section, the qualitative study of the systems (I.7)–(I.11) from Theorem 1.1 will be done. For this purpose, to determine the topological behavior of trajectories, the finite and the infinite singular points will be examined. This information and the information provided by the existence of invariant straight lines, will be taken into account when the phase portraits of systems (I.7)–(I.11) on the Poincaré disk will be constructed.

We set the abbreviations: SP for a *singular point* and TSP for *type* of SP. We use here the following symbols: λ_1 and λ_2 for eigenvalues of SP; S for a saddle ($\lambda_1\lambda_2 < 0$); TS for a topological saddle; N^s for a stable node ($\lambda_1, \lambda_2 < 0$); N^u for a unstable node ($\lambda_1, \lambda_2 > 0$); $DN^{s(u)}$ for a "decritic" stable (unstable) node ($\lambda_1 = \lambda_2 \neq 0$); $TN^{s(u)}$ for a stable (unstable) topological node; $S - N^{s(u)}$ for a saddle-node with a stable (unstable) parabolic sector; $P^{s(u)}$ for a stable (unstable) parabolic sector; H for a hyperbolic sector, $F^{s(u)}$ for a stable (instable) focus.

Systems (I.7), (I.8), (I.11). In the first column of Tables 3.1, 3.2 and 3.3 we indicate the real singular points (finite and infinite) of the systems (I.7), (I.8), (I.11), respectively; in the second column the eigenvalues corresponding to these singular points and in the third column the types of the singularities. All these points are simple and together with the invariant straight lines, complectly determine the phase portrait of each of the systems (I.7), (I.8) and (I.11).

Table 3.1

System (I.7) (Fig. 1.7)					
SP	$\lambda_1;\ \lambda_2$	TSP	SP	$\lambda_1;\ \lambda_2$	TSP
$O_1(-1,-1)$	$1+a;\ 1+a$	DN^u	$O_8(-1, \frac{1}{a})$	$1+a;\ \frac{1+a}{a}$	N^u
$O_2(-1,0)$	$-1;\ 1+a$	S	$O_9(a,a)$	$a(1+a);\ a^2(1+a)$	N^u
$O_3(0,-1)$	$-a;\ 2a$	S	$X_{1_\infty}(1,0,0)$	$-1;\ -1$	DN^s
$O_4(0,0)$	$-a;\ -a$	DN^s	$X_{2_\infty}(1,1,0)$	$-1;\ 1+a$	S
$O_5(a,-1)$	$a+a^2;\ a+a^2$	DN^u	$X_{3_\infty}(1,-\frac{1}{a},0)$	$-1;\ \frac{1+a}{a}$	S
$O_6(a,0)$	$-a^2;$	S	$Y_\infty(0,1,0)$	$-a;\ -a$	DN^s
$O_7(0,1)$	$-a;\ 2a$	S			

Table 3.2

System (I.8) (Fig. 1.8)					
SP	$\lambda_1;\ \lambda_2$	TSP	SP	$\lambda_1;\ \lambda_2$	TSP
$O_1(-1,-1)$	$1+a;\ 1+a$	DN^u	$O_8(a,a)$	$a+a^2;$ $-a(1+a)^2$	S
$O_2(-1,0)$	$-2(1+a);\ 1+a$	S	$O_9(0,-\frac{a}{1+a})$	$-a;\ \frac{a}{1+a}$	S
$O_3(0,-1)$	$-1;\ -a$	N^s	$X_{1_\infty}(1,0,0)$	$-1;\ -1$	DN^s
$O_4(0,0)$	$-a;\ -a$	DN^s	$X_{2_\infty}(1,1,0)$	$-1;\ -a$	N^s
$O_5(a,-1)$	$-(1+a)^2;\ a(1+a)$	S	$X_{3_\infty}(1,\frac{1}{1+a},0)$	$-1;\ \frac{a}{1+a}$	S
$O_6(a,0)$	$a(1+a);$	DN^u	$Y_\infty(0,1,0)$	$1+a;\ 1+a$	DN^u
$O_7(-1,-2)$	$-2(1+a);\ 1+a$	S			

