Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 26, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

A FIXED POINT METHOD FOR NONLINEAR EQUATIONS INVOLVING A DUALITY MAPPING DEFINED ON PRODUCT SPACES

JENICĂ CRÎNGANU, DANIEL PAŞCA

Abstract. The aim of this paper is to obtain solutions for the equation

$$J_{q,p}(u_1,u_2) = N_{f,g}(u_1,u_2),$$

where $J_{q,p}$ is the duality mapping on a product of two real, reflexive and smooth Banach spaces X_1, X_2, corresponding to the gauge functions $\varphi_1(t) = t^{q-1}$, $\varphi_2(t) = t^{p-1}$, $1 < q, p < \infty$, $N_{f,g}$ being the Nemytskii operator generated by the Carathéodory functions f, g which satisfies some appropriate conditions. To prove the existence solutions we use a topological method via Leray-Schauder degree. As applications, we obtained in a unitary manner some existence results for Dirichlet and Neumann problems for systems with (q,p)-Laplacian, with (q,p)-pseudo-Laplacian or with (A_q, A_p)-Laplacian.

1. Introduction

In this article we study the existence of solutions for the equation

$$J_{q,p}(u_1,u_2) = N_{f,g}(u_1,u_2) \tag{1.1}$$

in the following functional framework:

(H1) $1 < q, p < \infty$; $\Omega \subset \mathbb{R}^N, N \geq 2$, is a bounded domain with smooth boundary

(H2) X_1, X_2 are real reflexive and smooth Banach spaces, X_1 compactly embedded in $L^{q_1}(\Omega)$ and X_2 compactly embedded in $L^{p_1}(\Omega)$, where

$$1 < q_1 < q^* = \begin{cases} \frac{Nq}{N-q} & \text{if } N > q \\ +\infty & \text{if } N \leq q \end{cases}$$

and

$$1 < p_1 < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } N > p \\ +\infty & \text{if } N \leq p \end{cases}$$

and q^*, p^* are the critical Sobolev exponents of q, p respectively;

(H3) Let $i = 1, 2$. For any gauge functions $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$, the corresponding duality mapping $J_{\varphi_i} : X_i \to X_i^*$ (see the precise definition in Section 2.1 below) is continuous and satisfies the (S_+) condition: if $x_{in} \rightharpoonup x_i$ (weakly)

2000 *Mathematics Subject Classification.* 58C15, 35J20, 35J60, 35J65.

Key words and phrases. Duality mapping; Leray-Schauder degree; (q,p)-Laplacian.

Submitted July 31, 2012. Published January 27, 2013.

in X_i and $\limsup_{n\to\infty}\langle J_{\varphi_i}x_{in}, x_{in}-x_i\rangle \le 0$ then $x_{in}\to x_i$ (strongly) in X_i;

(H4) $J_{q,p}: X_1\times X_2 \to X_1^*\times X_2^*$, $J_{q,p}=(J_q,J_p)$, where J_q, J_p are the duality mappings corresponding to the gauge functions $\varphi_1(t)=t^{q-1}, t\ge 0$, $\varphi_2(t)=t^{p-1}, t\ge 0$ respectively;

(H5) $N_{f,g}: L^{q_1}(\Omega)\times L^{p_1}(\Omega)\to L^{q_1'}(\Omega)\times L^{p_1'}(\Omega)$, where $\frac{1}{q_1}+\frac{1}{q_1'}=1$, $\frac{1}{p_1}+\frac{1}{p_1'}=1$ defined by $N_{f,g}(u_1,u_2)(x)=\big(f(x,u_1(x),u_2(x)), g(x,u_1(x),u_2(x))\big)$, is the Nemytskii operator generated by the Carathéodory functions $f,g:\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, which satisfies the growth conditions

$$|f(x,s,t)|\le c_1|s|^{q_1-1}+c_2|t|^{(q_1-1)\frac{p_1}{q_1}}+b_1(x),\quad \text{for } x\in\Omega, (s,t)\in\mathbb{R}\times\mathbb{R}, \tag{1.2}$$

$$|g(x,s,t)|\le c_3|s|^{(p_1-1)\frac{q_1}{p_1}}+c_4|t|^{p_1-1}+b_2(x),\quad \text{for } x\in\Omega, (s,t)\in\mathbb{R}\times\mathbb{R}, \tag{1.3}$$

where $c_1,c_2,c_3,c_4>0$ are constants, $b_1\in L^{q_1'}(\Omega), b_2\in L^{p_1'}(\Omega)$, $\frac{1}{q_1}+\frac{1}{q_1'}=1$, $\frac{1}{p_1}+\frac{1}{p_1'}=1$.

