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DECAY OF NON-OSCILLATORY SOLUTIONS FOR A SYSTEM OF NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. In this article we study the non-oscillatory solutions for a system of neutral functional differential equation. We give sufficient conditions for all non-oscillatory solutions to tend to zero as t approaches infinity. Our results are illustrated with an example.

1. Introduction

In this article we study the non-oscillatory solutions to the homogeneous system of neutral differential equations

$$\begin{gathered}
\big[|y_1(t)-a(t)y_1(g(t))|^{\beta-1}\big(y_1(t)-a(t)y_1(g(t))\big)\big]'=p_1(t)y_2(t),\\
y_i'(t)=p_i(t)y_{i+1}(t),\quad i=2,3,\dots,n-1,\\
y_n'(t)=\sigma p_n(t)f(y_1(h(t))),\quad t\ge t_0,
\end{gathered}\tag{1.1}$$

where β is a positive constant, $n\ge 3$, $\sigma=\pm 1$, and a,g,h,f,p_i are continuous functions that satisfy the condition specified below.

Asymptotic properties of solutions to systems of functional differential equations with deviating arguments have been studied by many authors; see for example the references in this article and their references. When the coefficients p_i are positive, (1.1) can be written as n-order differential equation. In which case there are many results available, including for non-homogeneous and more general equations; see for example [2, 9].

The existence of oscillatory solutions to (1.1), with $\beta=1$, and such that $\lim_{t\to\infty} y_i(t)=0$ or $\lim_{t\to\infty}|y_i(t)|=\infty$ were established in [19]. The existence of non-oscillatory solutions to (1.1) has been shown among others by Marušiak [8] and Erbe-Kong-Zhang [1].

Non-oscillatory solutions for equation of the type (1.1), with $\beta=1$, have been grouped in to classes by Marušiak [7]. The authors in [18] expanded this classification, and used it for showing that if the function $y_1(t)$ is bounded, then non-oscillatory solutions decay to zero as $t\to\infty$. The goal in this article is to show the decay of non-oscillatory solutions, without such assumption.

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The results here are different from those in [2] in the sense that our coefficients p_i are allowed to have zeros, and our delayed arguments $g(t)$ and $h(t)$ are allowed to exceed t, while in [2] the delay arguments are bounded by t. However, in [2] the differential equation has a forcing term that (1.1) does not have. In this article, as in [18], we use the following assumptions:

- (A1) the coefficient $a : [t_0, \infty) \to (0, \infty]$ is a continuous function;
- (A2) the advanced arguments $g, h : [t_0, \infty) \to \mathbb{R}$ are continuous and strictly increasing functions, with $\lim_{t\to\infty} g(t) = \infty$ and $\lim_{t\to\infty} h(t) = \infty$;
- (A3) The coefficients $p_i : [t_0, \infty) \to [0, \infty)$ are continuous functions, p_n is not identically zero in any neighborhood of infinity, and $\int_{t_0}^{\infty} p_i(t)\,dt = \infty$ for $i = 1, 2, \dots, n-1$;
- (A4) the function $f : \mathbb{R} \to \mathbb{R}$ is continuous, with $uf(u) > 0$ for $u \neq 0$, and there is a positive constant M such that $|f(u)| \geq M|u|^{\beta}$.

The inverse of the functions in (A2) will be denoted by $g^{-1}(t)$ and $h^{-1}(t)$. For simplifying of notation, we define the function

$$z_1(t) = |y_1(t) - a(t)y_1(g(t))|^{\beta-1}\big(y_1(t) - a(t)y_1(g(t))\big). \tag{1.2}$$

Note if β is the quotient of odd integers, then $|x|^{\beta-1}x = x^{\beta}$. Also note that for $\beta > 0$ and x a differentiable function, $(|x|^{\beta-1}x)'$ and x' have the same sign. This is proved by considering the possible signs of x.

A function $y = (y_1, \dots, y_n)$ is a solution of (1.1) if there is a $t_1 \geq t_0$ such that y is continuous for $t \geq \min\{t_1, g(t_1), h(t_1)\}$; the functions $z_1(t)$ and $y_i(t)$, $i = 2, 3, \dots, n$ are continuously differentiable on $[t_1, \infty)$; and y satisfies (1.1) on $[t_1, \infty)$. In this article, we consider only solutions that are eventually non-trivial; i.e., solutions such that

$$\sup_{t\geq t_1} \max_{1\leq i\leq n} |y_i(t)| > 0\,.$$

A solution is non-oscillatory if there exist i and $T_y \geq t_0$ such that $y_i(t) \neq 0$ for all $t \geq T_y$. Otherwise, a solution y is said to be oscillatory.