Table 3.3

System (I.11) (Fig. 1.11)					
SP	$\lambda_1;\ \lambda_2$	TSP	SP	$\lambda_1;\ \lambda_2$	TSP
$O_1(0,0)$	$a^2+1;\ a^2+1$	DN^u	$X_\infty(1,0,0)$	$a^2+1;\ a^2+1$	DN^u
$O_2(-1,0)$	$a^2+1;\ -2(a^2+1)$	S	$Y_\infty(0,1,0)$	$-1;\ -1$	DN^s
$O_3(1,0)$	$a^2+1;\ a^2+1$	DN^u			

System (I.9) (Table 3.4). The origin of coordinates is a non-hyperbolic singular point for (I.9). We will study the behavior of the trajectories in a neighborhood of this point using blow-up method. In the polar coordinates $x = \rho\cos\theta$, $y = \rho\sin\theta$

the system (I.9) takes the form

$$
\begin{aligned}
\frac{d\rho}{d\tau} &= \rho(\cos^4\theta + (1-a)\sin^4\theta + a\cos\theta\sin^3\theta),\\
\frac{d\theta}{d\tau} &= \sin\theta\cos\theta(\sin\theta-\cos\theta)(\cos\theta+(1-a)\sin\theta),
\end{aligned}
\tag{3.11}
$$

where $\tau = \rho^2 t$. Taking into account that the system (I.9) is symmetric with respect to the origin, it is sufficient to consider $\theta \in [0,\pi)$. The singular points of the system (3.11) with first coordinate $\rho = 0$ and the second $\theta \in [0,\pi)$, and their eigenvalues respectively are: $\{M_1(0,0)$: $\lambda_{1,2} = \pm 1 \to$ saddle $\}$; $\{M_2(0,\frac{\pi}{2})$: $\lambda_{1,2} = \pm(1-a) \to$ saddle$\}$; $\{M_3(0,\frac{\pi}{4})$: $\lambda_{1,2} = \frac{1}{2}$, $\lambda_2 = \frac{2-a}{2} \to$ unstable node, if $a < 2$, and saddle, if $a > 2\}$; $\{M_4(0, \operatorname{arctg}\frac{1}{a-1})$: $\lambda_1 = \frac{(a-1)(a-2)}{a^2-2a+2}$, $\lambda_2 = \frac{(a-1)^2}{a^2-2a+2} \to$ unstable node, if $a < 1$ or $a > 3$; and saddle, if $1 < a < 3\}$. We obtain Fig. 1.9a if $a < 1$ and Fig. 1.9b if $a > 1$. In Fig. 3.12a, 3.12b, it is illustrated the case $a < 1$, i.e. the singular point $(0,0)$ is TN^u, and in Fig. 3.12c, 3.12d we have the case $a > 1$ with following partition in sectors of the neighborhood of the origin: P^uHHP^uHH.

Table 3.4

System (I.9) (Fig. 1.9a)			System (I.9) (Fig. 1.9b)		
SP	$\lambda_1;\lambda_2$	TSP	SP	$\lambda_1;\lambda_2$	TSP
$O(0,0)$	0; 0	TN^u	$O(0,0)$	P^uHHP^uHH	P^uHHP^uHH
$X_{1_\infty}(1,0,0)$	-1; -1	DN^s	$X_{1_\infty}(1,0,0)$	DN^s	DN^s
$X_{2,\infty}(1,1,0)$	-1; $2-a$	S	$X_{2,\infty}(1,1,0)$	S	N^s
$X_{3,\infty}(1,\frac{1}{a-1},0)$	-1; $\frac{a-2}{a-1}$	S	$X_{3,\infty}(1,\frac{1}{a-1},0)$	N^s	S
$Y_\infty(0,1,0)$	$a-1$; $a-1$	DN^s	$Y_\infty(0,1,0)$	DN^u	DN^u

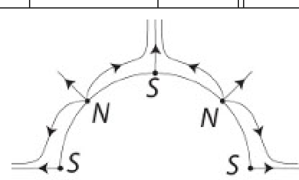

Figure 3.12a

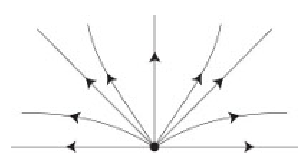

Figure 3.12b

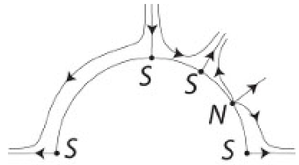

Figure 3.12c

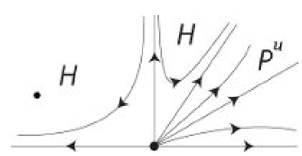

Figure 3.12d

System (I.10) Table 3.5.

