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PERSISTENCE AND EXTINCTION FOR A STOCHASTIC LOGISTIC MODEL WITH INFINITE DELAY

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ABSTRACT. This article, studies a stochastic logistic model with infinite delay. Using a phase space, we establish sufficient conditions for the extinction, nonpersistence in the mean, weak persistence, and stochastic permanence. A threshold between weak persistence and extinction is obtained. Our results state that different types of environmental noises have different effects on the persistence and extinction, and that the delay has no impact on the persistence and extinction for the stochastic model in the autonomous case. Numerical simulations illustrate the theoretical results.

1. INTRODUCTION

For the previous decades, the logistic equation with delays have received great attention due to their extensive application as models in a variety of scientific areas, such as population dynamics, biology and epidemiology. A classical logistic model with infinite delay can be expressed as follows

$$dx(t)/dt = x(t)\Big[r(t) - a(t)x(t) + b(t)x(t-\tau) + c(t)\int_{-\infty}^{0} x(t+\theta)d\mu(\theta)\Big]dt, \quad (1.1)$$

where $\tau \geq 0$ represents time delay and $\mu(\theta)$ is a probability measure on $(-\infty, 0]$. There is an extensive literature concerned with systems similar to (1.1). Regarding persistence, extinction, global attractivity and other dynamics, mention among others: Golpalsamy [1, 2], Kuang and Smith [3], Freedman and wu [4], Kuang [5] and Lisena [6]. Particularly, the book by Gopalsamy [2] is a good reference in this area.

However, population models are always affected by environmental noises. Therefore stochastic population models have been recently investigated by many authors (see e.g.,[7]-[15]). In particular, May [10] has revealed that due to environmental noises, the growth rate, interaction coefficient and so on should be stochastic. Suppose that the growth rate $r(t)$ and the competition coefficient $a(t)$ are affected by environmental noises, with

$$r(t) \to r(t) + \sigma_1(t)\dot{\omega}_1(t), \quad -a(t) \to -a(t) + \sigma_2(t)\dot{\omega}_2(t),$$

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where $\sigma_i(t)$ are continuous positive bounded function on $\overline{R}_+$ and $\sigma_i^2(t)$ represents the intensity of the white noise at time t; $\dot{\omega}_i(t)$ are the white noise, namely $w_i(t)$ is a Brownian motion defined on a complete probability space $(\Omega, \mathbb{F}, \mathcal{P})$ with a filtration $\{\mathbb{F}_t\}_{t\in\overline{R}_+}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathbb{F}_0$ contains all $\mathcal{P}$-null sets), where $\overline{R}_+ = [0, +\infty)$, $i = 1, 2$. Then the corresponding stochastic system takes the form

$$\begin{aligned} dx(t)/dt = x(t)\Big[r(t) - a(t)x(t) + b(t)x(t-\tau) + c(t)\int_{-\infty}^{0} x(t+\theta)d\mu(\theta)\Big]dt \\ + \sigma_1(t)x(t)d\omega_1(t) + \sigma_2(t)x^2(t)d\omega_2(t). \end{aligned} \tag{1.2}$$

Here, we let the initial data ξ be positive and belong to the phase space C_g (see [3, 16, 17]) which is defined as

$$C_g = \{\varphi \in C((-\infty, 0]; R) : \|\varphi\|_{c_g} = \sup_{-\infty<s\le 0} e^{rs}|\varphi(s)| < +\infty\},$$

where $g(s) = e^{-rs}$, $r > 0$. Furthermore, C_g is an admissible Banach space (see [3, 17]).

Model (1.2) describes population dynamics; so it is very important to investigate the survival of the logistic population which involve extinction, nonpersistence in the mean, weak persistence, stochastic permanence and threshold between nonpersistence in the mean and weak persistence. Liu and Wang [18] also pointed out that it is a interesting problem to consider the persistence and extinction of logistic model with infinite delay. As far as we know, there are few results of this aspect for model (1.2). The aims of this paper are to investigate the problems above. In addition, we investigate them at the phase space C_g, which is one of the most important phase space in discussing functional differential equations with infinite delay and can avoid the usual well-posedness questions related to functional equations of unbounded delay (see e.g.,[3, 16, 19]).

To study model (1.2) we assume the following:

- (A1) the functions $r(t)$, $a(t)$, $b(t)$ and $c(t)$ are bounded and continuous on $\overline{R}_+$, and $\inf_{t\in\overline{R}_+} a(t) > 0$.
- (A2) μ satisfies
$$\mu_r = \int_{-\infty}^{0} e^{-2r\theta} d\mu(\theta) < +\infty.$$

Assumption **A2** is satisfied when $\mu(\theta) = e^{kr\theta}(k > 2)$ for $\theta \le 0$, so there exists a large number of these probability measures.

For simplicity, we define the following symbols:

$$f^u = \sup_{t\in\overline{R}_+} f(t), \quad f^l = \inf_{t\in\overline{R}_+} f(t), \quad \langle x(t)\rangle = \frac{1}{t}\int_0^t x(s)ds,$$
$$x_* = \liminf_{t\to+\infty} x(t), \quad x^* = \limsup_{t\to+\infty} x(t), \quad R_+ = (0, +\infty),$$
$$\bar{d} = \limsup_{t\to+\infty} t^{-1}\int_0^t \Big(r(s) - \frac{\sigma_1^2(s)}{2}\Big)ds.$$

The following definitions are commonly used and we list them here.

(1) The population $x(t)$ is said to have extinction if $\lim_{t\to+\infty} x(t) = 0$.
(2) The population $x(t)$ is said to have nonpersistence in the mean [20] if $\limsup_{t\to+\infty}\langle x(t)\rangle = 0$.

(3) The population $x(t)$ is said to have weak persistence if $\limsup_{t\to+\infty} x(t) > 0$, see [21, 22].
(4) Population $x(t)$ is said to have stochastic permanence [23] if for arbitrary $\varepsilon > 0$, there are constants $\beta > 0$, $M > 0$ such that

$$\liminf_{t\to+\infty} \mathcal{P}\{x(t) \geq \beta\} \geq 1-\varepsilon \quad \text{and} \quad \liminf_{t\to+\infty} \mathcal{P}\{x(t) \leq M\} \geq 1-\varepsilon.$$

From the above definitions it follows that: stochastic permanence implies stochastic weak persistence, extinction implies stochastic non-persistence in the mean. But generally, their reverses are not true.

The rest of this article is arranged as follows. In Section 2, we show that model (1.2) has a unique positive global solution. Afterward, sufficient criteria for extinction, nonpersistence in the mean, weak persistence and stochastic permanence are established in Section 3. Section 4 presents some figures to illustrate the main results. We close this article with some conclusions and remarks.

2. Positive and global solutions

As the state $x(t)$ of model (1.2) is the size of the population in the system, it should be nonnegative. In this section we shall show that the solution of model (1.2) has a unique global positive solution.

