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SOLITON SOLUTIONS FOR A QUASILINEAR SCHRÖDINGER EQUATION

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Abstract. In this article, critical point theory is used to show the existence of nontrivial weak solutions to the quasilinear Schrödinger equation

$$-\Delta_p u - \frac{p}{2^{p-1}} u\Delta_p(u^2) = f(x,u)$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with Dirichlet boundary conditions.

1. Introduction

In this article, we study the soliton solutions for the quasilinear Schrödinger equation

$$\begin{gathered} -\Delta_p u - \frac{p}{2^{p-1}} u\Delta_p(u^2) = f(x,u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < N$.

When $p = 2$, equation (1.1) is a special case for some physical phenomena, see e.g. [17, 19, 24]. In fact, solutions for the problem (1.1) for $p = 2$ are the existence of standing wave solutions for the following quasilinear Schrödinger equations

$$i\partial_t z = -\Delta z + Wz - f(|z|^2)z - \kappa\Delta h(|z|^2)h'(|z|^2)z, \tag{1.2}$$

where $W(x)$, $x \in \mathbb{R}^N$ is a given potential, κ is a real constant and f, h are real functions of essentially pure power forms. The semi linear case corresponding to $\kappa = 0$ has been studied extensively in recent years, see [2, 9, 29]. Quasilinear equations of form (1.2) appear more naturally in mathematical physics and have been derived as model of several physical phenomena corresponding to various types of h, the superfluid film equation in plasma physics by Kurihara in [13] for $h(s) = s$. In the case $h(s) = (1+s)^{1/2}$, (1.2) models the self-channeling of a high-power ultra short laser in matter, see [3, 4, 5, 27] and the references in [7]. Equation (1.2) also appears in plasma physics and fluid mechanics [13, 14, 16, 23, 25], in the theory of Heisenberg ferromagnets and magnons [1, 12, 15, 26, 30], in dissipative quantum mechanics [11], and in condensed matter theory [22]. In the mathematical literature very few results are known about equations of the form (1.2) before Liu's research

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team [19, 17], in which, the existence of positive solution has been proved in [19] by using a constrained minimization argument. It is worthy of attention that another earlier paper [18] deals with a more general type equations without using the change of variables developed in the later literature. The problem (1.2) was transformed into a semilinear one by a change of variables and an Orlicz space framework was used in [17]. Since then several papers appear in the mathematical literature for the equation defined in the domain $\mathbb{R}^N$. For example, see [6, 10, 31, 20, 28] and a very recent paper [8], in which the authors established the existence of ground states of soliton type solutions by a minimization argument. But to our best knowledge, there is no one considering this problem for the p-Laplacian case in a bounded domain.

We consider soliton solutions for the following quasilinear Schrödinger equations of a more general form than (1.2), in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with the Dirichlet boundary condition

$$i\partial_t z = -\Delta_p z + Wz - f(|z|^2)z - \kappa\Delta_p h(|z|^2)h'(|z|^2)z,$$

in which $\kappa = \frac{p}{2^{p-1}} > 0$, $h(s) = s$ and $f = f(x,s)$ is a Caratheodory function under some power growth with respect to s. At the same time we assume $W(x) \equiv W$ (a constant) to indicate that the solution stays at a constant potential level. Putting $z(x,t) = \exp(-iWt)u(x)$ we obtain the corresponding equation (1.1) of elliptic type which has a formal variational structure, see Section 2.

For a deep insight into this problem one can find that a major difficulty of the problem (1.1) is that the functional corresponding to the equation is not well defined for all $u \in W_0^{1,p}(\Omega)$ if $N \geq p$. We generalized the method of a change of variables developed in [6] to overcome this difficulty, and make a slight different definition of weak solutions. Then by a standard argument from critical point theory, we develop the existence of nontrivial solutions to our problem.

This article is organized as follows. In Section 2, we give the definition of our weak solutions for our problem; in section 3, we give some existence theorems of solutions and some remarks for our theorems.

2. Definition of weak solution

We assume the perturbation $f(x,t)$ is a Caratheodory function. Firstly we introduce a variational framework of problem (1.1). Under some increasing conditions on f about the item u, we observe that (1.1) is the Euler-Lagrange equation associated with the energy functional

$$J(u) := \frac{1}{p}\int_\Omega (1 + p|u|^p)|\nabla u|^p \,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x, \tag{2.1}$$

where $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}x$.

It is difficult to apply variational methods to the functional J directly. Unless $N = 1$, the functional J is not well-defined for all $u \in W_0^{1,p}(\Omega)$. To overcome this difficulty, we generalized the method of changing variables developed in [6, 17]. That is

$$v := g^{-1}(u),$$

where g is defined by

$$g'(t) = \frac{1}{(1 + p|g(t)|^p)^{1/p}}, \quad \forall t \in [0, +\infty];$$

$$g(t) = -g(-t), \quad \forall t \in (-\infty, 0].$$

We summarize the properties of g as following.