2. Preliminaries

Our first lemma is a simplified version of [2, lemma 2.1], [6, lemma 5.2.1], [18, lemma 2.2].

Lemma 2.1. *Let $y = (y_1, \dots, y_n)$ be a solution of (1.1). Assume that (A3) holds and $y_n'(t)$ is eventually of one sign.*

- (i) *There exists $t_2 \geq t_0$ such that $z_1, y_2, \dots, y_n$ are monotonic and of constant sing on $[t_2, \infty)$.*
- (ii) *There exists an index ℓ such that $z_1, y_2, \dots, y_\ell$ have the same sign, and $y_\ell(t)y_{\ell+1}(t) < 0$, $y_{\ell+1}(t)y_{\ell+2}(t) < 0$, $\dots$, $y_n(t)y_n'(t) < 0$. When $z_1(t)y_2(t) < 0$ we set $\ell = 1$, and when $z_1, y_2, \dots, y_n'$ have the same sign, we set $\ell = n+1$.*

Lemma 2.2. *Under the assumptions on Lemma 2.1, for $s, t, x_k \in [t_2, \infty)$, we have: If there is a k in $\{2, 3, \dots, n\}$ for which $y_k(t)y_{k+1}(t) < 0$ (with $y_{n+1} = y_n'$), then*

$$\begin{aligned}
&|y_k(x_k)| \\
&\geq \int_{x_k}^{t} p_k(x_{k+1}) \int_{x_{k+1}}^{t} \dots \int_{x_{n-1}}^{t} p_{n-1}(x_n) \int_{x_n}^{t} |y_n'(x_{n+1})|\, dx_{n+1} \dots dx_{k+1} \qquad (2.1)\\
&:= J_{n+1-k}(x_k, t; p_k, \dots, p_{n-1}, 1; |y_n'|), \quad \forall x_k \leq t\,.
\end{aligned}$$

Note that t can be arbitrarily large, thus we can use the limit as $t \to \infty$. Also note that when $k = 1$, the above estimate has the form $|z_1(x_1)| \ge J_n(\dots)$.

If there is a k in $\{2, 3, \dots, n+1\}$ for which $z_1(t), y_2(t), \dots, y_k(t)$ have the same sign (with the convention $y_{n+1} = y_n'$), then

$$\begin{aligned} |z_1(t)| &\ge \int_s^t p_1(x_2) \int_s^{x_2} p_2(x_3) \dots \int_s^{x_{k-1}} p_{k-1}(x_k) |y_k(x_k)|\, dx_k \dots dx_2 \\ &:= I_{k-1}(s, t; p_1, \dots, p_{k-1}; |y_k|) \quad \forall s \le t\,. \end{aligned} \tag{2.2}$$

When $k = \ell$ as defined by Lemma 2.1, and $2 \le \ell \le n+1$, we have

$$\begin{aligned} &|z_1(t)| \\ &\ge \int_s^t p_1(x_2) \int_s^{x_2} \dots \int_s^{x_{\ell-1}} p_{\ell-1}(x_\ell) \int_{x_\ell}^t p_\ell(x_{\ell+1}) \\ &\quad \times \int_{x_{\ell+1}}^t p_{\ell+1}(x_{\ell+2}) \int_{x_{\ell+2}}^t \dots \int_{x_{n-1}}^t p_{n-1}(x_n) \int_{x_n}^t |y_n'(x_{n+1})|\, dx_{n+1} \dots dx_2 \\ &= I_{\ell-1}\Big(s, t; p_1, \dots, p_{\ell-1}; J_{n+1-\ell}\big(x_\ell, t; p_\ell, \dots, p_{n-1}, 1; |y_n'|\big)\Big), \quad \forall s \le t\,. \end{aligned} \tag{2.3}$$

Proof. Assuming that y_m and y_{m+1} have opposite signs, we have

$$\begin{aligned} |y_m(x_m)| &= |y_m(t)| + \int_{x_m}^t p_m(x_{m+1}) |y_{m+1}(x_{m+1})|\, dx_{m+1} \\ &\ge \int_{x_m}^t p_m(x_{m+1}) |y_{m+1}(x_{m+1})|\, dx_{m+1}. \end{aligned}$$

Inequality (2.1) follows by applying this inequality for $y_k, y_{k+1}, \dots, y_n$ (with the convention $y_{n+1} = y_n'$).