Table 3.5

System (I.10) (Fig. 1.10)		
SP	$\lambda_1;\lambda_2$	TSP
$O_1(0,0)$	0; 0	$HHHH$
$X_\infty(1,0,0)$	-1; -1	DN^s
$Y_\infty(0,1,0)$	1; 1	DN^u

We will study the behavior of the trajectories in a neighborhood of the origin of coordinates. We note that all trajectories are symmetric with respect to the point

$(0,0)$. Using polar coordinate, we write:

$$\begin{gathered}\dot{\rho} = \rho(\cos^4\theta + 2a\cos\theta\sin^3\theta - \sin^4\theta),\\ \dot{\theta} = \sin\theta\cos\theta(a\sin 2\theta - 1).\end{gathered} \tag{3.12}$$

The coordinates of the singular points $M_i(0,\theta_i)$ of the system (3.12) are given by the equation

$$\sin\theta\cos\theta(a\sin 2\theta - 1) = 0.$$

Since $|a| < 1$ (see (I.10)) we get $a\sin 2\theta - 1 < 0$ and therefore we obtain the singular points $M_1(0,0)$, $M_2(0,\pi/2)$, $M_3(0,\pi)$ and $M_4(0,3\pi/2)$, which are saddles with the same eigenvalues: $\lambda_{1,2} = \pm 1$ (see Fig. 3.13a). Therefore after blow-up we arrive at the topological structure of the vicinity of the origin of coordinates given by Fig. 3.13b.

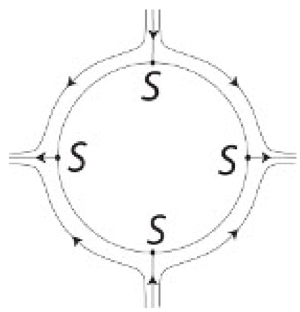

Figure 3.13a

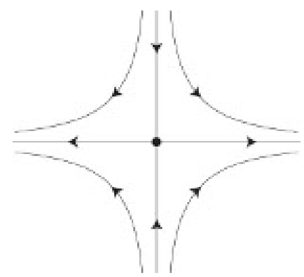

Figure 3.13b

3.2. **B. Cases of cubic systems without triplets of parallel invariant straight lines.** We have the following 15 configurations of 7 straight lines that do not contain a triplet of parallel invariant straight lines:

(B1) $(2r, 2r, 2r, 1r)$;
(B2) $(\mathbf{2(2)r}, \mathbf{2r}, \mathbf{2r}, \mathbf{1r})$;
(B3) $(\mathbf{2(2)r}, \mathbf{2(2)r}, \mathbf{2r}, \mathbf{1r})$;
(B4) $(2(2)r, 2(2)r, 2(2)r, 1r)$;
(B5) $(\mathbf{2r}, \mathbf{2r}, \mathbf{2c_0}, \mathbf{1r})$;
(B6) $(\mathbf{2(2)r}, \mathbf{2r}, \mathbf{2c_0}, \mathbf{1r})$;
(B7) $(\mathbf{2(2)r}, \mathbf{2(2)r}, \mathbf{2c_0}, \mathbf{1r})$;
(B8) $(2r, 2c_1, 2c_1, 1r)$;
(B9) $(\mathbf{2r}, \mathbf{2c_0}, \mathbf{2c_0}, \mathbf{1r})$;
(B10) $(\mathbf{2(2)r}, \mathbf{2c_1}, \mathbf{2c_1}, \mathbf{1r})$;
(B11) $(\mathbf{2(2)r}, \mathbf{2c_0}, \mathbf{2c_0}, \mathbf{1r})$;
(B12) $(2(2)r, 2(2)c_1, 2(2)c_1, 1r)$;
(B13) $(2c_0, 2c_0, 2c_0, 1r)$;
(B14) $(2c_0, 2c_1, 2c_1, 1r)$;
(B15) $(\mathbf{2c_0}, \mathbf{2(2)c_1}, \mathbf{2(2)c_1}, \mathbf{1r})$.

3.2.1. The classification of the cubic systems.

Remark 3.1. The properties (II.2), (II.7), (II.16), (II.26) and (II.27) do not allow realization of the configurations (B2), (B3), (B5)–(B7), (B9)–(B11) and (B15).