Wei [24, 25, 26] and Xu [27, 28] proved that, in order for a stochastic functional differential equations with infinite delay to have a unique global solution for any given initial data $\xi \in C_g$, the coefficients of the equation are generally required to satisfy the linear growth condition and the local Lipschitz condition. The local Lipschitz condition guarantees that the unique solution exists in $(-\infty, \tau_e)$, where τ_e is the explosion time (see [11]). Clearly, the coefficients of (1.2) satisfy the local Lipschitz condition, but do not satisfy the linear growth condition.

Theorem 2.1. *Let* (A1) *and* (A2) *hold. Then, for any given positive initial value $\xi \in C_g$, there is a unique global solution $x(t)$ to model* (1.2) *for $t \in \mathbb{R}$ and the solution will remain in R_+ with probability 1, namely $x(t) \in \mathbb{R}_+$ for all $t \in \mathbb{R}$ almost surely.*

Proof. Since the coefficients of the equation are locally Lipschitz continuous, for any given positive initial value $\xi \in C_g$, there is a unique local solution $x(t)$ on $t \in (-\infty, \tau_e)$, where τ_e is the explosion time. To show this solution is global, we need to show that $\tau_e = +\infty$ a.s. Let $k_0 > 0$ be sufficiently large such that

$$\frac{1}{k_0} < \min_{-\infty<\theta\leq 0} \xi(\theta) \leq \max_{-\infty<\theta\leq 0} \xi(\theta) < k_0.$$

For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\big\{t \in (-\infty, \tau_e) : x(t) \leq \frac{1}{k} \text{ or } x(t) \geq k\big\}.$$

Clearly, τ_k is increasing as $k \to +\infty$. Set $\tau_{+\infty} = \lim_{k\to+\infty} \tau_k$, whence $\tau_{+\infty} \leq \tau_e$ a.s. for all $t \geq 0$. If we can show that $\tau_{+\infty} = +\infty$ a.s., then $\tau_e = +\infty$ a.s. and $x(t) \in \mathbb{R}_+$ a.s. In other words, to complete the proof all we need to show is that $\tau_{+\infty} = +\infty$ a.s. To show this statement, let us define a C^2-function $V : R_+ \to R_+$ by $V(x) = \sqrt{x} - 1 - 0.5 \ln x$. Let $k \geq k_0$ and $T > 0$ be arbitrary. For $0 \leq t \leq \tau_k \wedge T$, we can apply the Itô's formula to $\int_{t-\tau}^t x^2(s)ds + V(x(t))$ to obtain

$$
\begin{aligned}
&d\Big[\int_{t-\tau}^t x^2(s)ds + V(x(t))\Big]\\
&= \big[x^2(t) - x^2(t-\tau)\big]dt + 0.5\big[x^{-0.5}(t) - x^{-1}(t)\big]x(t)\\
&\times\Big[\Big(r(t) - a(t)x(t) + b(t)x(t-\tau) + c(t)\int_{-\infty}^0 x(t+\theta)d\mu(\theta)\Big)dt\\
&\quad + \sigma_1(t)dw_1(t) + \sigma_2(t)x(t)dw_2(t)\Big] + 0.5\big[-0.25x^{-1.5}(t) + 0.5x^{-2}(t)\big]\sigma_1^2(t)x^2(t)dt\\
&\quad + 0.5\big[-0.25x^{-1.5}(t) + 0.5x^{-2}(t)\big]\sigma_2^2(t)x^4(t)dt\\
&= \big[x^2(t) - x^2(t-\tau)\big]dt + 0.5r(t)[x^{0.5}(t) - 1]dt - 0.5a(t)[x^{0.5}(t) - 1]x(t)dt\\
&\quad + 0.5b(t)x(t-\tau)[x^{0.5}(t) - 1]dt + 0.5c(t)[x^{0.5}(t) - 1]\int_{-\infty}^0 x(t+\theta)d\mu(\theta)dt\\
&\quad + 0.5\big[-0.25x^{-1.5}(t) + 0.5x^{-2}(t)\big]\sigma_1^2(t)x^2(t)dt + 0.5\big[-0.25x^{-1.5}(t)\\
&\quad + 0.5x^{-2}(t)\big]\sigma_2^2(t)x^4(t)dt + 0.5[x^{0.5}(t) - 1]\sigma_1(t)dw_1(t) + 0.5\big[x^{1.5}(t)\\
&\quad - x(t)\big]\sigma_2(t)dw_2(t)\\
&= \big[x^2(t) - x^2(t-\tau)\big]dt + 0.5r(t)[x^{0.5}(t) - 1]dt - 0.5a(t)[x^{0.5}(t) - 1]x(t)dt\\
&\quad + 0.0625b^2(t)[x^{0.5}(t) - 1]^2dt + 0.0625c^2(t)[x^{0.5}(t) - 1]^2dt\\
&\quad + \int_{-\infty}^0 x^2(t+\theta)d\mu(\theta)dt + x^2(t-\tau)dt\\
&\quad + 0.5\big[-0.25x^{-1.5}(t) + 0.5x^{-2}(t)\big]\sigma_1^2(t)x^2(t)dt\\
&\quad + 0.5\big[-0.25x^{-1.5}(t) + 0.5x^{-2}(t)\big]\sigma_2^2(t)x^4(t)dt\\
&\quad + 0.5[x^{0.5}(t) - 1]\sigma_1(t)dw_1(t) + 0.5\big[x^{1.5}(t) - x(t)\big]\sigma_2(t)dw_2(t)\\
&= \Big[x^2(t) + 0.25\sigma_1^2(t) + 0.25\sigma_2^2(t)x^2(t) + 0.5r(t)\big(x^{0.5}(t) - 1\big)\\
&\quad - 0.5a(t)\big(x^{0.5}(t)dt - 1\big)x(t) + 0.0625b^2(t)\big(x^{0.5}(t) - 1\big)^2 - 0.125\sigma_2^2(t)x^{2.5}(t)\\
&\quad + 0.0625c^2(t)\big(x^{0.5}(t) - 1\big)^2 + \int_{-\infty}^0 x^2(t+\theta)d\mu(\theta) - 0.125\sigma_1^2(t)x^{1.5}(t)\Big]dt\\
&\quad + 0.5[x^{0.5}(t) - 1]\sigma_1(t)dw_1(t) + 0.5[x^{1.5}(t) - x(t)]\sigma_2(t)dw_2(t),\\
&= F(x)dt + \int_{-\infty}^0 x^2(t+\theta)d\mu(\theta)dt - x^2(t)dt + 0.5[x^{0.5}(t) - 1]\sigma_1(t)x(t)dw_1(t)\\
&\quad + 0.5[x^{0.5}(t) - 1]\sigma_2(t)x^2(t)dw_2(t)
\end{aligned}
$$

where

$$
\begin{aligned}
F(x) &= 0.25\sigma_1^2(t) + 0.25\sigma_2^2(t)x^2(t) + 0.5r(t)[x^{0.5}(t) - 1] - 0.5a(t)[x^{0.5}(t) - 1]x(t)\\
&\quad + 0.0625b^2(t)[x^{0.5}(t) - 1]^2 + 0.0625c^2(t)[x^{0.5}(t) - 1]^2\\
&\quad + 2x^2(t) - 0.125\sigma_1^2(t)x^{1.5}(t) - 0.125\sigma_2^2(t)x^{2.5}(t).
\end{aligned}
$$