Now assume that z_1 and y_2 are of the same sign. Then

$$|z_1(t)| = |z_1(s)| + \int_s^t p_1(x_2) |y_2(x_2)|\, dx_2 \ge \int_s^t p_1(x_2) |y_2(x_2)|\, dx_2.$$

Using this inequality for $y_2, y_3, \dots y_k$, we obtain (2.2). When $k = \ell$ in the two inequalities above, we have (2.3). □

The functionals similar to I_k and J_k have been defined recursively in [7, 18].

Lemma 2.3 ([5, Lemma 2.2]). *Assume* (A1)–(A2) *hold, $g(t) > t$, and*

$$1 \le a(t) \quad \textit{for } t \ge t_0.$$

Let $y_1(t)$ be a continuous non-oscillatory solution to the functional inequality

$$y_1(t)[y_1(t) - a(t) y_1(g(t))] > 0$$

defined in a neighborhood of infinity. Then $y_1(t)$ is bounded. Moreover, if there exist a constant a_* such that

$$1 < a_* \le a(t), \quad \forall t \ge t_0,$$

then $\lim_{t\to\infty} y_1(t) = 0$.

Lemma 2.4 ([5, Lemma 2.1]). *Assume* (A1)–(A2) *hold, $g(t) < t$, and*

$$0 < a(t) \le 1 \quad \textit{for } t \ge t_0.$$

Let $y_1(t)$ be a continuous non-oscillatory solution to the functional inequality

$$y_1(t)[y_1(t) - a(t)y_1(g(t))] < 0$$

defined in a neighborhood of infinity. Then $y_1(t)$ is bounded. Moreover, if there is a constant a^* such that

$$0 < a(t) \le a^* < 1, \quad \forall t \ge t_0,$$

then $\lim_{t\to\infty} y_1(t) = 0$.

Lemma 2.5 ([11, Lemma 4]). *Assume that $q : [t_0, \infty) \to [0, \infty)$ and $\delta : [t_0, \infty) \to \mathbb{R}$ are continuous functions, with $\delta(t) > t$ for $t \ge t_0$, and*

$$\liminf_{t\to\infty} \int_t^{\delta(t)} q(s)\,\mathrm{d}s > \frac{1}{e}.$$

Then the functional inequality

$$x'(t) - q(t)x(\delta(t)) \ge 0, \quad t \ge t_0$$

has no eventually positive solution, and the functional inequality

$$x'(t) - q(t)x(\delta(t)) \le 0, \quad t \ge t_0$$

has no eventually negative solution.

The next Lemma can be proved as in [7, Lemma 2].

Lemma 2.6. *Let $y = (y_1, y_2, \dots, y_n)$ be a non-oscillatory solution of (1.1), and let $\lim_{t\to\infty} |z_1(t)| = L_1$, $\lim_{t\to\infty} |y_k(t)| = L_k$ for $k = 2, \dots, n$. For $k \ge 2$,*

$$L_k > 0 \implies L_i = \infty, \quad \text{for } i = 1, \dots, k-1\,. \tag{2.4}$$

For $1 \le k < n$,

$$L_k < \infty \implies L_i = 0, \quad \text{for } i = k+1, \dots, n. \tag{2.5}$$

3. Main results

Theorem 3.1. *Assume (A1)–(A4), and let $\sigma = (-1)^n$. Also assume the following conditions hold: there exist constants a_*, a^* such that*

$$1 < a_* \le a(t) \le a^*, \quad \text{for } t \ge t_0; \tag{3.1}$$

$$t < g(t) < h(t) \quad \text{for } t \ge t_0; \tag{3.2}$$

for all k in $\{3, 4, \dots, n\}$, the functionals defined by (2.1)-(2.2) satisfy

$$\begin{aligned}\limsup_{s\to\infty} I_{k-1}\Big(&s, g^{-1}(h(s)); p_1, \dots, p_{k-1}; J_{n+1-k}\big(x_k, g^{-1}(h(s)); p_k, \\ &\dots, p_n; \frac{M}{a^\beta(g^{-1}(h))}\big)\Big) > 1\,;\end{aligned} \tag{3.3}$$

$$\liminf_{s\to\infty} \int_s^{g^{-1}(h(s))} p_1(x_2) J_{n-1}\Big(x_2, \infty; p_2, \dots, p_n; \frac{M}{a^\beta(g^{-1}(h))}\Big)\,dx_2 > \frac{1}{e}\,. \tag{3.4}$$

Then every non-oscillatory solution of (1.1) decays to zero; i.e., $\lim_{t\to\infty} y_i(t) = 0$ for $i = 1, 2, \dots, n$.