We make the convention that in the case of a Carathéodory function, the assertion "$x\in\Omega$" is understood in the sense "a.e. $x\in\Omega$".

To prove the existence of the solutions of the problem (1.1) we use topological methods via Leray-Schauder degree.

We note that equality (1.1) is understood in the sense of $X_1^*\times X_2^*$, where the norm on this product space is $\|(x_1^*,x_2^*)\|_{X_1^*\times X_2^*}=\|x_1^*\|_{X_1^*}+\|x_2^*\|_{X_2^*}$. More precisely, let $i_1: X_1\to L^{q_1}(\Omega)$ and $i_2: X_2\to L^{p_1}(\Omega)$ be the identity mappings on X_1, X_2 respectively and $i_1^*: L^{q_1'}(\Omega)\to X_1^*$ and $i_2^*: L^{p_1'}(\Omega)\to X_2^*$ be the corresponding dual:

$$i_1^*u_1^*=u_1^*\circ i_1 \text{ for } u_1^*\in L^{q_1'}(\Omega)\quad i_2^*u_2^*=u_2^*\circ i_2 \text{ for } u_2^*\in L^{p_1'}(\Omega).$$

We define $i: X_1\times X_2\to L^{q_1}(\Omega)\times L^{p_1}(\Omega)$ given by $i(u_1,u_2)=(i_1(u_1),i_2(u_2))$ and its dual $i^*: L^{q_1'}(\Omega)\times L^{p_1'}(\Omega)\to X_1^*\times X_2^*$ given by

$$i^*(u_1^*,u_2^*)=(i_1^*u_1^*, i_2^*u_2^*)=(u_1^*\circ i_1, u_2^*\circ i_2).$$

We say that $(u_1,u_2)\in X_1\times X_2$ is a *solution* of (1.1) if and only if

$$J_{q,p}(u_1,u_2)=i^*N_{f,g}(i(u_1,u_2)) \tag{1.4}$$

or equivalently

$$\begin{aligned}
&\langle J_{q,p}(u_1,u_2),(v_1,v_2)\rangle_{X_1^*\times X_2^*, X_1\times X_2}\\
&=\langle i^*N_{f,g}(i(u_1,u_2)), i(v_1,v_2)\rangle_{L^{q_1'}(\Omega)\times L^{p_1'}(\Omega), L^{q_1}(\Omega)\times L^{p_1}(\Omega)}\\
&=\int_\Omega\big[f(x,u_1(x),u_2(x))v_1(x)+g(x,u_1(x),u_2(x))v_2(x)\big]dx
\end{aligned} \tag{1.5}$$

for all $(v_1,v_2)\in X_1\times X_2$.

The rest of this article is organized as follows. The preliminary and abstract results are presented in Section 2. In Section 3 we prove the existence results for problem (1.1) using the method mentioned above. Section 4 provides some examples.

2. Preliminary results

2.1. **Duality mappings.** Let $i = 1, 2$, $(X_i, \|\cdot\|_{X_i})$ be real Banach spaces, X_i^* the corresponding dual spaces and $\langle \cdot, \cdot \rangle$ the duality between X_i^* and X_i. Let $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ be gauge functions, such that φ_i are continuous, strictly increasing, $\varphi_i(0) = 0$ and $\varphi_i(t) \to \infty$ as $t \to \infty$. The duality mapping corresponding to the gauge function φ_i is the set valued mapping $J_{\varphi_i} : X_i \to 2^{X_i^*}$, defined by

$$J_{\varphi_i} x = \big\{x_i^* \in X_i^* : \langle x_i^*, x_i \rangle = \varphi_i(\|x_i\|_{X_i})\|x_i\|_{X_i},\ \|x_i^*\|_{X_i^*} = \varphi_i(\|x_i\|_{X_i})\big\}.$$

If X_i are smooth, then $J_{\varphi_i} : X_i \to X_i^*$ is defined by

$$J_{\varphi_i} 0 = 0, \quad J_{\varphi_i} x_i = \varphi_i(\|x_i\|_{X_i})\|\ \|'_{X_i}(x_i), \quad x_i \neq 0,$$

and the following metric properties being consequent:

$$\|J_{\varphi_i} x_i\|_{X_i^*} = \varphi_i(\|x_i\|_{X_i}), \quad \langle J_{\varphi_i} x_i, x_i \rangle = \varphi_i(\|x_i\|_{X_i})\|x_i\|_{X_i}. \tag{2.1}$$