Under assumptions (A1)–(A2), it is easy to see that $F(x)$ is bounded, say by K, in R_+. We therefore obtain that

$$\begin{aligned} &d\Big[\int_{t-\tau}^{t} x^2(s)ds + V(x(t))\Big] \\ &\le Kdt + \int_{-\infty}^{0} x^2(t+\theta)d\mu(\theta)dt - x^2(t)dt + 0.5[x^{0.5}(t)-1]\sigma_1(t)dw_1(t) \\ &\quad + 0.5[x^{1.5}(t) - x(t)]\sigma_2(t)dw_2(t). \end{aligned}$$

Integrating both sides from 0 to t, and then taking expectations, yields

$$\begin{aligned} &E\Big[\int_{t-\tau}^{t} x^2(s)ds + V(x(t))\Big] \\ &\le \int_{-\tau}^{0} x^2(s)ds + V(x(0)) + Kt + E\int_0^t\int_{-\infty}^{0} x^2(s+\theta)d\mu(\theta)ds - E\int_0^t x^2(s)ds. \end{aligned}$$

Moreover, we obtain that

$$\begin{aligned} &\int_0^t\int_{-\infty}^{0} x^2(s+\theta)d\mu(\theta)ds \\ &= \int_0^t \Big[\int_{-\infty}^{-s} x^2(s+\theta)d\mu(\theta)ds + \int_{-s}^{0} x^2(s+\theta)d\mu(\theta)\Big]ds \\ &= \int_0^t ds\Big[\int_{-\infty}^{-s} e^{2r(s+\theta)}x^2(s+\theta)e^{-2r(s+\theta)}d\mu(\theta) + \int_{-t}^{0} d\mu(\theta)\int_{-\theta}^{t} x^2(s+\theta)ds \\ &\le \|\xi\|_{C_g}^2 \int_0^t e^{-2rs}ds\int_{-\infty}^{0} e^{-2r\theta}d\mu(\theta) + \int_{-\infty}^{0} d\mu(\theta)\int_0^t x^2(s)ds \\ &\le \|\xi\|_{C_g}^2\mu_r t + \int_0^t x^2(s)ds. \end{aligned}$$

Consequently,

$$E\Big[\int_{t-\tau}^{t} x^2(s)ds + V(x(t))\Big] \le \int_{-\tau}^{0} x^2(s)ds + V(x(0)) + Kt + \|\xi\|_{C_g}^2\mu_r t.$$

Letting $t = \tau_k \wedge T$, we obtain

$$\begin{aligned} &E\Big[\int_{\tau_k\wedge T-\tau}^{\tau_k\wedge T} x^2(s)ds + V(x(\tau_k\wedge T))\Big] \\ &\le \int_{-\tau}^{0} x^2(s)ds + V(x(0)) + KE(\tau_k\wedge T) + \|\xi\|_{C_g}^2\mu_r(\tau_k\wedge T). \end{aligned}$$

Therefore,

$$EV(x(\tau_k\wedge T)) \le \int_{-\tau}^{0} x^2(s)ds + V(x(0)) + KT + \|\xi\|_{C_g}^2\mu_r T. \tag{2.1}$$

Note that for every $\omega \in \{\tau_k \le T\}$, $x(\tau_k,\omega)$ equals either k or $\frac{1}{k}$, and hence $V(x(\tau_k,\omega))$ is no less than either

$$\sqrt{k} - 1 - 0.5\log(k)$$

or

$$\sqrt{\frac{1}{k}} - 1 - 0.5\log(\frac{1}{k}) = \sqrt{\frac{1}{k}} - 1 + 0.5\log(k).$$

Consequently,

$$V(x(\tau_k,\omega)) \geq [\sqrt{k} - 1 - 0.5\log(k)] \wedge [\sqrt{\frac{1}{k}} - 1 + 0.5\log(k)].$$

From (2.1) it follows that

$$\begin{aligned}
&\int_{-\tau}^{0} x^2(s)ds + V(x(0)) + KT + \|\xi\|^2_{C_g}\mu_r T \\
&\geq E[1_{\{\tau_k \leq T\}}(\omega)V(x(\tau_k,\omega))] \\
&\geq P\{\tau_k \leq T\}([\sqrt{k} - 1 - 0.5\log(k)] \wedge [\sqrt{\frac{1}{k}} - 1 + 0.5\log(k)]),
\end{aligned}$$

where $1_{\{\tau_k \leq T\}}$ is the indicator function of $\{\tau_k \leq T\}$. Letting $k \to \infty$ gives

$$\lim_{k\to+\infty} P\{\tau_k \leq T\} = 0$$

and hence $P\{\tau_{+\infty} \leq T\} = 0$. Since $T > 0$ is arbitrary, we must have

$$P\{\tau_{+\infty} < +\infty\} = 0,$$

so $P\{\tau_{+\infty} = +\infty\} = 1$ as required. □

3. Persistence and extinction for model (1.2)

From Theorem 2.1 we know that solutions of (1.2) will remain in the positive cone R_+. This nice property provides us with a great opportunity to construct different types of Lyapunov functions to discuss how the solutions vary in R_+ in more details. In this section, we shall study the persistence and extinction of model (1.2).

Theorem 3.1. *Let assumption* (A1) *and* (A2) *hold. If* $\bar{d} < 0$ *and* $\inf_{t\in\overline{R}_+}\{a(t) - b(t+\tau) - c^u\} \geq 0$, *then the population* $x(t)$ *modeled by* (1.2) *approaches extinction a.s.*

Proof. Applying Itô's formula to (1.2) leads to

$$\begin{aligned}
&d\int_{t-\tau}^{t} b(s+\tau)x(s)ds + d\ln x(t) \\
&= (b(t+\tau)x(t) - b(t)x(t-\tau))dt + \Big[r(t) - \frac{\sigma_1^2(t)}{2} - a(t)x(t) + b(t)x(t-\tau) \\
&\quad + c(t)\int_{-\infty}^{0} x(t+\theta)d\mu(\theta) - \frac{\sigma_2^2(t)(t)x^2(t)}{2}\Big]dt + \sigma_1(t)d\omega_1(t) + \sigma_2(t)x(t)d\omega_2(t).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\int_{t-\tau}^{t} b(s+\tau)x(s)ds - \int_{-\tau}^{0} b(s+\tau)x(s)ds + \ln x(t) - \ln x(0) \\
&= \int_0^t \Big[r(s) - \frac{\sigma_1^2(t)}{2} - (a(s) - b(s+\tau))x(s) + c(s)\int_{-\infty}^{0} x(s+\theta)d\mu(\theta) \qquad (3.1)\\
&\quad - \frac{\sigma_2^2(t)x^2(s)}{2}\Big]ds + \int_0^t \sigma_1(s)d\omega_1(s) + \int_0^t \sigma_2(s)x(s)d\omega_2(s).
\end{aligned}$$