Lemma 2.1. *The function g defined above satisfies the following conditions:*

(1) $g(0) = 0$;
(2) g *is uniquely defined, C^∞ and invertible;*
(3) $0 < g'(t) \le 1$ *for all* $t \in \mathbb{R}$;
(4) $\frac{1}{2} g(t) \le t g'(t) \le g(t)$ *for all* $t > 0$;
(5) $g(t)/t \nearrow 1$, *as* $t \to 0+$;
(6) $|g(t)| \le |t|$ *for all* $t \in \mathbb{R}$;
(7) $g(t)/\sqrt{t} \nearrow K_0 := \sqrt{2} p^{-1/(2p)}$, *as* $t \to +\infty$;
(8) $|g(t)| \le K_0 |t|^{1/2}$ *for all* $t \in \mathbb{R}$;
(9) $g^2(t) - g(t)g'(t)t \ge 0$ *for all* $t \in \mathbb{R}$;
(10) *There exists a positive constant C such that $|g(t)| \ge C|t|$ for $|t| \le 1$ and $|g(t)| \ge C|t|^{1/2}$ for $|t| \ge 1$;*
(11) $|g(t)g'(t)| < K_0^2$ *for all* $t \in \mathbb{R}$;
(12) $g''(t) < 0$ *when* $t > 0$ *and* $g''(t) > 0$ *when* $t < 0$.

Proof. The conclusions (1), (2) and (3) are trivial. To establish the left hand side of inequality (4), we need to show that, for all $t \ge 0$,

$$(1 + p|g(t)|^p)^{1/p} g(t) \le 2t.$$

To prove this we study the function $h : \mathbb{R}^+ \to \mathbb{R}$, defined by

$$h(t) := 2t - (1 + p|g(t)|^p)^{1/p} g(t).$$

We have $h(0) = 0$, and since $g'(t)(1 + p|g(t)|^p)^{1/p} = 1$ for all $t \in \mathbb{R}$, we have

$$h'(t) = |g'(t)|^p \ge 0.$$

Hence the left hand side inequality is proved. The right hand side inequality can be proved in a similar way.

It is easy to get (5) and (6) by (4). We give the proof of (7) by the Principle of L'Hospital. In fact, since $g(t) \to +\infty$ as $t \to +\infty$, we have

$$\begin{aligned}
\lim_{t\to+\infty} \frac{g(t)}{t^{1/2}} &= \lim_{t\to+\infty} \Big(\frac{(g(t))^2}{t}\Big)^{1/2} = \Big(\lim_{t\to+\infty} \frac{(g(t))^2}{t}\Big)^{1/2} \\
&= \Big(\lim_{t\to+\infty} \frac{2g(t)g'(t)}{t'}\Big)^{1/2} \\
&= \Big(\lim_{t\to+\infty} \frac{2g(t)}{(1 + p|g(t)|^p)^{1/p}}\Big)^{1/2} \\
&= \Big(\frac{2}{p^{1/p}}\Big)^{1/2} = K_0.
\end{aligned}$$

Then (7) is proved by (4). It is easy to get (8) by (4).

We can get (9) from (4). Inequalities in (10) are trivial and (11) is from (4) and (8).

For (12), it is easy to see

$$g''(t) = -p\big(1 + p|g(t)|^p\big)^{-\frac{1}{p}-1} |g(t)|^{p-2} g(t) g'(t).$$

So the conclusion of (12) is true. $\square$

We assume the following conditions on f:

(F1) $|f(x,t)| \le C(1+|t|^{2q-1})$ holds for some positive constant C, all $x \in \Omega$ and $t \in \mathbb{R}$, where $1 \le q < p^* := \frac{Np}{N-p}$.

Under condition (F1), consider the functional

$$\Phi(v) := \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x - \int_\Omega F(x, g(v))\,\mathrm{d}x. \tag{2.2}$$

Then Φ is well defined on the space $W_0^{1,p}(\Omega)$ (equipped with the norm $\|v\| := (\int_\Omega |\nabla v|^p \,\mathrm{d}x)^{1/p}$), and $\Phi \in C^1(W_0^{1,p}(\Omega);\mathbb{R})$ by assumption (F1) and Lemma 2.1. Thus for all $w \in W_0^{1,p}(\Omega)$, we have

$$\langle \Phi'(v), w\rangle = \int_\Omega |\nabla v|^{p-2}\nabla v \nabla w \,\mathrm{d}x - \int_\Omega f(x, g(v))g'(v) w\,\mathrm{d}x,$$

where $\langle \cdot,\cdot\rangle$ is the duality pairing between $W_0^{1,p}(\Omega)$ and $(W_0^{1,p}(\Omega))^*$. Then the critical points of Φ are weak solutions (in the usual sense) for the problem

$$\begin{gathered} -\Delta_p v = f(x, g(v))g'(v), \quad \text{in } \Omega, \\ v = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{2.3}$$

By setting $v = g^{-1}(u)$, it is easy to see that equation (2.3) is equivalent to our problem (1.1), which takes $u = g(v)$ as its solution.

Motivated by the above, we give the following definition of the weak solution for problem (1.1).