Proof. Let y be a non-oscillatory solution of (1.1). Note that if $y(t)$ is solution of (1.1), then $-y(t)$ is also a solution; therefore, we assume that $y_1(t)$ is positive, without loss of generality. Then by (A4), y_n' is one sign and, by Lemma 2.1, each of the functions z_1, y_2, …is of one sign (positive or negative); thus we have only the following cases:

Case 1p: $z_1(t) > 0$ for all $t \geq t_2$, and no restriction on $y_2, y_3, \dots$. Since $z_1(t)$ is positive so is $y_1(t) - a(t)y_1(g(t))$. By Lemma 2.3, $\lim_{t\to\infty} y_1(t) = 0$. Then $\lim_{t\to\infty} z_1(t) = 0$, because a is bounded. Then by Lemma 2.6, $\lim_{t\to\infty} y_i(t) = 0$ for $i = 1, 2, \dots, n$.

Case 1n2p: $z_1(t) < 0$, $y_2(t) > 0$ for all $t \geq t_2$, and no restriction on $y_3, y_4, \dots$. Then by (2.1) we have $\ell = 1$, and $y_2, y_4, y_6, \dots$ are positive, while $y_3, y_5, y_7, \dots$ are negative. However, (A4), the choice $\sigma = (-1)^n$, and the fact that $y_1 > 0$ do not allow this case to happen. See Theorem 3.2 below.

Case 1n2n3n: $z_1(t) < 0$, $y_2(t) < 0$ $y_3(t) < 0$ for all $t \geq t_2$, and no restriction on $y_4, y_5, \dots$. Then $\ell \geq 3$ in Lemma 2.1. By (2.3)

$$\begin{aligned} z_1(t) &\leq -I_{\ell-1}\Big(s, t; p_1, \dots, p_{\ell-1}; J_{n+1-\ell}\big(x_\ell, t; p_\ell, \dots, p_n; |\sigma f(y_1(h))|\big)\Big) \\ &\leq I_{\ell-1}\Big(s, t; p_1, \dots, p_{\ell-1}; J_{n+1-\ell}\big(x_\ell, t; p_\ell, \dots, p_n; -M(y_1(h))\big)\Big) \end{aligned} \tag{3.5}$$

for all $s \leq t$. Since $z_1(t)$ is negative so is $y_1(t) - a(t)y_1(g(t))$. Therefore, $(-z_1(t))^\beta = a(t)y_1(g(t)) - y_1(t) < a(t)y_1(g(t))$, and $z_1(t) > -a^\beta(t)y_1^\beta(g(t))$. Then for $t = g^{-1}(h(x_{n+1}))$,

$$-y_1^\beta\big(g^{-1}(h(x_{n+1}))\big) < \frac{z_1(g^{-1}(h(x_{n+1}))}{a^\beta\big(g^{-1}(h(x_{n+1}))\big)}\,. \tag{3.6}$$

Applying this inequality and that z_1 is non-decreasing, in (3.5), we have

$$\begin{aligned} &z_1(t) \\ &\leq z_1\big(g^{-1}(h(s))\big) I_{\ell-1}\Big(s, t; p_1, \dots, p_{\ell-1}; J_{n+1-\ell}\big(x_\ell, t; p_\ell, \dots, p_n; M/a^\beta(g^{-1}(h))\big)\Big)\,. \end{aligned}$$

Since $g(t) < h(t)$ and g is strictly increasing, $s < g^{-1}(h(s))$; thus we can set $t = g^{-1}(h(s))$. Dividing by $z_1(t)$ we have a contradiction to (3.3). Therefore, this case can not happen.

Case 1n2n3p: $z_1(t) < 0$, $y_2(t) < 0$, $y_3(t) > 0$ for all $t \geq t_2$, and no restriction on $y_4, y_5, \dots$. Using (2.1) for y_2, we obtain

$$y_2(s) \leq -J_{n-1}(s, t; p_2, \dots, p_n; |\sigma f(y_1(h))|) \leq J_{n-1}(s, t; p_2, \dots, p_n; -y_1^\beta(h)M)$$

Applying (3.6) and that z_1 is non-increasing we have

$$y_2(s) \leq z_1\big(g^{-1}(h(s))\big) J_{n-1}\big(s, t; p_2, \dots, p_n; M/a^\beta(g^{-1}(h))\big) \quad \forall s \leq t;$$

therefore,

$$y_2(s) \leq z_1\big(g^{-1}(h(s))\big) J_{n-1}\big(s, \infty; p_2, \dots, p_n; M/a^\beta(g^{-1}(h))\big)\,.$$