By hypothesis (A3), we obtain

$$\begin{aligned}
&\int_0^t c(s)\int_{-\infty}^0 x(s+\theta)d\mu(\theta)ds \\
&= \int_0^t c(s)\Big[\int_{-\infty}^{-s} x(s+\theta)d\mu(\theta)ds + \int_{-s}^0 x(s+\theta)d\mu(\theta)\Big]ds \\
&= \int_0^t c(s)ds\int_{-\infty}^{-s} e^{r(s+\theta)}x(s+\theta)e^{-r(s+\theta)}d\mu(\theta) + \int_{-t}^0 d\mu(\theta)\int_{-\theta}^t c(s)x(s+\theta)ds \\
&\le c^u\|\xi\|_{c_g}\int_0^t e^{-rs}ds\int_{-\infty}^0 e^{-r\theta}d\mu(\theta) + c^u\int_{-\infty}^0 d\mu(\theta)\int_0^t x(s)ds \\
&\le c^u\|\xi\|_{c_g}\int_0^t e^{-rs}ds\int_{-\infty}^0 e^{-2r\theta}d\mu(\theta) + c^u\int_{-\infty}^0 d\mu(\theta)\int_0^t x(s)ds \\
&\le \frac{1}{r}c^u\|\xi\|_{C_g}\mu_r(1-e^{-rt}) + c^u\int_0^t x(s)ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_{t-\tau}^t b(s+\tau)x(s)ds - \int_{-\tau}^0 b(s+\tau)x(s)ds + \ln x(t) - \ln x(0) \\
&\le \int_0^t \Big[r(s) - \frac{\sigma_1^2(s)}{2} - \Big(a(s) - b(s+\tau) - c^u\Big)x(s) - \frac{\sigma_2^2(s)x^2(s)}{2}\Big]ds \\
&\quad + \frac{1}{r}c^u\|\xi\|_{C_g}\mu_r(1-e^{-rt}) + M_1(t) + M_2(t),
\end{aligned} \tag{3.2}$$

where

$$M_1(t) = \int_0^t \sigma_1(s)d\omega_1(s), \quad M_2(t) = \int_0^t \sigma_2(s)x(s)d\omega_2(s).$$

The quadratic variation of $M_1(t)$ is

$$\langle M_1(t), M_1(t)\rangle = \int_0^t \sigma_1^2(t)(s)ds \le (\sigma_1^u)^2 t.$$

Using the strong law of large numbers for martingales (see e.g. [29, page 16]) leads to

$$\lim_{t\to+\infty}\frac{M_1(t)}{t} = 0, \quad \text{a.s.} \tag{3.3}$$

The quadratic variation of $M_2(t)$ is $\langle M_2(t), M_2(t)\rangle = \int_0^t \sigma_2^2(s)x^2(s)ds$. By the exponential martingale inequality, for any positive constants T_0, α and β, we have

$$P\Big\{\sup_{0\le t\le T_0}\Big[M_2(t) - \frac{\alpha}{2}\langle M_2(t), M_2(t)\rangle\Big] > \beta\Big\} \le e^{-\alpha\beta}. \tag{3.4}$$

Choose $T_0 = k$, $\alpha = 1$, $\beta = 2\ln k$. Then it follows that

$$P\Big\{\sup_{0\le t\le k}\Big[M_2(t) - \frac{1}{2}\langle M_2(t), M_2(t)\rangle\Big] > 2\ln k\Big\} \le \frac{1}{k^2}.$$

Using Borel-Cantelli's lemma yields that for almost all $\omega \in \Omega$, there is a random integer $k_0 = k_0(\omega)$ such that for $k \ge k_0$,

$$\sup_{0\le t\le k}\Big[M_2(t) - \frac{1}{2}\langle M_2(t), M_2(t)\rangle\Big] \le 2\ln k.$$

This is to say

$$M_2(t) \le 2\ln k + \frac{1}{2}\langle M_2(t), M_2(t)\rangle = 2\ln k + \frac{1}{2}\int_0^t \sigma_2^2(s)x^2(s)ds$$

for all $0 \le t \le k, k \ge k_0$ a.s. Substituting this inequality into (3.1), we obtain

$$\begin{aligned} \ln x(t) - \ln x(0) \le \int_{-\tau}^0 b(s)x(s)ds + \int_0^t \Big[r(s) - \frac{\sigma_1^2(s)}{2} - (a(s) - b(s) - c^u)x(s)\Big]ds \\ + 2\ln k + \frac{1}{r}c^u\|\xi\|_{C_g}\mu_r(1-e^{-rt}) + M_1(t) \end{aligned} \tag{3.5}$$

for all $0 \le t \le k$, $k \ge k_0$ a.s. In other words, we have shown that for $0 < k-1 \le t \le k, k \ge k_0$,

$$\begin{aligned} &t^{-1}\{\ln x(t) - \ln x(0)\} \\ &\le t^{-1}\int_{-\tau}^0 b(s)x(s)ds + t^{-1}\int_0^t \Big[r - \frac{\sigma_1^2(s)}{2} - ((a(s) - b(s) - c^u)x(s)\Big]ds \\ &\quad + 2(k-1)^{-1}\ln k + t^{-1}\frac{1}{r}c^u\|\xi\|_{C_g}\mu_r(1-e^{-rt}) + M_1(t)/t. \end{aligned}$$

Taking the limit superior on both sides and using (3.3) yields $\limsup_{t\to+\infty}\frac{\ln x(t)}{t} \le \bar{d}$. That is to say, if $\bar{d} < 0$, one sees that $\lim_{t\to+\infty} x(t) = 0$ a.s. □

Theorem 3.2. *Let* (A1) *and* (A2) *hold, if* $\bar{d} = 0$ *and* $\inf_{t\in\overline{R}_+}\{a(t) - b(t+\tau) - c^u\} \ge 0$*, then the population* $x(t)$ *represented by* (1.2) *is nonpersistent in the mean a.s.*

Proof. From $\bar{d} = 0$ and (3.3), then for all $\varepsilon > 0$, there exists T, such that

$$\begin{aligned} &t^{-1}\int_0^t (r(s) - \frac{\sigma_1(s)}{2})ds + t^{-1}\int_{-\tau}^0 b(s)x(s)ds \\ &+ t^{-1}\frac{1}{r}c^u\|\xi\|_{C_g}\mu_r(1-e^{-rt}) + \frac{2\ln k}{t} + \frac{M_1(t)}{t} < \varepsilon \quad \forall t > T. \end{aligned}$$