Now we define $J_{\varphi_1,\varphi_2} : X_1 \times X_2 \to 2^{X_1^*} \times 2^{X_2^*}$ by $J_{\varphi_1,\varphi_2}(x_1, x_2) = (J_{\varphi_1} x_1, J_{\varphi_2} x_2)$. From (2.1) we obtain

$$\begin{aligned} \|J_{\varphi_1,\varphi_2}(x_1, x_2)\|_{X_1^* \times X_2^*} &= \|J_{\varphi_1} x_1\|_{X_1^*} + \|J_{\varphi_2} x_2\|_{X_2^*} \\ &= \varphi_1(\|x_1\|_{X_1}) + \varphi_2(\|x_2\|_{X_2}), \end{aligned} \tag{2.2}$$

$$\begin{aligned} \langle J_{\varphi_1,\varphi_2}(x_1, x_2), (x_1, x_2) \rangle &= \langle J_{\varphi_1} x_1, x_1 \rangle + \langle J_{\varphi_2} x_2, x_2 \rangle \\ &= \varphi_1(\|x_1\|_{X_1})\|x_1\|_{X_1} + \varphi_2(\|x_2\|_{X_2})\|x_2\|_{X_2}. \end{aligned} \tag{2.3}$$

In what follows we consider the particular case when $J_{\varphi_i} : X_i \to X_i^*$ are the duality mappings, assumed to be single-valued, corresponding to the gauge functions $\varphi_1(t) = t^{q-1}$, $\varphi_2(t) = t^{p-1}$, $1 < q, p < \infty$. In this case we denote $J_{q,p} : X_1 \times X_2 \to X_1^* \times X_2^*$ given by $J_{q,p} = (J_q, J_p)$.

Other properties of the duality mapping are contained in the following propositions:

Proposition 2.1. *$J_{\varphi_i} : X_i \to 2^{X_i^*}$ is single valued if and only if X_i is smooth, if and only if the norm of X_i is Gâteaux differentiable on $X_i \setminus \{0\}$.*

Proposition 2.2. *If X_i is reflexive and $J_{\varphi_i} : X_i \to X_i^*$, then J_{φ_i} is demicontinuous (i.e. if $x_n \to x$ (strongly) in X_i, then $J_{\varphi_i} x_n \rightharpoonup J_{\varphi_i} x$ (weakly) in X_i^*.*

Let us recall that X_i has the *Kadeč-Klee property* ((K-K) for short) if it is strictly convex and for any sequence $(x_{in}) \subset X_i$ such that $x_{in} \rightharpoonup x_i$ (weakly) in X_i and $\|x_{in}\| \to \|x_i\|$ it follows that $x_{in} \to x_i$ (strongly) in X_i.

Proposition 2.3. *If X_i has the (k-k) property and J_{φ_i} is single valued then J_{φ_i} satisfies the (S_+) condition.*

Let us remark that if X_i is locally uniformly convex then X_i has the (k-k) property and then, if in addition, J_{φ_i} is single valued it results that J_{φ_i} satisfies the (S_+) condition. Also, if X_i is reflexive and X_i^* has the (k-k) property then $J_{\varphi_i} : X_i \to X_i^*$ is continuous.

Proposition 2.4. *J_{φ_i} is single valued and continuous if and only if the norm of X_i is Fréchet differentiable.*

Proposition 2.5. *If X_i is reflexive and $J_{\varphi_i} : X_i \to X_i^*$ then J_{φ_i} is surjective. If, in addition X_i is locally uniformly convex then J_{φ_i} is bijective, with its inverse $J_{\varphi_i}^{-1}$ bounded, continuous and monotone.*

For the details and the proofs of he above propositions, see [1, 2, 4]. Clearly, Propositions 2.3 and 2.4, offer sufficient conditions ensuring that hypothesis (H3) be satisfied.

Let i_1 and i_2 the compactly embedded injections of X_1, X_2 in $L^{q_1}(\Omega)$ and $L^{p_1}(\Omega)$ respectively:

$$\begin{aligned} \|i_1(u_1)\|_{L^{q_1}(\Omega)} \le C_1\|u_1\|_{X_1} \quad \text{for all } u_1 \in X_1, \\ \|i_2(u_2)\|_{L^{p_1}(\Omega)} \le C_2\|u_2\|_{X_2} \quad \text{for all } u_2 \in X_2. \end{aligned} \tag{2.4}$$

We introduce

$$\lambda_1 = \inf\Big\{\frac{\|u_1\|_{X_1}^{q_1}}{\|i_1(u_1)\|_{L^{q_1}(\Omega)}^{q_1}} : u_1 \in X_1 \setminus \{0\}\Big\} > 0,$$
$$\lambda_2 = \inf\Big\{\frac{\|u_2\|_{X_2}^{p_1}}{\|i_2(u_2)\|_{L^{p_1}(\Omega)}^{p_1}} : u_2 \in X_2 \setminus \{0\}\Big\} > 0.$$

Proposition 2.6. *λ_1, λ_2 are attained and λ_1^{-1/q_1} and λ_2^{-1/p_1} are the best constants C_1 and C_2, respectively in the writing of the embeddings of X_1 into $L^{q_1}(\Omega)$ and X_2 into $L^{p_1}(\Omega)$, respectively.*

For a proof of the above proposition, see [6, Proposition 4].