Further we will study the configurations (B1), (B4), (B8), (B12), (B13) and (B14).

Configuration (B1): $(2r, 2r, 2r, 1r)$. For this configuration the properties (II.2) and (II.7) allow only the cases (a) and (b) from Fig. 3.14. We consider $l_1 = x$, $l_2 = x+1$, $l_3 = y$, $l_4 = y+1$. In the case (a) we have $l_5 = x+y+1$, $l_6 = x-y$ and $l_7 = x-y+1$. The cubic system with these invariant affine straight lines has the form:

$$\dot{x} = x(x+1)(1-x+3y), \quad \dot{y} = y(y+1)(1+3x-y). \tag{3.13}$$

It is easy to check that for (3.13) the straight line $l_8 = x-y-1$ is also invariant.

In the case of Fig. 3.14b we have the straight lines (1.13) and the system (I.12) from Theorem 1.1. If $a = 1$, then after the time rescaling $t \to -t$ this system coincide with the system (3.13).

Configuration (B4): $(2(2)r, 2(2)r, 2(2)r, 1r)$. We can consider $l_{1,2} = x$ and $l_{3,4} = y$. The property (II.27) impose to other straight lines of this configuration to pass through the origin of coordinate (see Fig. 3.16). Rescaling Ox axis we can write $l_{5,6} = x - y$. The conditions imposed to a cubic system to have the invariant straight lines $l_1, \dots l_6$ leads to the system (I.13) from Theorem 1.1, and we observe that this system has the seventh invariant affine straight line: $l_7 = ax - y - 2ay$.

If $a(a+1)(2a+1) = 0$, then $\gcd(P, Q) \neq const$, and if $3a + 2 = 0$ $(3a + 1 = 0)$, then the invariant straight line $y = 0$ $(x - y = 0)$ has parallel multiplicity equal to three.

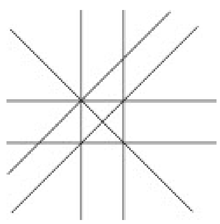
Figure 3.14a

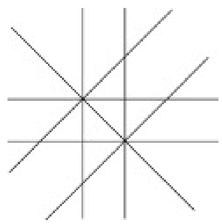
Figure 3.14b

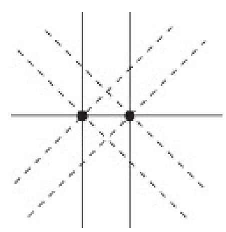
Figure 3.15a

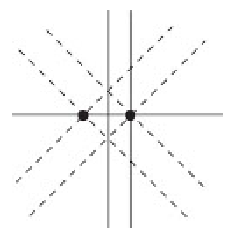
Figure 3.15b

Configuration (B8): $(2r, 2c_1, 2c_1, 1r)$. Let $l_1, \dots, l_7$ be the straight lines of this configuration, where $l_{1,2,7}$ are real, $l_3, \dots, l_6$ are relative complex and $l_1 \parallel l_2$, $l_3 \parallel l_4$, $l_5 \parallel l_6$, $l_5 = \overline{l_3}$, $l_6 = \overline{l_4}$, $l_j \nparallel l_k$, $(j, k) \in \{(1,3), (1,5), (1,7), (3,7), (5,7)\}$. According to properties (II.2), (II.7) and (II.16), the only cases illustrated in Fig. 3.15 can occur. Let $O_{3,5} = l_3 \cap l_5 \in l_1$. Via an affine transformation of the phase plane we can make the straight line l_3 to be written into form $y - ix = 0$ and then $l_5 = y + ix$. We rotate the phase plane such that the straight line l_1 coincides with the Oy axis, and apply rescaling $x \to kx$, $y \to ky$, $\quad k \neq 0$. We choose k such that l_2 passes through the point $(-1, 0)$. Finally, we obtain: $l_1 = x$, $l_2 = x + 1$, $l_{3,5} = y \mp ix$, $l_{4,6} = y \mp ix - a \mp bi$, $a, b \in \mathbb{R}$, $b \neq 0$. In the case Fig. 3.15a we have $b = 1$, $l_4 \cap l_6 = (-1, a)$, $l_7 = y + ax$, and the system (I.14) from the Theorem 1.1. We note that if $a = 0$, then the system (I.14) has an additionally invariant affine straight line $l_8 = 2x + 1$.