In view of (3.5), we have

$$\begin{aligned} &\ln x(t) - \ln x(0) \\ &\le \int_{-\tau}^0 b(s)x(s)ds + \int_0^t \Big[r(s) - \frac{\sigma_1^2(s)}{2} - (a(s) - b(s+\tau) - c^u)x(s)\Big]ds \\ &\quad + t^{-1}\frac{1}{r}c^u\|\xi\|_{C_g}\mu_r(1-e^{-rt}) + 2\ln k + M_1(t) \\ &< \varepsilon t - \int_0^t (a(s) - b(s+\tau) - c^u)x(s)ds \end{aligned}$$

for all $T \le k-1 \le t \le k$, $k \ge k_0$ a.s. Define $h(t) = \int_0^t x(s)ds$ and $N = \inf_{s\in\mathbb{R}}[a(s) - b(s+\tau) - c^u]$, then we have

$$\ln(dh/dt) < \varepsilon t - Nh(t) + \ln x(0), \quad t > T.$$

The rest of proof is similar to [14, Theorem 3] and is hence omitted. □

Theorem 3.3. *Let* (A1) *and* (A2) *hold. If* $\bar{d} > 0$*, then the population* $x(t)$ *modeled by* (1.2) *is weakly persistent a.s.*

Proof. Now suppose that $\bar{d} > 0$, we will prove $\limsup_{t\to+\infty} x(t) > 0$ a.s. If this assertion is not true, let $F = \{\limsup_{t\to+\infty} x(t) = 0\}$ and suppose $P(F) > 0$. In view of (3.1),

$$\begin{aligned} &t^{-1}\int_{t-\tau}^{t} bx(s)ds - t^{-1}\int_{-\tau}^{0} bx(s)ds + t^{-1}\ln x(t) \\ &= t^{-1}\ln x(0) + t^{-1}\int_0^t \Big[r(s) - \frac{\sigma_1^2(s)}{2} - (a(s)-b(s))x(s) \\ &\quad + c(s)\int_{-\infty}^{0} x(s+\theta)d\mu(\theta) - \frac{\sigma_2^2(s)x^2(s)}{2}\Big]ds + M_1(t)/t + M_2(t)/t. \end{aligned} \tag{3.6}$$

On the other hand, for all $\omega \in F$, we have $\lim_{t\to+\infty} x(t,\omega) = 0$, then the law of large numbers for local martingales indicates that $\lim_{t\to+\infty} M_2(t)/t = 0$. Substituting this equality and (3.3) into (3.6) results in the contradiction

$$0 \ge \limsup_{t\to+\infty}[t^{-1}\ln x(t,\ \omega)] = \bar{d} > 0.$$

□

It is well known that in the study of population system, stochastic permanence, which means that the population will survive forever, is one of the most important and interesting topics due to its theoretical and practical significance. So we show that $x(t)$ modeled by (1.2) is stochastic permanent in some cases.

Theorem 3.4. *Let* (A1) *and* (A2) *hold. If* $\big(r(t) - \frac{\sigma_1^2(t)}{2}\big)_* > 0$, $b(t) \ge 0$, *and* $c(t) \ge 0$, *then the population* $x(t)$ *modeled by* (1.2) *will be stochastic permanent.*

Proof. First, we prove that for arbitrary $\varepsilon > 0$, there are constants $M > 0$ such that $\liminf_{t\to+\infty} \mathcal{P}\{x(t) \le M\} \ge 1 - \varepsilon$.

Let $0 < p < 1$ and $\varepsilon_1 \in (0, 2r)$, we compute

$$\begin{aligned} &dx^p(t) \\ &= px^{p-1}(t)dx(t) + \frac{1}{2}p(p-1)x^{p-2}(t)(dx(t))^2 \\ &= px^{p-1}(t)\Big[\Big(x(t)\Big(r(t) - a(t)x(t) + b(t)x(t-\tau) + c(t)\int_{-\infty}^{0} x(t+\theta)d\mu(\theta)\Big)\Big)dt \\ &\quad + \sigma_1(t)x(t)d\omega_1(t) + \sigma_2(t)x^2 d\omega_2(t)\Big] + \frac{1}{2}p(p-1)\sigma_1^2(t)x^p(t)dt \\ &\quad + \frac{1}{2}p(p-1)\sigma_2^2(t)x^{p+2}(t)dt \\ &\le \Big[r(t)px^p(t) + \frac{p^2b^2(t)x^{2p}(t)}{4} + x^2(t-\tau) + \frac{p^2c^2(t)x^{2p}(t)}{4} \\ &\quad + \int_{-\infty}^{0} x^2(t+\theta)d\mu(\theta)\Big]dt + p\sigma_1(t)x^p(t)d\omega_1(t) + p\sigma_2(t)x^{p+1}(t)d\omega_2(t) \\ &\quad - \frac{1}{2}p(1-p)\sigma_1^2(t)x^p(t)dt - \frac{1}{2}p(1-p)\sigma_2^2(t)x^{p+2}(t)dt \\ &= F(x)dt - \Big[\varepsilon_1 x^p(t) + e^{\varepsilon_1\tau}x^2(t) - x^2(t-\tau) - \int_{-\infty}^{0} x^2(t+\theta)d\mu(\theta) + \mu_r x^2(t)\Big]dt \\ &\quad + p\sigma_1(t)x^p(t)d\omega_1(t) + p\sigma_2(t)x^{p+1}(t)d\omega_2(t) \end{aligned}$$

where

$$F(x) = e^{\varepsilon_1\tau}x^2(t) + \mu_r x^2(t) + (\varepsilon_1 + r(t)p)x^p(t) + \frac{p^2b^2(t)x^{2p}(t)}{4} + \frac{p^2c^2(t)x^{2p}(t)}{4} - \frac{1}{2}p(1-p)\sigma_1^2(t)(t)x^p(t) - \frac{1}{2}p(1-p)\sigma_2^2(t)x^{2+p}(t).$$

From (A1)–(A2) and $0 < p < 1$, we have that $F(x)$ is bounded in R_+, namely

$$\sup_{x\in\mathbb{R}_+} F(x) = M_3 < +\infty.$$

Therefore,

$$\begin{aligned} dx^p(t) &= [M_3 - \varepsilon_1 x^p(t) - e^{\varepsilon_1\tau}x^2(t) + x^2(t-\tau)]dt \\ &\quad + \int_{-\infty}^0 x^2(t+\theta)d\mu(\theta)dt - \mu_r x^2(t)dt \\ &\quad + p\sigma_1(t)x^p(t)d\omega_1(t) + p\sigma_2(t)x^{p+1}(t)d\omega_2(t). \end{aligned}$$

Once again by Itŏ's formula we have

$$\begin{aligned} d[e^{\varepsilon_1 t}x^p(t)] &= e^{\varepsilon_1 t}[\varepsilon_1 x^p(t)dt + dx^p(t)] \\ &\le e^{\varepsilon_1 t}\Big[M_3 - e^{\varepsilon_1\tau}x^2(t) + x^2(t-\tau) + \int_{-\infty}^0 x^2(t+\theta)d\mu(\theta) - \mu_r x^2(t)\Big]dt \\ &\quad + e^{\varepsilon_1 t}\Big(p\sigma_1(t)x^p(t)d\omega_1(t) + p\sigma_2(t)x^{p+1}(t)d\omega_2(t)\Big). \end{aligned}$$