Definition 2.2. We say u is a weak solution for (1.1), if $v = g^{-1}(u) \in W_0^{1,p}(\Omega)$ is a critical point of the following functional corresponding to problem (2.3):

$$\Phi(v) := \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x - \int_\Omega F(x, g(v))\,\mathrm{d}x.$$

3. Existence of weak solutions

For simplicity, we make a use of the following notation. X denotes Sobolev space $W_0^{1,p}(\Omega)$ with the norm $\|\cdot\| := (\int_\Omega |\nabla \cdot|^p \,\mathrm{d}x)^{1/p}$; X^* denotes the conjugate space for X; $L^p(\Omega)$ denotes Lebesgue space with the norm $|\cdot|_p$; $\langle\cdot,\cdot\rangle$ is the dual pairing on the space X^* and X; by $\to$ (resp. $\rightharpoonup$) we mean strong (resp. weak) convergence. $|\Omega|$ denotes the Lebesgue measure of the set $\Omega \subset \mathbb{R}^N$; $C, C_1, C_2, \dots$ denote (possibly different) positive constants.

Let $\varphi(v) := \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x$ for all $v \in X$. It is obvious that the functional φ is a continuously Gâteaux differentiable whose Gâteaux derivative at the point $v \in X$ is the functional $\varphi'(u) \in X^*$, given by

$$\langle \varphi'(v), u\rangle = \int_\Omega |\nabla v|^{p-2}\nabla v \nabla u \,\mathrm{d}x;$$

Let $\mathcal{F}(v) = \int_\Omega F(x, g(v))\,\mathrm{d}x$. Then by the notation of section 2, $\Phi(v) = \varphi(v) - \mathcal{F}(v)$. It is well known that the following lemma holds for the functional φ.

Lemma 3.1. (i) *$\varphi' : X \to X^*$ is a continuous and strictly monotone operator;*
(ii) *φ' is a mapping of type (S_+), i.e. if $v_n \rightharpoonup v$ in X and $\limsup_{n\to+\infty}\langle \varphi'(v_n) - \varphi'(v), v_n - v\rangle \le 0$, then $v_n \to v$ in X;*
(iii) *$\varphi'(v) : X \to X^*$ is a homeomorphism;*
(iv) *φ is weakly lower semicontinuous.*

If f is independent of u, we have the following result.

Theorem 3.2. *If $f(x,u) = f(x), f \in L^r(\Omega)$ in which $\frac{1}{r} + \frac{1}{p^*} < 1$, then* (1.1) *has a unique weak solution.*

Proof. It is clear that $(f,u) := \int_\Omega f(x)u\,\mathrm{d}x, \quad \forall u \in X$ defines a continuous linear functional on X. By Lemma 3.1 (iii), (1.1) has a unique weak solution. $\square$

Next we assume the following conditions on f,

(F2) There exists $p^* > \theta > p$, $M > 0$ such that $|t| \geq M$ implies
$$0 < \theta F(x,t) \leq \frac{1}{2} t f(x,t).$$
(F3) $f(x,t) = o(|t|^{p-1}), t \to 0$, for $x \in \Omega$ uniformly.
(F4) $f(x,-t) = -f(x,t), x \in \Omega, t \in \mathbb{R}$.

Lemma 3.3. *Under assumption* (F1),
(i) *the functional $\mathcal{F}$ is sequentially weak-strong continuous, i.e., $v_n \rightharpoonup v$ in X implies $\mathcal{F}(v_n) \to \mathcal{F}(v)$;*
(ii) *$\mathcal{F}'(v_n) \to \mathcal{F}(v)$ in X^* as $v_n \rightharpoonup v$ in X.*

Proof. (i) By (F1) and Lemma 2.1, we have
$$|F(x,g(t))| = \int_0^{g(t)} |f(x,s)|\,\mathrm{d}s \leq C(1 + |g(t)|^{2q}) \leq C(1 + |t|^q).$$
Then the Caratheodory mapping $F(x,g(\cdot)) : L^q(\Omega) \to L^1(\Omega)$ is continuous. Since $v_n \rightharpoonup v$ in X, by the Sobolev compact imbedding, it is east to see $v_n \to v$ in $L^q(\Omega)$. Then $F(x, v_n(x)) \to F(x, v(x))$ in $L^1(\Omega)$, which means $\mathcal{F}(v_n) \to \mathcal{F}(v)$.

(ii) By (F1) and Lemma 2.1, we have
$$|f(x,g(t))g'(t)| \leq C(1 + |g(t)|^{2q-1}|g'(t)|) \leq C(1 + |g(t)|^{2q-2}) \leq C(1 + |t|^{q-1}).$$
Hence, the mapping $L^q(\Omega) \to L^{q'}(\Omega)$: $v \mapsto f(x,g(v))g'(v)$ is continuous. Then it is easy to see that $\mathcal{F} \in C^1(X)$ and $\mathcal{F}' : X \to X^*$ defined by
$$\langle \mathcal{F}'(v), u\rangle = \langle \mathcal{F}'(v), u\rangle_{L^{q'},L^q} = \int_\Omega f(x, g(v(x)))g'(v(x))u(x)\,\mathrm{d}x,$$
for all $v, u \in X \subset L^q(\Omega)$, is completely continuous. In fact, we have the following decomposition for the operator
$$\mathcal{F}' : X \xrightarrow{i} L^\Psi(\Omega) \xrightarrow{f(x,g(\cdot))g'(\cdot)} L^{\Psi'}(\Omega) \xrightarrow{j} (L^\Psi(\Omega))^* \xrightarrow{k} X^*,$$
i.e.,
$$\mathcal{F}'(u) = k \circ j \circ f \circ i(u), \quad \forall u \in X,$$
in which, i is compact, j is homeomorphic, and k means restriction on X^* of functionals in $(L^\Psi(\Omega))^*$. Then it is clear that $\mathcal{F}'$ is completely continuous. $\square$

Remark 3.4. Under assumption (F1), by lemma 3.3 and Lemma 3.1, we know that $\Phi' = \varphi' - \mathcal{F}'$ is of type (S_+).