In the case Fig. 3.15b we have the straight lines $l_1 = x$, $l_2 = x + 1$, $l_{3,5} = y \mp ix$, $l_{4,6} = y \mp i(x + 2)$, $l_7 = y$, and the cubic system with these invariant affine straight lines looks:

$$\dot{x} = 2x(x+1)(x+2), \quad \dot{y} = y(4 + 6x + 3x^2 + y^2).$$

The obtained system has also the eighth invariant affine straight line: $l_8 = x + 2$.

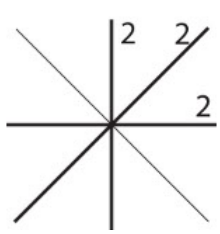

Figure 3.16

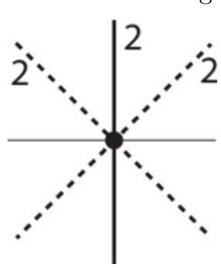

Figure 3.17

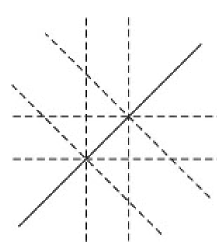
Figure 3.18

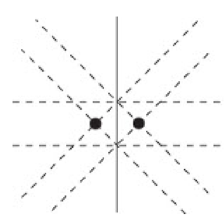
Figure 3.19

Configuration (B12): $(2(2)r, 2(2)c_1, 2(2)c_1, 1r)$. Let $l_1 = l_2$, $l_3 = l_4$, $l_5 = l_6$, $l_5 = \overline{l_3}$, $l_7 \not\parallel l_1$, $l_7 \not\parallel l_3$. Properties (II.7) and (II.27) allow these straight lines to have only reciprocal position illustrated in Fig. 3.17.

Via affine transformations similar to those applied to the configuration (B8), we can write $l_{1,2} = x$, $l_{3,4} = y - ix$, $l_{5,6} = y + ix$. Then, $l_7 = y - ax$, $a \geq 0$. These straight lines are invariant for the cubic system (I.15) from the Theorem 1.1. If $a = 0$, then the straight line l_1 has parallel multiplicity equal to three, which is not allowed in this configuration.

Configuration (B13): $(2c_0, 2c_0, 2c_0, 1r)$ (Fig. 3.18). We can consider $l_1 = x - i$, $l_2 = x + i$, $l_3 = y - i$, $l_4 = y + i$. Then $l_5 = y - a(x - i) - i$, $l_6 = y - a(x + i) + i$, $a \in \mathbb{R}$, $a(a - 1) \neq 0$, $l_7 = y - x$. Forcing a generic cubic system to possess these invariant straight lines, we arrive at the system (I.16) from the Theorem 1.1. If $a = 1/2$ ($a = 2$), then (I.16) has one more invariant affine straight line: $l_8 = y$ ($l_8 = x$).

Configuration (B14): $(2c_0, 2c_1, 2c_1, 1r)$ (Fig. 3.19). We can take $l_1 = y - i$, $l_2 = y + i$ and $l_7 = x$. The cubic system (2.1) with invariant straight lines l_1, l_2 and l_7 has the form

$$\begin{aligned} \dot{x} &= x(a_{10} + a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2), \\ \dot{y} &= (1 + y^2)(b_{00} + b_{10}x + b_{01}y). \end{aligned} \tag{3.14}$$

We denote by $l_3, \dots, l_6$ the relatively complex straight lines and assume $l_3 \parallel l_4$, $l_5 \parallel l_6$, $l_5 = \overline{l_3}$, $l_6 = \overline{l_4}$. Two of these straight lines pass through the point $O_{1,7}(0, i)$, and other two - through the point $O_{2,7}(0, -i)$. Let l_3 pass through the point $O_{1,7}$, i.e. it is described by an equation of the form $y = (a+bi)x+i$. Then the straight line l_4 passes through the point $O_{2,7}$ and it is described by the equation $y = (a+bi)x-i$, a, $b \in \mathbb{R}$, $b \neq 0$. Via the rescaling $x \to x/b$ we can make $b = 1$. Therefore we obtain the straight lines $l_{3,4} = y - (a + i)x \mp i$ and $l_{5,6} = y - (a - i)x \pm i$. If these straight lines are invariant for (3.14), then we get the system (I.17) from Theorem 1.1.