Hence, we have

$$\begin{aligned} & e^{\varepsilon_1 t}Ex^p(t) \\ &\le \xi^p(0) + \frac{e^{\varepsilon_1 t}M_3}{\varepsilon_1} - \frac{M_3}{\varepsilon_1} - E\int_0^t e^{\varepsilon_1 s+\varepsilon_1\tau}x^2(s)ds + E\int_0^t e^{\varepsilon_1 s}x^2(s-\tau)ds \\ &\quad + E\int_0^t e^{\varepsilon_1 s}\int_{-\infty}^0 x^2(s+\theta)d\mu(\theta)ds - E\int_0^t \mu_r e^{\varepsilon_1 s}x^2(s)ds \\ &\le \xi^p(0) + \frac{e^{\varepsilon_1 t}M_3}{\varepsilon_1} - \frac{M_3}{\varepsilon_1} - E\int_0^t e^{\varepsilon_1 s+\varepsilon_1\tau}x^2(s)ds + E\int_{-\tau}^{t-\tau} e^{\varepsilon_1 s+\varepsilon_1\tau}x^2(s)ds \\ &+ E\int_0^t e^{\varepsilon_1 s}\int_{-\infty}^0 x^2(s+\theta)d\mu(\theta)ds - E\int_0^t \mu_r e^{\varepsilon_1 s}x^2(s)ds \\ &\le \xi^p(0) + \frac{e^{\varepsilon_1 t}M_3}{\varepsilon_1} - \frac{M_3}{\varepsilon_1} + \int_{-\tau}^0 e^{\varepsilon_1 s+\varepsilon_1\tau}x^2(s)ds \\ &\quad + E\int_0^t e^{\varepsilon_1 s}\int_{-\infty}^0 x^2(s+\theta)d\mu(\theta)ds - E\mu_r\int_0^t e^{\varepsilon_1 s}x^2(s)ds \end{aligned}$$

From (A2), we have

$$\begin{aligned} &\int_0^t e^{\varepsilon_1 s}\int_{-\infty}^0 x^2(s+\theta)d\mu(\theta)ds \\ &= \int_0^t e^{\varepsilon_1 s}\Big[\int_{-\infty}^{-s} x^2(s+\theta)d\mu(\theta) + \int_{-s}^0 x^2(s+\theta)d\mu(\theta)\Big]ds \\ &= \int_0^t e^{\varepsilon_1 s}ds\int_{-\infty}^{-s} e^{2r(s+\theta)}x^2(s+\theta)e^{-2r(s+\theta)}d\mu(\theta) + \int_{-t}^0 d\mu(\theta)\int_{-\theta}^t e^{\varepsilon_1(s)}x^2(s+\theta)ds \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t e^{\varepsilon_1 s} ds \int_{-\infty}^{-s} e^{2r(s+\theta)} x^2(s+\theta) e^{-2r(s+\theta)} d\mu(\theta) + \int_{-t}^0 d\mu(\theta) \int_0^{t+\theta} e^{\varepsilon_1(s-\theta)} x^2(s) ds \\
&\le \|\xi\|_{C_g}^2 \int_0^t e^{(\varepsilon_1 - 2r)s} ds \int_{-\infty}^0 e^{-2r\theta} d\mu(\theta) + \int_{-\infty}^0 e^{-\varepsilon_1 \theta} d\mu(\theta) \int_0^t e^{\varepsilon_1 s} x^2(s) ds \\
&\le \|\xi\|_{C_g}^2 \mu_r t + \mu_r \int_0^t e^{\varepsilon_1 s} x(s) ds.
\end{aligned}
$$

This implies

$$
\limsup_{t \to +\infty} E[x^p(t)] \le \frac{M_3}{\varepsilon_1}.
$$

Now, for any $\varepsilon > 0$ and $M = \big(\frac{M_3}{\varepsilon_1}\big)^{1/p} / \varepsilon^{1/p}$, by Chebyshev's inequality,

$$
\mathcal{P}\{x(t) > M\} = \mathcal{P}\{x^p(t) > M^p\} \le E[x^p(t)]/M^p.
$$

Hence

$$
\limsup_{t \to +\infty} \mathcal{P}\{x(t) > M\} \le \varepsilon.
$$

This implies

$$
\liminf_{t \to +\infty} \mathcal{P}\{x(t) \le M\} \ge 1 - \varepsilon.
$$

Next, we claim that for arbitrary $\varepsilon > 0$, there is a constant $\beta > 0$ such that $\liminf_{t \to +\infty} \mathcal{P}\{x(t) \ge \beta\} \ge 1 - \varepsilon$. Define $V_1(x) = 1/x^2$ for $x \in \mathbb{R}_+$. Applying Itô's formula to(1.2) we obtain

$$
\begin{aligned}
dV_1(x(t)) &= -2x^{-3} dx + 3x^{-4} (dx)^2 \\
&= 2V_1(x)[1.5\sigma_2^2(t) x^2 + a(t)x - r(t) + 1.5\sigma_1^2(t) - b(t)x(t-\tau) \\
&\quad - c(t) \int_{-\infty}^0 x(t+\theta) d\mu(\theta)] dt - 2\sigma_1(t) x^{-2} d\omega_1(t) - 2\sigma_2(t) x^{-1} d\omega_2(t).
\end{aligned}
$$

Since $(r(t) - \frac{\sigma_1^2(t)}{2})_* > 0$, we can choose a sufficient small constant $0 < \kappa < 1$ such that $\big(r(t) - \frac{\sigma_1^2(t)}{2}\big)_* - \kappa(\sigma_1^u)^2 > 0$.

Define $V_2(x) = (1 + V_1(x))^\kappa$. Using Itô's formula again leads to

$$
\begin{aligned}
dV_2 &= \kappa(1 + V_1(x(t)))^{\kappa-1} dV_1 + 0.5\kappa(\kappa - 1)(1 + V_1(x(t)))^{\kappa-2} (dV_1)^2 \\
&= \kappa(1 + V_1(x))^{\kappa-2} \{(1 + V_1(x)) 2V_1(x)[1.5\sigma_2^2(t) x^2 + a(t)x - r(t) + 1.5\sigma_1^2(t) \\
&\quad - b(t)x(t-\tau) - c(t) \int_{-\infty}^0 x(t+\theta) d\mu(\theta)] + 2\sigma_1^2(t)(\kappa - 1) V_1^2(x) \\
&\quad + 2\sigma_2^2(t)(\kappa - 1) V_1(x)\} dt - 2\kappa(1 + V_1(x))^{\kappa-1} x^{-2} \sigma_1(t) d\omega_1(t) \\
&\quad - 2\kappa(1 + V_1(x))^{\kappa-1} x^{-1} \sigma_2(t) d\omega_2(t) \\
&\le \kappa(1 + V_1(x))^{\kappa-2} \{(-2r(t) + 3\sigma_1^2(t) + 2\sigma_1^2(t)(\kappa - 1)) V_1^2(x) + 2a(t) V_1^{1.5}(x) \\
&\quad + [3\sigma_1^2(t) - 2r(t) + (2\kappa + 1)\sigma_2^2(t)] V_1(x) + 2a(t) V_1^{0.5}(x) + 3\sigma_2^2(t)\} dt \\
&\quad - 2\kappa(1 + V_1(x))^{\kappa-1} x^{-2} \sigma_1(t) d\omega_1(t) - 2\kappa(1 + V_1(x))^{\kappa-1} x^{-1} \sigma_2(t) d\omega_2(t) \\
&= \kappa(1 + V_1(x))^{\kappa-2} \{(-2r(t) + \sigma_1^2(t) + 2\kappa\sigma_1^2(t)) V_1^2(x) + 2a(t) V_1^{1.5}(x) + [3\sigma_1^2(t) \\
&\quad - 2r + (2\kappa + 1)\sigma_2^2(t)] V_1(x) + 2a(t) V_1^{0.5}(x) + 3\sigma_2^2(t)\} dt \\
&\quad - 2\kappa(1 + V_1(x))^{\kappa-1} x^{-2} \sigma_1(t) d\omega_1(t) - 2\kappa(1 + V_1(x))^{\kappa-1} x^{-1} \sigma_2(t) d\omega_2(t)
\end{aligned}
$$