Theorem 3.5. *If* (F1) *holds and $q < p$, then* (1.1) *has a weak solution.*

Proof. By (F1), we have the estimate
$$|F(x,t)| \leq C(1 + |t|^{2q}).$$

Then Φ is coercive because of the inequality

$$\begin{aligned}\Phi(v) &= \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x - \int_\Omega F(x, g(v))\,\mathrm{d}x \\ &\geq \frac{1}{p}\|v\|^p - C\int_\Omega |g(v)|^{2q}\,\mathrm{d}x - C \\ &\geq \frac{1}{p}\|v\|^p - C\int_\Omega |v|^q\,\mathrm{d}x - C \\ &\geq \frac{1}{p}\|v\|^p - C|v|_q^q - C \\ &\geq \frac{1}{p}\|v\|^p - C\|v\|^q - C \to +\infty, \quad \text{as } \|v\| \to +\infty.\end{aligned}$$

By Lemma 3.1 and Lemma 3.3, it is easy to verify that Φ is weakly lower semicontinuous. Then Φ has a minimum point v in X and v is a weak solution of (1.1), which completes the proof. □

Lemma 3.6. *Under assumptions* (F1) *and* (F2)*, the functional* Φ *satisfies the (PS) condition.*

Proof. Suppose that $\{v_n\} \subset X$, $|\Phi(v_n)| \leq B$ for some $B \in \mathbb{R}$, and $\Phi'(v_n) \to 0$ in X^* as $n \to \infty$.

After integrating, we obtain from the assumption (F2) that there exists C_1 such that

$$C_1(|t|^{2\theta} - 1) \leq F(x, t) \quad \forall x \in \Omega, t \in \mathbb{R}. \tag{3.1}$$

Let $c := \sup_n \Phi(v_n)$ and $\beta \in (\frac{1}{\theta}, \frac{1}{p})$ for large n. From Lemma 2.1 (4) and (10), (F6) and the inequality (3.1) we have

$$\begin{aligned}&c + 1 + \|v_n\| \\ &\geq \Phi(v_n) - \beta\langle \Phi'(v_n), v_n\rangle \\ &= \frac{1}{p}\|v_n\|^p - \beta\|v_n\|^p + \int_\Omega \big(\beta f(x, g(v_n))g'(v_n)v_n - F(x, g(v_n))\big)\,\mathrm{d}x \\ &\geq \frac{1}{p}\|v_n\|^p - \beta\|v_n\|^p + \int_\Omega \Big(\frac{1}{2}\beta f(x, g(v_n))g(v_n) - F(x, g(v_n))\Big)\,\mathrm{d}x \\ &\geq \Big(\frac{1}{p} - \beta\Big)\|v_n\|^p + (\theta\beta - 1)\int_\Omega F(x, g(v_n))\,\mathrm{d}x \\ &\geq \Big(\frac{1}{p} - \beta\Big)\|v_n\|^p + C_1(\theta\beta - 1)\int_\Omega |g(v_n)|^{2\theta}\,\mathrm{d}x - C_3 \\ &\geq \Big(\frac{1}{p} - \beta\Big)\|v_n\|^p + C_2(\theta\beta - 1)|v_n|_\theta^\theta - C_3, \\ &\geq \Big(\frac{1}{p} - \beta\Big)\|v_n\|^p - C_3,\end{aligned}$$

noticing that $\frac{1}{p} - \beta > 0$, and $\theta\beta - 1 > 0$, we obtain the boundedness of $\{v_n\}$ in X. Without of loss of generality, we assume $v_n \rightharpoonup v$, then $\langle \Phi'(v_n) - \Phi'(v), v_n - v\rangle \to 0$. Since Φ' is of type (S_+), we have $v_n \to v$ in X. □

Theorem 3.7. *Under assumption* (F1), (F2), (F3) *and* $q > p$*, problem* (1.1) *has a nontrivial solution.*

Proof. We will show that the functional Φ satisfies the Mountain Pass Theorem. By Lemma 3.6, Φ satisfies (PS) condition in X. Since $p < q < p^*$, $X \subset L^p(\Omega)$; i.e., there exists a $C > 0$ such that

$$|v|_p \leq C\|v\|, \quad \forall v \in X.$$

By assumption (F3) and Lemma 2.1, for small $\epsilon > 0$, we have

$$F(x, g(t)) \leq \epsilon |g(t)|^p + C|g(t)|^{2q-1} \leq \epsilon |t|^p + C|t|^q, \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

So we have

$$\begin{aligned}
\Phi(v) &\geq \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x - \epsilon \int_\Omega |v|^p \,\mathrm{d}x - C\int_\Omega |v|^q \,\mathrm{d}x \\
&\geq \frac{1}{p}\|v\|^p - C\|v\|^p - C\|v\|^q \\
&\geq \frac{1}{2p}\|v\|^p - C\|v\|^q, \text{ when } \|v\| \leq 1.
\end{aligned}$$

So there exist $r > 0$ and $\delta > 0$ such that $\Phi(v) \geq \delta > 0$ for every $\|v\| = r$.