3.2.2. Qualitative study of systems (I.12)-(I.17).

Systems (I.12), (I.14), (I.16) and (I.17). The behavior of trajectories in systems (I.12), (I.14), (I.16) and (I.17) from Theorem 1.1 it is completely determined by information from (1.13), (1.15), (1.17), (1.18) and Tables 3.6 - 3.9.

Table 3.6

System (I.12) (Fig. 1.12)					
SP	λ_1; λ_2	TSP	SP	λ_1; λ_2	TSP
$O_1(0,0)$, $O_2(-1,-1)$	-1; $-a$	DN^s	$X_{1_\infty}(1,0,0)$	$-a$; $-a$	DN^s
$O_3(-1,0)$, $O_4(0,-1)$	$a+1$; $a+1$	DN^u	$X_{2,\infty}(1,-1,0)$	$-2(a+1)$; $a+1$	S
$O_5(-\frac{1}{2},-\frac{1}{2})$	$\frac{a+1}{4}$; $-\frac{a+1}{2}$	S	$X_{3,\infty}(1,a,0)$	$a(a+1)$; $a(a+1)$	DN^u
$O_6(\frac{1}{a},0)$, $O_7(-\frac{a+1}{a},-1)$	$\frac{a+1}{a}$; $-\frac{(a+1)^2}{a}$	S	$Y_\infty(0,1,0)$	-1; -1	DN^u
$O_8(-1,-a-1)$, $O_9(0,a)$	a^2+a; $-(a+1)^2$	S			

Table 3.7

System (I.14) (Fig. 1.14)					
SP	$\lambda_1;\ \lambda_2$	TSP	SP	$\lambda_1;\ \lambda_2$	TSP
$O_1(0,0)$, $O_2(-1,a)$	$a^2+1;\ a^2+1$	DN^u	$X_\infty(1,-a,0)$	$-2(a^2+1);$ a^2+1	S
$O_3(-\frac{1}{2},\frac{a}{2})$	$-\frac{1}{2}(a^2+1);\ \frac{1}{4}(a^2+1)$	S	$Y_\infty(0,1,0)$	$-1;\ -1$	DN^s

Table 3.8

System (I.16) (Fig. 1.16)		
SP	$\lambda_1;\ \lambda_2$	TSP
$O_1(0,0)$	$1-a;\ -2(1-a)$	S
$X_{1_\infty}(1,0,0)$	$-a;\ -a$	DN^u if $a<0$; DN^s if $a>0$
$X_{2_\infty}(1,1,0)$	$-2(1-a);\ -(1-a)$	S
$X_{3_\infty}(1,a,0)$	$a(1-a);\ a(1-a)$	DN^u if $a<0$ or $a>1$; DN^s if $a\in(0,1)$
$Y_\infty(0,1,0)$	$1;\ 1$	DN^u

Table 3.9

System (I.17) (Fig. 1.17)		
SP	$\lambda_1;\ \lambda_2$	TSP
$O_1(0,0)$	$-2;\ 1$	S
$O_2(-1,-a)$, $O_3(1,a)$	$-2(1+ia);\ -2(1-ia)$	F^s
$X_\infty(1,0,0)$	$1+a^2;\ 1+a^2$	DN^u
$Y_\infty(0,1,0)$	$-1;\ 2$	S

System (I.13) For this system we have Table 3.10.

Table 3.10

System (I.13) (Fig. 1.13)		
SP	$\lambda_1;\ \lambda_2$	TSP
$O_1(0,0)$	$0;\ 0$	$P^{s(i)}HHP^{s(i)}HH-$ if $a(a+1)(2a+1)<0\,(>0)$
$X_{1_\infty}(1,0,0)$	$-a;\ -a$	DN^u if $a<0$; DN^s if $a>0$
$X_{2_\infty}(1,1,0)$	$-a-1;\ -a-1$	DN^u if $a<-1$; DN^s if $a>-1$
$X_{3_\infty}(1,\frac{a}{2a+1},0)$	$-\frac{2a(a+1)}{2a+1};\ \frac{a(a+1)}{2a+1}$	S
$Y_\infty(0,1,0)$	$2a+1;\ 2a+1$	DN^u if $a<-1/2$; DN^s if $a>-1/2$