2.2. **Nemytskii operators.** Let Ω be an open subset in $\mathbb{R}^N$, $N \ge 1$ and $f, g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions, i.e.:

- (i) for each $(s,t) \in \mathbb{R} \times \mathbb{R}$, the functions $x \mapsto f(x,s,t)$, $x \mapsto g(x,s,t)$ are Lebesgue measurable in Ω;
- (ii) for a.e. $x \in \Omega$, the functions $(s,t) \mapsto f(x,s,t)$, $(s,t) \mapsto g(x,s,t)$ are continuous in $\mathbb{R} \times \mathbb{R}$.

Let $\mathcal{M}$ be the set of all measurable functions $u : \Omega \to \mathbb{R}$. If $f, g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions and $(v_1, v_2) \in \mathcal{M} \times \mathcal{M}$ then the function $x \mapsto (f(x, v_1(x), v_2(x)), g(x, v_1(x), v_2(x)))$ is measurable in Ω. So, we can define the operator $N_{f,g} : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ by

$$N_{f,g}(v_1, v_2)(x) = (f(x, v_1(x), v_2(x)), g(x, v_1(x), v_2(x)))$$

which we will be the Nemytskii operator.

We need the following result:

Lemma 2.7. *Let $r_1, r_2, k_1, k_2 > 0$. Then there are the constants $k_3, k_4 > 0$ such that*

$$k_1 a^{r_1} + k_2 b^{r_2} \le k_3(a+b)^{\max(r_1,r_2)} + k_4, \quad \textit{for all } a, b > 0.$$

Proof. If $a, b \ge 1$ we have

$$\begin{aligned} k_1 a^{r_1} + k_2 b^{r_2} &\le k_1 a^{\max(r_1,r_2)} + k_2 b^{\max(r_1,r_2)} \\ &\le \max(k_1,k_2)(a^{\max(r_1,r_2)} + b^{\max(r_1,r_2)}) \\ &\le \max(k_1,k_2)(a+b)^{\max(r_1,r_2)}, \end{aligned}$$

and the proof is ready with $k_3 = \max(k_1, k_2)$ and $k_4 > 0$ arbitrary.

If $a, b < 1$ then

$$k_1 a^{r_1} + k_2 b^{r_2} \le k_1 + k_2$$

and we may take $k_4 = k_1 + k_2, k_3 > 0$, arbitrary.

If $a \geq 1, b < 1$,

$$k_1 a^{r_1} + k_2 b^{r_2} \leq k_1 a^{r_1} + k_2 \leq k_1(a+b)^{r_1} + k_2 \leq k_1(a+b)^{\max(r_1,r_2)} + k_2,$$

and similarly if $a < 1, b \geq 1$. □

Some properties of the Nemytskii operator that will be used in the sequel are contained in the following proposition.

Proposition 2.8. *Let $p_1, q_1 > 1$, $f, g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory functions which satisfy the growth conditions:*

$$|f(x,s,t)| \leq c_1|s|^{q_1-1} + c_2|t|^{(q_1-1)\frac{p_1}{q_1}} + b_1(x), \quad \text{for } x \in \Omega, (s,t) \in \mathbb{R}\times\mathbb{R}, \tag{2.5}$$

$$|g(x,s,t)| \leq c_3|s|^{(p_1-1)\frac{q_1}{p_1}} + c_4|t|^{p_1-1} + b_2(x), \quad \text{for } x \in \Omega, (s,t) \in \mathbb{R}\times\mathbb{R}, \tag{2.6}$$

where $c_1, c_2, c_3, c_4 > 0$ are constants, $b_1 \in L^{q_1'}(\Omega)$, $b_2 \in L^{p_1'}(\Omega)$, $\frac{1}{q_1} + \frac{1}{q_1'} = 1$, $\frac{1}{p_1} + \frac{1}{p_1'} = 1$.