From the assumption (F2) and Lemma 2.1, there exists a constant $C_1 > 0$ such that

$$F(x, g(t)) \geq C_1 |g(t)|^{2\theta} \geq C_2 |t|^\theta, \quad \text{for } |t| \geq M.$$

For $w \in X\backslash\{0\}$ and $t > 1$, in view of the above in equality, we have

$$\begin{aligned}
\Phi(tw) &= \frac{1}{p}\int_\Omega |t\nabla w|^p \,\mathrm{d}x - \int_\Omega F(x, tw)\,\mathrm{d}x \\
&\leq Ct^p\|w\|^p - C\int_\Omega |tw|^\theta \,\mathrm{d}x - C \\
&\leq Ct^p\|w\|^p - Ct^\theta |w|_\theta^\theta - C \to -\infty, \quad \text{as } t \to +\infty.
\end{aligned}$$

Obviously we have $\Phi(0) = 0$, so Φ satisfies the geometry conditions of the Mountain Pass Theorem in [32]. Then Φ admits at least one nontrivial critical which corresponds to the weak solution of (1.1). □

Thanks to Lemma 2.1, the translation g is strictly increasing and g is odd, which means that the functional $\mathcal{F}$ is even. This allows us to make an application of Fountain theorem and Dual Fountain theorem to obtain infinitely many solutions to (1.1).

Theorem 3.8. *Let* (F1), (F2), (F4) *hold and* $p^* > q > p$, *then* (1.1) *has a sequence of weak solutions* $\{\pm u_k\}_{k=1}^\infty$ *such that* $\Phi(\pm u_k) \to +\infty$ *as* $k \to +\infty$.

We will use the fountain theorem to prove Theorem 3.8. Since X is a reflexive and separable Banach space, there exist $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\operatorname{span}\{e_j : j = 1, 2, \dots\}}, \quad X^* = \overline{\operatorname{span}\{e_j^* : j = 1, 2, \dots\}}$$

in which

$$\langle e_i, e_j^* \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

We will write $X_j = \operatorname{span}\{e_j\}$, $Y_k = \oplus_{j=1}^k X_j$, $X_k = \overline{\oplus_{j=k}^\infty X_j}$.

Lemma 3.9 ([32]). *Let $q < p^*$, and denote*

$$\beta_k = \sup\{|v|_q : \|v\| = 1, v \in Z_k\}.$$

Then $\lim_{k\to+\infty} \beta_k = 0$.

Next, we have the Fountain Theorem, see [32].

Lemma 3.10. *Assume*

(A1) *X is a Banach space, $\Phi \in C^1(X, \mathbb{R})$ is an even functional.*

For each $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that

(A2) $\inf_{v\in Z_k,\|v\|=r_k} \Phi(v) \to +\infty$ *as* $k \to +\infty$.

(A3) $\max_{v\in Y_k,\|v\|=\rho_k} \Phi(v) \le 0$

(A4) *Φ satisfies $(PS)_c$ condition for every $c > 0$.*

Then Φ admits a sequence of critical values tending to $+\infty$.

Proof of Theorem 3.8. By assumption (F4) and since the translation of g defined in section 2 is odd and increasing, $\mathcal{F}$ is even, which implies $\Phi = \varphi - \mathcal{F}$ is also even. Further more, by Lemma 3.6, Φ satisfies the $(PS)_c$ condition. We need only to prove that there exist $\rho_k > r_k > 0$ such that condition (A2) and (A3) in Lemma 3.10 hold.

(A2) Let $v \in Z_k$, $\|v\| = r_k := (C_1 q K_0^{2q} \beta_k^q)^{1/(p-q)}$, in which K_0 is the same one in Lemma 2.1. By (F1) and Lemma 2.1, we have

$$\begin{aligned}
\Phi(v) &= \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x - \int_\Omega F(x, g(v))\,\mathrm{d}x \\
&\ge \frac{1}{p}\|v\|^p - C_1\int_\Omega |g(v)|^{2q}\,\mathrm{d}x - C_2 \\
&\ge \frac{1}{p}\|v\|^p - C_1 K_0^{2q}\int_\Omega |v|^q\,\mathrm{d}x - C_2 \\
&\ge \frac{1}{p}\|v\|^p - C_1 K_0^{2q}\beta_k^q \|v\|^q\,\mathrm{d}x - C_2 \\
&= \frac{1}{p}(C_1 q K_0^{2q}\beta_k^q)^{\frac{p}{p-q}} - C_1 K_0^{2q}\beta_k^q (C_1 q K_0^{2q}\beta_k^q)^{\frac{q}{p-q}} - C_2 \\
&= \Big(\frac{1}{p} - \frac{1}{q}\Big)(C_1 q K_0^{2q}\beta_k^q)^{\frac{p}{p-q}} - C_2 \to +\infty, \quad \text{as } k \to +\infty
\end{aligned}$$

since $p^* > q > p$ and $\beta_k \to 0$.