Multiplying by $p_1(s)$ in both sides, we note that z_1 is a negative solution of the differential inequality

$$z_1'(s) - z_1\big(g^{-1}(h(s))\big) p_1(s) J_{n-1}\big(s, \infty; p_2, \dots, p_n; M/a^\beta(g^{-1}(h))\big) \leq 0\,.$$

Since $g(s) < h(s)$ and g is strictly increasing, $s < g^{-1}(h(s))$. This inequality is one of the conditions needed for applying Lemma 2.5. The other condition is

$$\liminf_{s\to\infty} \int_s^{g^{-1}(h(s))} p_1(x_2) J_{n-1}\big(x_2, \infty; p_1, \dots, p_n; M/a^\beta(g^{-1}(h))\big)\, dx_2 > \frac{1}{e}$$

which is provided by (3.4). The fact that z_1 is negative and is a solution of the differential inequality contradicts Lemma 2.5. Therefore, this case can not happen. The proof is complete. □

Next we remove the condition $\sigma = (-1)^n$ in Theorem 3.1, at the cost of restricting the coefficient p_n.

Theorem 3.2. *Assume* (A1)–(A4), (3.1)–(3.2), *and that p_n is bounded below by a positive constant. Then every non-oscillatory solution of* (1.1) *decays to zero; i.e.,* $\lim_{t\to\infty} y_i(t) = 0$ *for* $i = 1, 2, \dots, n$.

Proof. Let y be a non-oscillatory solution of (1.1), and without loss of generality assume that $y_1(t)$ is positive. The proofs of the various cases are the same as in Theorem 3.1, except for one case.

Case 1n2p: $z_1(t) < 0$, $y_2(t) > 0$ for all $t \ge t_2$. Then by (2.1) we have $\ell = 1$.

First we show that $\liminf_{t\to\infty} y_1(t) = 0$. The function y_n being monotonic and having its derivative with opposite sign imply the existence of $\lim_{t\to\infty} y_n(t)$. From (1.1) it follows that

$$\int_{t_2}^\infty p_n(t)\sigma f(y_1(h(t)))\, dt < \infty\,.$$

Recall that $|f(y)| \ge M|y|^\beta$ and that p_n is bounded below by a positive constant. Using a contradiction argument, we can show that $\liminf_{t\to\infty} y_1(t) = 0$. Then by (A2),

$$\liminf_{t\to\infty} y_1(g(t)) = 0, \quad \liminf_{t\to\infty} y_1(h(t)) = 0\,.$$

Next we show that $\lim_{t\to\infty} z_1(t) = 0$. Since z_1 is negative and non-decreasing, there exists L_1 such that $0 \ge L_1 = \lim_{t\to\infty} z_1(t) > -\infty$. Let $\{t_k\}$ be a sequence such that

$$\lim_{k\to\infty} y_1(g(t_k)) = \liminf_{t\to\infty} y_1(g(t)) = 0\,.$$

Since $z_1(t)$ is negative so is $y_1(t) - a(t)y_1(g(t))$. From y_1 being positive,

$$-(-z_1(t_k))^{1/\beta} + a(t_k)y(g(t_k)) = y_1(t_k) > 0.$$

In the limit as $k \to \infty$, and using that a is bounded function, we have

$$0 \ge -(-L_1)^{1/\beta} + 0 = \liminf_{k\to\infty} y_1(g(t_k)) \ge 0\,.$$

Thus $L_1 = 0$.

Next we show that $y_1(g)$ is bounded from above, which implies y_1 being bounded from above. Suppose that $y_1(g)$ is unbounded, then there is a sequence $\{t_k\}$ such that $\lim_{k\to\infty} y_1(g(t_k)) = \infty$, and $y_1(g(s)) \le y_1(g(t_k))$ for all $s \le t_k$. Since g is strictly increasing, $y_1(g(s)) \le y_1(g(t_k))$ for all s for which $g(s) \le g(t_k)$. By (A2), for each t_k, there exists an s such that $t_k = g(s)$. Then by (3.2), $t_k < g(t_k)$ and $y_1(t_k) = y_1(g(s)) \le y_1(g(t_k))$. From $z_1(t)$ and $y_1(t) - a(t)y_1(g(t))$ being negative, and (3.1),

$$-(-z_1(t_k))^{1/\beta} = y(t_k) - a(t_k)y_1(g(t_k)) \le (1 - a_*)y_1(g(t_k)) < 0\,.$$

In the limit as $k \to \infty$, the left-hand side approaches zero, while the right-hand side approaches $-\infty$. This contradiction shows that y_1 is bounded.