$$
\begin{aligned}
&\le \kappa(1+V_1(x))^{\kappa-2}\Big\{-2\Big(\Big(r(t)-\frac{\sigma_1^2(t)}{2}\Big)_* - \kappa(\sigma_1^u)^2\Big)V_1^2(x) + 2a^u V_1^{1.5}(x) \\
&\quad + [3(\sigma_1^u)^2 - 2r^l + (2\kappa+1)(\sigma_2^u)^2]V_1(x) + 2a^u V_1^{0.5}(x) + 3(\sigma_2^u)^2\Big\}dt \\
&\quad - 2\kappa(1+V_1(x))^{\kappa-1}x^{-2}\sigma_1(t)d\omega_1(t) - 2\kappa(1+V_1(x))^{\kappa-1}x^{-1}\sigma_2(t)d\omega_2(t)
\end{aligned}
$$

for sufficiently large $t \ge T$. Now, let $\eta > 0$ be sufficiently small satisfying

$$
0 < \frac{\eta}{2\kappa} < (r(t) - \frac{\sigma_1^2(t)}{2})_* - \kappa(\sigma_1^u)^2.
$$

Define $V_3(x) = e^{\eta t}V_2(x)$. By Itŏ's formula

$$
\begin{aligned}
&dV_3(x(t)) \\
&= \eta e^{\eta t}V_2(x) + e^{\eta t}dV_2(x) \\
&\le \kappa e^{\eta t}(1+V_1(x(t)))^{\kappa-2}\Big\{\eta(1+V_1(x))^2/\kappa - 2\Big((r(t)-\frac{\sigma_1^2(t)}{2})_* - \kappa(\sigma_1^u)^2\Big)V_1^2(x) \\
&\quad + 2a^u V_1^{1.5}(x) + (3(\sigma_1^u)^2 - 2r_* + (2\kappa+1)(\sigma_1^u)^2)V_1(x) + 2a^u V_1^{0.5}(x) + 3(\sigma_2^u)^2\Big\}dt \\
&\quad - 2\kappa e^{\eta t}(1+V_1(x))^{\theta-1}x^{-2}\sigma_1(t)d\omega_1(t) - 2\kappa e^{\eta t}(1+V_1(x))^{\kappa-1}x^{-1}\sigma_2(t)d\omega_2(t) \\
&\le \kappa e^{\eta t}(1+V_1(x(t)))^{\kappa-2}\Big\{-2\Big(\Big(r(t)-\frac{\sigma_1^2(t)}{2}\Big)_* - \kappa(\sigma_1^u)^2 - \frac{\eta}{2\kappa}\Big)V_1^2(x) + 2a^u V_1^{1.5}(x) \\
&\quad + [3(\sigma_1^u)^2 - 2r^l + (2\kappa+1)(\sigma_2^u)^2 + 2\eta/\kappa]V_1(x) + 2a^u V_1^{0.5}(x) + 3(\sigma_2^u)^2 + \eta/\kappa\Big\}dt \\
&\quad - 2\theta e^{\eta t}(1+V_1(x))^{\kappa-1}x^{-2}\sigma_1(t)d\omega_1(t) - 2\kappa e^{\eta t}(1+V_1(x))^{\kappa-1}x^{-1}\sigma_2(t)d\omega_2(t) \\
&= e^{\eta t}H(x)dt - 2\kappa e^{\eta t}(1+V_1(x))^{\kappa-1}x^{-2}\sigma_1(t)d\omega_1(t) \\
&\quad - 2\kappa e^{\eta t}(1+V_1(x))^{\kappa-1}x^{-1}\sigma_2(t)d\omega_2(t)
\end{aligned}
$$

for $t \ge T$. Note that $H(x)$ is bounded from above in R_+, namely $\sup_{x\in\mathbb{R}_+} H(x) = H < +\infty$. Consequently,

$$
\begin{aligned}
dV_3(x(t)) = He^{\eta t}dt &- 2\kappa e^{\eta t}(1+V_1(x(t)))^{\kappa-1}x^{-2}(t)\sigma_1(t)d\omega_1(t) \\
&- 2\kappa e^{\eta t}(1+V_1(x))^{\kappa-1}x^{-1}\sigma_2(t)d\omega_2(t)
\end{aligned}
$$

for sufficiently large t. Integrating both sides of the above inequality and then taking expectations gives

$$
E[V_3(x(t))] = E[e^{\eta t}(1+V_1(x(t)))^{\kappa}] \le e^{\eta T}(1+V_1(x(T)))^{\kappa} + \frac{H}{\eta}\big(e^{\eta t} - e^{\eta T}\big).
$$

That is to say

$$
\limsup_{t\to+\infty} E[V_1^{\kappa}(x(t))] \le \limsup_{t\to+\infty} E[(1+V_1(x(t)))^{\kappa}] < \frac{H}{\eta}.
$$

In other words, we have shown that

$$
\limsup_{t\to+\infty} E\big[\frac{1}{x^{2\kappa}(t)}\big] \le \frac{H}{\eta} = M_4.
$$

So for any $\varepsilon > 0$, set $\beta = \varepsilon^{1/2\kappa}/M_4^{1/2\kappa}$, by Chebyshev's inequality, one can derive that

$$
\mathcal{P}\{x(t) < \beta\} = \mathcal{P}\Big\{\frac{1}{x^{2\kappa}(t)} > \frac{1}{\beta^{2\kappa}}\Big\} \le \frac{E[\frac{1}{x^{2\kappa}(t)}]}{\frac{1}{\beta^{2\kappa}}}.
$$

This is to say

$$\limsup_{t\to+\infty} \mathcal{P}\{x(t)<\beta\} \le \beta^{2\kappa} M_4 = \varepsilon.$$

Consequently

$$\liminf_{t\to+\infty} \mathcal{P}\{x(t)\ge\beta\} \ge 1-\varepsilon.$$

This completes the whole proof. □

Remark 3.5. Theorems 3.1–3.3 have an obvious and interesting biological interpretation. Under assumption (A1) and (A2), if $\bar{d}>0$, the population $x(t)$ will be weakly persistent. Under assumption (A1) and (A2), if $\bar{d}<0$ and $\inf_{t\in\overline{R}_+}\{a(t)-b(t+\tau)-c^u\}\ge 0$, the population $x(t)$ will go extinct. That is to say, if assumption (A1) and (A2) hold and $\inf_{t\in\overline{R}_+}\{a(t)-b(t+\tau)-c^u\}\ge 0$, then $\bar{d}$ is the threshold between weak persistence and extinction for the population $x(t)$.