(A3) From assumption (F2) and Lemma 2.1, there exists a constant $C_1 > 0$ such that

$$F(x, g(t)) \ge C_1|g(t)|^{2\theta} \ge C_2|t|^\theta, \quad \text{for } |t| \ge M.$$

For any $w \in Y_k$ with $\|w\| = 1$ and $\rho_k = t > 1$, we have

$$\begin{aligned}
\Phi(tw) &= \frac{1}{p}\int_\Omega |t\nabla w|^p\,\mathrm{d}x - \int_\Omega F(x, g(tw))\,\mathrm{d}x \\
&\le \frac{1}{p}\|tw\|^p - C\int_\Omega |tw|^\theta\,\mathrm{d}x + C \\
&\le \frac{t^p}{p} - Ct^\theta |w|_\theta^\theta + C.
\end{aligned}$$

Since all norms in a finite dimensional space Y_k are equivalent, we have $\Phi(tw) \to -\infty$ by $\theta > p$. The conclusion of Theorem 3.8 is obtained by Lemma 3.10. $\square$

Also by the fine properties of the g, we give the solution existence result for that the nonlinear term is "concave and convex nonlinearities" by Dual Fountain Theorem. More precisely we have the following theorem.

Theorem 3.11. *Assume $\gamma, \beta > 0$ such that $p < \gamma < p^*$, $\beta < p$ and $f(x,t) = \lambda|t|^{2\gamma-2}t + \delta|t|^{2\beta-2}t$. Then*

(i) *for every $\lambda > 0$, $\delta \in \mathbb{R}$, (1.1) has a sequence of weak solutions $\{v_k\}_{k=1}^{+\infty}$, such that $\Phi(\pm v_k) \to +\infty$ as $k \to +\infty$;*

(ii) *for every $\delta > 0$, $\lambda \in \mathbb{R}$, (1.1) has a sequence of weak solutions $\{w_k\}_{k=1}^{+\infty}$, such that $\Phi(\pm w_k) \to 0$ as $k \to +\infty$.*

To prove Theorem 3.11, we will need the following "Dual Fountain Theorem", see [32].

Lemma 3.12. *Assume* (A1) *is satisfied, and there is a $k_0 > 0$ such that for each $k \geq k_0$, there exist $\rho_k > r_k > 0$ such that*

(B1) $\inf_{v\in Z_k, \|v\|=\rho_k} \Phi(v) \geq 0$.
(B2) $b_k := \max_{v\in Y_k, \|v\|=r_k} \Phi(v) < 0$.
(B3) $d_k := \inf_{v\in Z_k, \|v\|\leq\rho_k} \Phi(v) \to 0$ *as* $k \to +\infty$.
(B4) Φ *satisfies the* $(PS)_c^*$ *condition for every* $c \in [d_{k_0}, 0)$.

Then Φ has a sequence of negative critical values converging to 0.

Definition 3.13. We say that Φ satisfies the $(PS)_c^*$ condition with respect to $\{Y_n\}_{n=1}^{\infty}$, if any sequence $\{v_{n_j}\} \subset X$ such that $n_j \to +\infty, v_{n_j} \in Y_{n_j}$, $\Phi(v_{n_j}) \to c$ and $\Phi|'_{Y_{n_j}}(v_{n_j}) \to 0$, contains a subsequence converging to a critical point of Φ.

Proof of Theorem 3.11. The proof of this part (i) is similar to that of Theorem 3.8, if we specify $f(x,t) := \lambda|t|^{2\gamma-2}t + \delta|t|^{2\beta-2}t$ and $F(x,t) := \frac{\lambda}{2\gamma}|t|^{2\gamma} + \frac{\delta}{2\beta}|t|^{2\beta}$. We only verify the (PS) condition here. Suppose

$$\{v_n\} \subset X, \quad |\Phi(v_n)| \leq C, \quad \Phi'(v_n) \to 0 \quad \text{as } n \to +\infty.$$

for $\|v\| > 1$ and large n, by Lemma 2.1, we have

$$\begin{aligned}
c + 1 + \|v_n\| &\geq \Phi(v_n) - \frac{1}{\gamma}\langle \Phi'(v_n), v_n\rangle \\
&= \frac{1}{p}\|v_n\|^p - \frac{1}{\gamma}\|v_n\|^p + \int_\Omega \big(\frac{1}{\gamma} f(x, g(v_n))g'(v_n)v_n - F(x, g(v_n))\big)\,\mathrm{d}x \\
&\geq \frac{1}{p}\|v_n\|^p - \frac{1}{\gamma}\|v_n\|^p + \int_\Omega \Big(\frac{1}{2}\frac{1}{\gamma} f(x, g(v_n))g(v_n) - F(x, g(v_n))\Big)\,\mathrm{d}x \\
&\geq \Big(\frac{1}{p} - \frac{1}{\gamma}\Big)\|v_n\|^p + \frac{\delta}{2}(\frac{1}{\gamma} - \frac{1}{\beta})\int_\Omega |g(v_n)|^{2\beta}\,\mathrm{d}x \\
&\geq \Big(\frac{1}{p} - \frac{1}{\gamma}\Big)\|v_n\|^p - C_1\int_\Omega |v_n|^\beta\,\mathrm{d}x, \\
&\geq \Big(\frac{1}{p} - \frac{1}{\gamma}\Big)\|v_n\|^p - C_1|v_n|_\beta^\beta, \\
&\geq \Big(\frac{1}{p} - \frac{1}{\gamma}\Big)\|v_n\|^p - C_2\|v_n\|^\beta,
\end{aligned}$$

since $\gamma > p > \beta$, we know that $\{v_n\}$ is bounded in X.