Next we show that $\limsup_{t\to\infty} y_1(t) = 0$. Let

$$\alpha := \limsup_{t\to\infty} y_1(t) = \limsup_{t\to\infty} y_1(g(t)) \geq 0,$$

and let $\{t_k\}$ be a sequence such that $\lim_{k\to\infty} y_1(g(t_k)) = \alpha$. Let us recall that $\lim_{k\to\infty} z_1(t_k) = 0$, $\liminf_{k\to\infty} a(t_k) \geq a_* > 1$, and

$$\liminf_{k\to\infty} y_1(t_k) \leq \limsup_{k\to\infty} y_1(t_k) \leq \limsup_{t\to\infty} y_1(t) = \alpha$$

From $z_1(t)$ and $y_1(t) - a(t)y_1(g(t))$ being negative, we have

$$-(-z_1(t_k))^{1/\beta} + a(t_k)y(g(t_k)) = y(t_k),$$

which by taking the limit inferior yields

$$0 + a_*\alpha \leq \liminf_{k\to\infty} y_1(t_k) \leq \alpha\,.$$

Since $a_* > 1$, the only choice for α is being zero.

Therefore, $\lim_{t\to\infty} y_1(t) = 0$. By Lemma 2.6, $\lim_{t\to\infty} y_i(t) = 0$ for $i = 2, 3, \dots, n$, which completes the proof. □

Theorem 3.3. *Assume* (A1)–(A4)*, and let* $\sigma = (-1)^{n+1}$. *Also assume the following conditions: there exist a constant* a^* *such that*

$$0 < a(t) \leq a^* < 1, \quad \text{for } t \geq t_0; \tag{3.7}$$

$$g(t) < t < h(t) \quad \text{for } t \geq t_0\,; \tag{3.8}$$

the functionals defined by (2.1) *satisfy*

$$\liminf_{s\to\infty} I_{k-1}\Big(s, h(s); p_1, \dots, p_{k-1}; J_{n+1-k}\big(x_k, t; p_k, \dots, p_n; M\big)\Big) < 1\,; \tag{3.9}$$

$$\liminf_{s\to\infty} \int_s^{h(s)} p_1(x_2) J_{n-1}\Big(x_2, \infty; p_2, \dots, p_n, M\Big)\, dx_2 > \frac{1}{e}\,. \tag{3.10}$$

Then every non-oscillatory solution of (1.1) *decays to zero; i.e.,* $\lim_{t\to\infty} y_i(t) = 0$ *for* $i = 1, 2, \dots, n$.

Proof. Let y be a non-oscillatory solution of (1.1), and without loss of generality, assume that $y_1(t)$ is positive. Then by (A4), y_n' is one sign and, by Lemma 2.1, each of the functions z_1, y_2, ... is of one sign (positive or negative); thus we have only the following cases:

Case 1n: $z_1(t) < 0$ for all $t \geq t_2$, and no restriction on $y_2, y_3, \dots$. Since $z_1(t)$ is negative, so is $y_1(t) - a(t)y_1(g(t))$. By Lemma 2.4, $\lim_{t\to\infty} y_1(t) = 0$. Then $\lim_{t\to\infty} z_1(t) = 0$, because a is bounded. Then by Lemma 2.6, $\lim_{t\to\infty} y_i(t) = 0$ for $i = 1, 2, \dots, n$.

Case 1p2n: $z_1(t) > 0$, $y_2(t) < 0$ for all $t \geq t_2$, and no restriction on $y_3, y_4, \dots$. Then by (2.1) we have $\ell = 1$, and $y_2, y_4, y_6, \dots$ are negative, while $y_3, y_5, y_7, \dots$ are positive. However, (A4), the choice $\sigma = (-1)^{n+1}$, and the fact that $y_1 > 0$ do not allow this case to happen. See Theorem 3.4 below.