Remark 3.6. With $\bar{d}=\limsup_{t\to+\infty} t^{-1}\int_0^t \big(r(s)-\frac{\sigma_1^2(s)}{2}\big)ds$ in Theorem 3.1–3.3, we note that the stochastic noise on $r(t)$ is detrimental to the survival of the population but the stochastic noise on $a(t)$ has little effect on the persistence or extinction of the population. Thus, in true ecological modeling, the stochastic noise on $r(t)$ should be considered, but the stochastic noise on $a(t)$ could be overlooked in some cases.

Remark 3.7. From Theorem 3.1–3.3, we found that the delay has no effect on the persistence and extinction of the stochastic model in autonomous case.

Remark 3.8. If $b(t)\le 0$ and $c(t)\le 0$ hold, then the condition $\inf_{t\in\overline{R}_+}\{a(t)-b(t+\tau)-c^u\}\ge 0$ in Theorem 3.1–3.2 can be omitted.

Remark 3.9. Liu and Wang [14] studied the persistence and extinction of two stochastic logistic model. Our work extends their results to stochastic population with infinite delay.

4. Examples and numerical simulations

In this section, we explore the behavior of the model (1.2) using numerical solutions . For convenience, we let the probability measure $\mu(\theta)$ be e^θ on $(-\infty,0]$. So the model (1.2) will be written as

$$\begin{aligned} dx(t) = x(t)\Big[&r(t)-a(t)x(t)+b(t)x(t-\tau)+c(t)e^{-t}\int_{-\infty}^0 e^s\xi(s)ds \\ &+c(t)e^{-t}\int_0^t e^s x(s)ds\Big]dt+\sigma_1(t)x(t)dw_1(t)+\sigma_2(t)x^2(t)dw_2(t). \end{aligned} \tag{4.1}$$

By employing the Euler scheme to discretize this equation, where the integral term is approximated by using the composite $\mathscr{K}$-rule as a quadrature [30] and taking the initial values as $\xi(s)=e^{-0.5s}$, $\tau=0.8$. We obtain the discrete approximate solution

$$\begin{aligned} x_{k+1} = x_k + x_k\Big[&r(k\Delta t)-a(k\Delta t)x_k+b(k\Delta t)x_{k-800}+c(k\Delta t)e^{-k\Delta t}\int_{-\infty}^0 e^{0.5\theta}d\theta \\ &+c(k\Delta t)e^{-k\Delta t}\sum_{j=0}^k \omega_j^{(k)} e^{j\Delta t}x_j\Big]\Delta t + x_k(\Delta B_1)_k + x_k^2(\Delta B_2)_k, \end{aligned}$$

where $(\Delta B_i)_k = B_i((k+1)\Delta t) - B_i(k\Delta t)$, $k = 0,1,2,\dots$, $i = 1,2$. The general composite $\mathscr{K}$-rule has weights

$$\{\omega_0^{(k)}, \omega_1^{(k)}, \dots, \omega_k^{(k)}\} = \{\mathscr{K}, 1, \dots, 1-\mathscr{K}\}, \quad \mathscr{K} \in [0,1]$$

and $\sum_{j=0}^{k} \omega_j^{(k)} = k$, $k \geq 0$.

Here, we choose $r(t) = 0.2 + 0.02\sin t$, $a(t) = 0.09$, $b(t) = 0.01$, $c(t) = 0.005$, $\sigma_2(t) = 0.08$, $\mathscr{K} = 0$ and step size $\Delta t = 0.001$. The only difference between conditions of Figure .1(a), Figure 1(b), Figure 1(c) and Figure 1(d) is that the representations of $\sigma_1(t)$ are different. In Figure 1(a), we choose $\sigma_1^2(t) \equiv 0.5$, then $\bar{d} = -0.05$. In view of Theorem 3.1, population $x(t)$ will go to extinction. In Figure 1(b), we consider $\sigma_1^2(t) = 0.4 + 0.01\sin t$, then $\bar{d} = 0$. By Theorem 3.2, population $x(t)$ will be nonpersistent in the mean. In Figure 1(c), we choose $\sigma_1^2(t) \equiv 0.38$, then $\bar{d} = 0.01 > 0$. From Theorem 3.3, the population $x(t)$ will be weakly persistent. In Figure 1(d), we consider $\sigma_1^2(t) \equiv 0.32$, then $(r(t) - \frac{\sigma_1^2(t)}{2})_* = 0.02 > 0$. Using Theorem 3.4, the population $x(t)$ will be stochastic permanence. By using numerical simulations, we find that the stochastic noise on $r(t)$ can change the properties of the population models significantly.

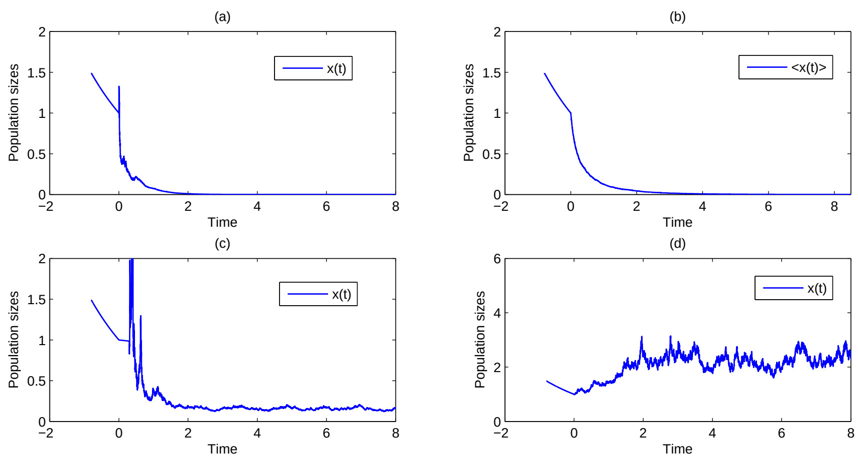

Figure 1. The horizontal axis is time and the vertical axis is the population size $x(t)$ (step size $\Delta t = 0.001$)

Conclusions and future directions. In the real world, the natural growth of population is inevitably affected by stochastic disturbances. In this paper, a stochastic logistic model with infinite delay is proposed and analyzed. With space C_g as phase space, sufficient conditions for extinction are established and nonpersistent in the mean, weak persistence and stochastic permanence. Furthermore, we obtain the threshold between weak persistence and extinction.

Some interesting topics merit further consideration. It is interesting to study what happens if $c(t)$ is stochastic. Another significant problem is devoted to multidimensional stochastic model with infinite delay, and these investigations are in progress.

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