(ii) From the odd and increasing properties of the function g in Lemma 2.1, we know the functional Φ is even, i.e. (A1) is satisfied.

To verify (B1), we define

$$\beta_k := \sup\{|v|_\beta : \|v\| = 1, v \in Z_k\}$$

For any $v \in Z_k$, $\|v\| = 1$ and $0 < t < 1$, we have

$$\begin{aligned}
\Phi(tv) &= \frac{t^p}{p}\|v\|^p - \int_\Omega F(x, g(tv))\,\mathrm{d}x \\
&\geq \frac{t^p}{p}\|v\|^p - \frac{|\lambda|}{2\gamma}\int_\Omega |g(tv)|^{2\gamma}\,\mathrm{d}x - \frac{\delta}{2\beta}\int_\Omega |g(tv)|^{2\beta}\,\mathrm{d}x \\
&\geq \frac{t^p}{p} - K_0^{2\gamma}\frac{|\lambda|}{2\gamma}\int_\Omega |tv|^\gamma\,\mathrm{d}x - K_0^{2\beta}\frac{\delta}{2\beta}\int_\Omega |tv|^\beta\,\mathrm{d}x \qquad (3.2) \\
&\geq \frac{t^p}{p} - CK_0^{2\gamma}\frac{|\lambda|}{2\gamma}t^\gamma\|v\|^\gamma - K_0^{2\beta}\beta_k^\beta\frac{\delta}{2\beta}t^\beta\|v\|^\beta \\
&\geq \frac{t^p}{p} - CK_0^{2\gamma}\frac{|\lambda|}{2\gamma}t^\gamma - K_0^{2\beta}\beta_k^\beta\frac{\delta}{2\beta}t^\beta,
\end{aligned}$$

for big k such that $\beta_k \leq 1$, where C is the Sobolev constant. Since $\gamma > p$, there exists a $0 < \rho_1 < 1$ such that $\frac{\rho_1^p}{2p} \geq CK_0^{2\gamma}\frac{|\lambda|}{2\gamma}\rho_1^\gamma$. Let $0 < t \leq \rho_1$. From the inequality (3.2), we have

$$\Phi(tv) \geq \frac{t^p}{2p} - K_0^{2\beta}\beta_k^\beta\frac{\delta}{2\beta}t^\beta. \qquad (3.3)$$

Let $t = \rho_k = \big(\frac{p\delta K_0^{2\beta}\beta_k^\beta}{\beta}\big)^{\frac{1}{p-\beta}}$ for big k such that $\rho_k \leq \rho_1$ and $\beta_k \leq 1$, we have $\Phi(tv) \geq 0$, i.e. for big $k \in \mathbb{N}$,

$$\inf_{v \in Z_k, \|v\| = \rho_k} \Phi(tv) \geq 0,$$

which implies (B1) holds.

(B2) For $v \in Y_k$ such that $\|v\| \leq 1$, by Lemma 2.1 we have

$$\begin{aligned}
\Phi(v) &= \frac{1}{p}\|v\|^p - \int_\Omega F(x, g(v))\,\mathrm{d}x \\
&\leq \frac{1}{p}\|v\|^p + \frac{|\lambda|}{2\gamma}\int_\Omega |g(v)|^{2\gamma}\,\mathrm{d}x - \frac{\delta}{2\beta}\int_\Omega |g(v)|^{2\beta}\,\mathrm{d}x \\
&\leq \frac{1}{p}\|v\|^p + \frac{|\lambda|}{2\gamma}K_0\int_\Omega |v|^\gamma\,\mathrm{d}x - \frac{\delta}{2\beta}C\int_\Omega |v|^\beta\,\mathrm{d}x.
\end{aligned}$$

Then $\dim Y_k < \infty$, $\beta < p$ and $\gamma > p$ imply that there exists a $0 < r_k < \rho_k$ small enough such that $\Phi(v) < 0$ for $\|u\| = r_k$, i.e.,

$$b_k := \max_{v \in Y_k, \|v\| = r_k} \Phi(v) < 0,$$

which implies (B2).

(B3) Since $Y_k \cup Z_k \neq \emptyset$, and $r_k < \rho_k$, we have

$$d_k := \inf_{v \in Z_k, \|u\| \leq \rho_k} \Phi(v) \leq b_k = \max_{v \in Y_k, \|v\| = r_k} \Phi(v) < 0.$$

By (3.3), for $v \in Z_k$, $\|v\| = 1$, $0 \leq t \leq \rho_k$, we have

$$\Phi(tv) \geq \frac{t^p}{2p} - K_0^{2\beta}\beta_k^\beta\frac{\delta}{2\beta}t^\beta \geq -K_0^{2\beta}\beta_k^\beta\frac{\delta}{2\beta}t^\beta \to 0, \quad \text{as } k \to +\infty,$$

which implies that $d_k \to 0$, i.e., (B3) holds.