As we can see from Table 3.10, system (I.13) has a nilpotent singular point in the finite part of the phase plane and four hyperbolic singular points at the infinity. We can find the type of the nilpotent singular point by using blow-up method. Therefore, applying to system (I.13) the transformation $x=\rho\cos\theta$, $y=\rho\sin\theta$, we obtain

$$\begin{gathered}
\frac{d\rho}{d\tau}=\rho(a\cos^4\theta-(1+2a)\sin^4\theta+\sin\theta\cos^3\theta+(2+3a)\sin^3\theta\cos\theta),\\
\frac{d\theta}{d\tau}=\sin\theta\cos\theta(\sin\theta-\cos\theta)(a\cos\theta-(1+2a)\sin\theta)),
\end{gathered}\tag{3.15}$$

where $\tau = \rho^2 t$. The vector field associated to the system (I.13) is symmetric with respect to the origin of the coordinates. This allows us to consider the angle θ from (3.15) to be between 0 and π. The singular points M_k of the system (3.15) with the first coordinate $\rho = 0$ and the second coordinate $\theta \in [0, \pi)$, their eigenvalues λ_1, λ_2 and their types are, respectively: $\{M_1(0,0) : \lambda_{1,2} = \pm a \to \text{saddle}\}$; $\{M_2(0, \frac{\pi}{2}) : \lambda_{1,2} = \pm(1+2a) \to \text{saddle}\}$; $\{M_3(0, \frac{\pi}{4}) : \lambda_{1,2} = \pm\frac{1+a}{2} \to \text{saddle}\}$; $\{M_4(0, \arctan\frac{a}{1+2a}) : \lambda_1 = \frac{a(a+1)(2a+1)}{5a^2+4a+1}, \lambda_2 = \frac{2a(a+1)(2a+1)}{5a^2+4a+1} \to$ stable node, if $a(a+1)(ab+1) < 0$ and unstable node, if $a(a+1)(2a+1) > 0\}$. Depending on the values of the parameter a, the neighborhood of the singular point $(0,0)$ consists from sectors of the type P^sHHP^sHH or of the type P^uHHP^uHH (see Fig. 3.12c, 3.12d). From the topological point of view, the cubic system (I.13) has the same phase portrait as the system (I.9) in the case $a > 1$, $a \neq 3/2, 2, 3$ (see Fig. 1.9b).

System (I.15). This system has only one non-hyperbolic finite singular point and two hyperbolic singular points at the infinity (Table 3.11). Using blow-up method we investigate the neighborhood of the origin of the coordinates. In polar coordinates we can write (I.15) as:

$$\begin{aligned} \dot\rho &= \rho(2\cos^2\theta + a\cos\theta\sin\theta + \sin^2\theta), \\ \dot\theta &= \cos\theta(\sin\theta - a\cos\theta). \end{aligned} \tag{3.16}$$

The singular points M_k of the system (3.16) with first coordinate $\rho = 0$ and the second coordinate $\theta \in [0, \pi)$, their characteristic values λ_1, λ_2 and their type are: $\{M_1(0, \frac{\pi}{2}), M_2(0, -\frac{\pi}{2})$: $\lambda_1 = -1; \lambda_2 = 1 -$ saddle;$\{M_3(0, \arctan a), M_3(0, \arctan a + \pi) : \lambda_1 = 1; \lambda_2 = 2 -$ unstable node. The behavior of the trajectories near $(0,0)$ is illustrated in Fig. 3.20.

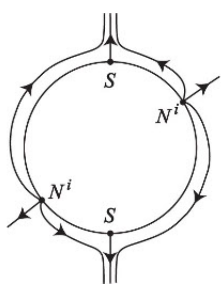

Figure 3.20a

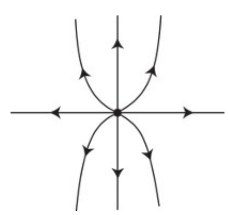

Figure 3.20b

Table 3.11

System (I.15) (Fig. 1.15)		
SP	λ_1; λ_2	TSP
$O_1(0,0)$	0; 0	P^uP^u
$X_\infty(1,0,0)$	$-2(a^2+1)$; a^2+1	S
$Y_\infty(0,1,0)$	-1; -1	DN^s

As all of the cases mentioned above are considered, therefore Theorem 1.1 is proved.

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