Case 1p2p3p: $z_1(t) > 0$, $y_2(t) > 0$ $y_3(t) > 0$ for all $t \ge t_2$, and no restriction on $y_4, y_5, \dots$. Then $\ell \ge 3$ in Lemma 2.1. By (2.3)

$$\begin{aligned} z_1(t) &\ge I_{\ell-1}\Big(s, h(s); p_1, \dots, p_{\ell-1}; J_{n+1-\ell}\big(x_\ell, t; p_\ell, \dots, p_n; |\sigma f(y_1(h))|\big)\Big) \\ &\ge I_{\ell-1}\Big(s, h(s); p_1, \dots, p_{\ell-1}; J_{n+1-\ell}\big(x_\ell, t; p_\ell, \dots, p_n; M y_1^\beta(h)\big)\Big) \end{aligned} \tag{3.11}$$

for all $s \le t$. Since $z_1(t)$ is positive, so is $y_1(t) - a(t)y_1(g(t))$. Using the inequality $z_1^{1/\beta}(t) = y_1(t) - a(t)y_1(g(t)) < y_1(t)$, we have $z_1(t) < y_1^\beta(t)$. Using that z_1 is non-decreasing, in (3.11), we have

$$z_1(t) \ge z_1(h(s)) I_{\ell-1}\Big(s, h(s); p_1, \dots, p_{\ell-1}; J_{n+1-\ell}\big(x_\ell, t; p_\ell, \dots, p_n; M\big)\Big).$$

Since $s < h(s)$ we can set $t = h(s)$. Dividing by $z_1(t)$ we have a contradiction to (3.9). Therefore, this case can not happen.

Case 1p2p3n: $z_1(t) > 0$, $y_2(t) > 0$, $y_3(t) < 0$ for all $t \ge t_2$, and no restriction on $y_4, y_5, \dots$. Using (2.1) for y_2, we obtain

$$y_2(s) \ge J_{n-1}(s, t; p_2, \dots, p_n; |\sigma f(y_1(h))|) \ge J_{n-1}(s, t; p_2, \dots, p_n; y_1(h)M).$$

Using that $z_1(t)$ and $y_1(t) - a(t)y_1(g(t))$ are positive, we have the inequalities $(z_1(t))^{1/\beta} = y_1(t) - a(t)y_1(g(t)) < y_1(t)$ and $z_1(t) < y_1^\beta(t)$. Using that z_1 is non-decreasing we have

$$y_2(s) \ge z_1(h(s)) J_{n-1}\big(s, t; p_2, \dots, p_n; M)\big) \quad \forall s \le t\,;$$

therefore,

$$y_2(s) \ge z_1(h(s)) J_{n-1}\big(s, \infty; p_2, \dots, p_n; M)\big).$$

Multiplying by $p_1(s)$ in both sides, we note that z_1 is a positive solution of the differential inequality

$$z_1'(s) - z_1(h(s)) J_{n-1}\big(s, \infty; p_2, \dots, p_n; M\big) \ge 0.$$

Since $s < h(s)$, we have one of the conditions needed for applying Lemma 2.5. The other condition is

$$\liminf_{s\to\infty} \int_s^{h(s)} p_1(x_2) J_{n-1}\big(x_2, \infty; p_2, \dots, p_n; M\big)\, dx_2 > \frac{1}{e},$$

which is provided by (3.10). The fact that z_1 is positive and is a solution of the differential inequality contradicts Lemma 2.5. Therefore, this case can not happen. The proof is complete. □

Next we remove the condition $\sigma = (-1)^n$ in Theorem 3.3, but we need to restrict the coefficient p_n.

Theorem 3.4. *Assume* (A1)–(A4), (3.7)–(3.8)*, and that p_n is bounded below by a positive constant. Then every non-oscillatory solution of* (1.1) *decays to zero; i.e.,* $\lim_{t\to\infty} y_i(t) = 0$ *for* $i = 1, 2, \dots, n$.

Proof. Let y_1 be a non-oscillatory solution of (1.1), and without loss of generality assume that $y_1(t)$ is positive. The proofs of the various cases are the same as in Theorem 3.3, except for one case.

Case 1p2n: $z_1(t) > 0$, $y_2(t) < 0$ for all $t \ge t_2$. Then by (2.1) we have $\ell = 1$. Since $z_1(t)$ is positive, so is $y_1(t) - a(t)y_1(g(t))$. The proof of $\liminf_{t\to\infty} y_1(t) = 0$ is the same as in Theorem 3.3.