Finally we verify the $(PS)_c^*$ condition. Consider a sequence $v_{n_j} \in Y_{n_j}$ such that
$$\Phi(v_{n_j}) \to c, \quad \Phi|'_{Y_{n_j}}(v_{n_j}) \to 0 \quad \text{in } X^* \text{ as } n_j \to \infty.$$
If $\lambda > 0$, as in the proof Lemma 3.6, it is easy to get the boundedness of $\|v_{n_j}\|$. If $\lambda < 0$, for $\|v\| > 1$ and large n, by Lemma 2.1, we have
$$\begin{aligned}
&c + 1 + \|v_{n_j}\| \\
&\geq \Phi(v_{n_j}) - \frac{1}{\gamma}\langle \Phi'(v_{n_j}), v_{n_j}\rangle \\
&= \frac{1}{p}\|v_{n_j}\|^p - \frac{1}{\gamma}\|v_{n_j}\|^p + \int_\Omega \big(\frac{1}{\gamma} f(x, g(v_{n_j}))g'(v_{n_j})v_{n_j} - F(x, g(v_{n_j}))\big)\,\mathrm{d}x \\
&\geq \frac{1}{p}\|v_{n_j}\|^p - \frac{1}{\gamma}\|v_{n_j}\|^p + \int_\Omega \Big(\frac{1}{2}\frac{1}{\gamma} f(x, g(v_{n_j}))g(v_{n_j}) - F(x, g(v_{n_j}))\Big)\,\mathrm{d}x \\
&\geq \Big(\frac{1}{p} - \frac{1}{\gamma}\Big)\|v_{n_j}\|^p + \frac{\delta}{2}\big(\frac{1}{\gamma} - \frac{1}{\beta}\big)\int_\Omega |g(v_{n_j})|^{2\beta}\,\mathrm{d}x \\
&\geq \Big(\frac{1}{p} - \frac{1}{\gamma}\Big)\|v_{n_j}\|^p - C_1\int_\Omega |v_{n_j}|^\beta\,\mathrm{d}x, \\
&\geq \Big(\frac{1}{p} - \frac{1}{\gamma}\Big)\|v_{n_j}\|^p - C_1|v_{n_j}|_\beta^\beta, \\
&\geq \Big(\frac{1}{p} - \frac{1}{\gamma}\Big)\|v_{n_j}\|^p - C_2\|v_{n_j}\|^\beta.
\end{aligned}$$
Since $\gamma > p > \beta$, we can see $\{v_{n_j}\}$ is bounded in X. We can select, if necessary, a subsequence, we assume $v_{n_j} \rightharpoonup v$ in X. As $X = \overline{\cup_{n_j} Y_{n_j}}$, we can choose $w_{n_j} \in Y_{n_j}$ such that $w_{n_j} \to u$. Hence
$$\begin{aligned}
\lim_{n_j\to\infty}\langle \Phi'(v_{n_j}), v_{n_j} - v\rangle &= \lim_{n_j\to\infty}\langle \Phi'(v_{n_j}), v_{n_j} - w_{n_j}\rangle + \lim_{n_j\to\infty}\langle \Phi'(v_{n_j}), w_{n_j} - u\rangle \\
&= \lim_{n_j\to\infty}\langle \Phi|'_{Y_{n_j}}(v_{n_j}), v_{n_j} - w_{n_j}\rangle = 0.
\end{aligned}$$
Sice Φ' is of type (S_+), we get $v_{n_j} \to v$, which implies $\Phi' v_{n_j} \to \Phi'(v)$.

The last step is to verify $\Phi'(v) = 0$. For any $u_k \in Y_k$, when $n_j \geq k$ we have
$$\begin{aligned}
\langle \Phi'(v), u_k\rangle &= \langle \Phi'(v) - \Phi'(v_{n_j}), u_k\rangle + \langle \Phi'(v_{n_j}), u_k\rangle \\
&= \langle \Phi'(v) - \Phi'(v_{n_j}), u_k\rangle + \langle \Phi|'_{Y_{n_j}}(v_{n_j}), u_k\rangle.
\end{aligned}$$
Going to the limit in the right side of above equation we get
$$\langle \Phi'(v), u_k\rangle = 0, \quad \forall u_k \in Y_k,$$
which means $\Phi'(v) = 0$. Thus Φ satisfies the $(PS)_c^*$ condition. □

When $p = 2$, we can have the corresponding theorems in this paper for the existence of solutions to the following equation for more physical meanings as we mentioned in section 1:
$$\begin{gathered}
-\Delta u - u\Delta(u^2) = f(x, u), \quad \text{in } \Omega, \\
u = 0, \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.4}$$
In fact, in the most literature such as in [18, 19, 17, 6, 8], the authors consider problem (3.4) in $\mathbb{R}^N$, the technic they used there include Nehari method, Mountain Pass theorem and some other topological Mini-Max methods. In the case for $p = 2$, the existence results in this paper are problems considered in a bounded domain in

$\mathbb{R}^N$. We close this section by pointing out that, recently, in an interesting paper [21], the authors developed the existence of a positive solution by mountain pass theorem, and the existence of a sequence solutions by symmetric mountain pass theorem under similar odd condition (F4). The method they used there is an approximation of the original functional, but without changing of variables.

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