Next we show that $\lim_{t\to\infty} z_1(t) = 0$. Since z_1 is positive and non-increasing, $\lim_{t\to\infty} z_1(t)$ exists. From the inequalities $z_1^{1/\beta}(t) = y(t) - a(t)y(g(t)) < y_1(t_k)$ and $z_1(t) < y_1^{\beta}$, by taking the limit inferior,

$$0 \le \lim_{t\to\infty} z_1^{1/\beta}(t) \le \liminf_{t\to\infty} y_1(t) = 0\,.$$

Next we show that y_1 is bounded from above. Suppose that y_1 is unbounded, then there is a sequence $\{t_k\}$ such that $\lim_{k\to\infty} y_1(t_k) = \infty$, and $y(s) \le y(t_k)$ for all $s \le t_k$. In particular for $g(t_k) < t_k$, we have $y_1(g(t_k)) \le y_1(t_k)$, and

$$z_1^{1/\beta}(t_k) = y_1(t_k) - a(t_k)y_1(g(t_k)) \ge (1 - a^*)y_1(g(t_k)) > 0\,.$$

In the limit as $k \to \infty$, the left-hand side approaches zero, while the right-hand side approaches $+\infty$. This contradiction implies y_1 being bounded from above.

Next we show that $\limsup_{t\to\infty} y_1(t) = 0$. Let

$$\alpha := \limsup_{t\to\infty} y_1(t) = \limsup_{t\to\infty} y_1(g(t)) \ge 0,$$

and let $\{t_k\}$ be a sequence such that $\lim_{k\to\infty} y_1(t_k) = \alpha$. Note that $\lim_{k\to\infty} z_1(t_k) = 0$, $\limsup_{k\to\infty} a(t_k) \le a^* < 1$, and

$$\limsup_{k\to\infty} y_1(g(t_k)) \le \limsup_{t\to\infty} y_1(g(t)) = \alpha\,.$$

From $z_1(t)$ and $y(t) - a(t)y(g(t))$ being positive, we have $y_1(t_k) = z_1^{1/\beta}(t_k) + a(t_k)y(g(t_k))$, which by taking in the limit superior, yields

$$\alpha = \lim_{k\to\infty} y_1(t_k) \le 0 + a^* \limsup_{k\to\infty} y_1(gt_k)) \le a^*\alpha\,.$$

Since $a^* < 1$, the only choice for α is being zero. Therefore, $\lim_{t\to\infty} y_1(t) = 0$. By Lemma 2.6, $\lim_{t\to\infty} y_i(t) = 0$ for $i = 2, 3, \dots, n$, which completes the proof. $\square$

Example 3.5. To illustrate Theorem 3.1, we set $a(t) = 2$, $\beta = 1$, $f(y) = y$, $g(t) = 4t$, $h(t) = 8t$, $M = 1$, $n = 5$, $p_1(t) = t$, $p_2(t) = 3t$, $p_3(t) = 5t$, $p_4(t) = 7t$, $p_5 = 36t^{-9}$, and $\sigma = (-1)^5 = -1$. Then for $t \ge 1$, a solution of (1.1) has the form

$$y_1(t) = 2t^{-1}, \quad z_1(t) = y_2(t) = -1^{-3}, \quad y_3(t) = t^{-5}, \quad y_4(t) = -t^{-7}, \quad y_5(t) = t^{-9}.$$

Note that $z_1(t) = t^{-1}$, $g^{-1}(h(s)) = 2s$ and $p_5(x_6)M/a^{\beta}(g^{-1}(h(x_6))) = 18x_6^{-9}$. Then

$$\begin{aligned}
&\int_s^{2s} x_2 \int_{x_2}^{\infty} 3x_3 \int_{x_3}^{\infty} 5x_4 \int_{x_4}^{\infty} 7x_5 \int_{x_5}^{\infty} 18x_6^{-9}\,dx_6 \dots dx_2\\
&= \frac{18(1)(3)(5)(7)}{(2)(4)(6)(8)} \ln(2) > 1/e
\end{aligned}$$

which satisfies (3.4). To check (3.3), we compute the expression

$$I_{k-1}\Big(s, 2s; p_1, \dots, p_{k-1}; J_{6-k}\big(x_k, 2s; p_k,\, \dots, p_5; 1/2\big)\Big)$$

which has the following values: 2.00296 for $k = 2$, 6.17293 for $k = 3$, 14.8507 for $k = 4$, and 34.7885 for $k = 5$. Clearly all the conditions for Theorem 3.1 are satisfied and the solution decays to zero as $t \to \infty$.

We remark that the results in Theorems 3.1–3.4 when the coefficient $a(t)$ crosses, or approaches, the value 1 remains an open question. On the other hand, Theorems 3.1–3.4 can easily be extended to difference equation and to time scales; see the extensions indicated in [2].

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