Then $N_{f,g}$ is continuous from $L^{q_1}(\Omega) \times L^{p_1}(\Omega)$ into $L^{q_1'}(\Omega) \times L^{p_1'}(\Omega)$ and maps bounded sets into bounded sets. Moreover, it holds

$$\|N_{f,g}(v_1,v_2)\|_{L^{q_1'}(\Omega)\times L^{p_1'}(\Omega)} \leq c_8\|(v_1,v_2)\|^{R_1-1}_{L^{q_1}(\Omega)\times L^{p_1}(\Omega)} + c_9, \tag{2.7}$$

for all $(v_1,v_2) \in L^{q_1}(\Omega) \times L^{p_1}(\Omega)$, where $c_8, c_9 > 0$ are constants and $R_1 = \max(q_1,p_1)$.

Proof. From (2.5) and (2.6), for $(v_1,v_2) \in L^{q_1}(\Omega) \times L^{p_1}(\Omega)$ we have

$$\begin{aligned}
&\|N_{f,g}(v_1,v_2)\|_{L^{q_1'}(\Omega)\times L^{p_1'}(\Omega)} \\
&= \|N_f(v_1,v_2)\|_{L^{q_1'}(\Omega)} + \|N_g(v_1,v_2)\|_{L^{p_1'}(\Omega)} \\
&\leq c_1\||v_1|^{q_1-1}\|_{L^{q_1'}(\Omega)} + c_2\Big\||v_2|^{(q_1-1)\frac{p_1}{q_1}}\Big\|_{L^{q_1'}(\Omega)} + \|b_1\|_{L^{q_1'}(\Omega)} \\
&\quad + c_3\Big\||v_1|^{(p_1-1)\frac{q_1}{p_1}}\Big\|_{L^{p_1'}(\Omega)} + c_4\||v_2|^{p_1-1}\|_{L^{p_1'}(\Omega)} + \|b_2\|_{L^{p_1'}(\Omega)} \\
&= c_1\|v_1\|^{q_1-1}_{L^{q_1}(\Omega)} + c_2\|v_2\|^{(q_1-1)\frac{p_1}{q_1}}_{L^{p_1}(\Omega)} + K_1 + c_3\|v_1\|^{(p_1-1)\frac{q_1}{p_1}}_{L^{q_1}(\Omega)} + c_4\|v_2\|^{p_1-1}_{L^{p_1}(\Omega)} + K_2.
\end{aligned}$$

By Lemma 2.7 there are the constants $c_5, c_6, c_7 > 0$, such that

$$\begin{aligned}
&\|N_{f,g}(v_1,v_2)\|_{L^{q_1'}(\Omega)\times L^{p_1'}(\Omega)} \\
&\leq c_5(\|v_1\|_{L^{q_1}(\Omega)} + \|v_2\|_{L^{p_1}(\Omega)})^{\max(p_1-1,q_1-1)} \\
&\quad + c_6(\|v_1\|_{L^{q_1}(\Omega)} + \|v_2\|_{L^{p_1}(\Omega)})^{\max\left((q_1-1)\frac{p_1}{q_1},(p_1-1)\frac{q_1}{p_1}\right)} + c_7 \\
&= c_5\|(v_1,v_2)\|^{\max(p_1-1,q_1-1)}_{L^{q_1}(\Omega)\times L^{p_1}(\Omega)} + c_6\|(v_1,v_2)\|^{\max\left((q_1-1)\frac{p_1}{q_1},(p_1-1)\frac{q_1}{p_1}\right)}_{L^{q_1}(\Omega)\times L^{p_1}(\Omega)} + c_7.
\end{aligned}$$

Since

$$\max\Big((q_1-1)\frac{p_1}{q_1}, (p_1-1)\frac{q_1}{p_1}\Big) \leq \max(p_1-1, q_1-1)$$

we obtain

$$\|N_{f,g}(v_1,v_2)\|_{L^{q_1'}(\Omega)\times L^{p_1'}(\Omega)} \leq c_8\|(v_1,v_2)\|^{R_1-1}_{L^{q_1}(\Omega)\times L^{p_1}(\Omega)} + c_9,$$

for all $(v_1,v_2) \in L^{q_1}(\Omega) \times L^{p_1}(\Omega)$, where $c_8, c_9 > 0$ are constants and $R_1 = \max(q_1,p_1)$.

Now assume that $(v_{1n}, v_{2n}) \to (v_1, v_2)$ in $L^{q_1}(\Omega) \times L^{p_1}(\Omega)$ and claim that $N_{f,g}(v_{1n}, v_{2n}) \to N_{f,g}(v_1, v_2)$ in $L^{q_1'}(\Omega) \times L^{p_1'}(\Omega)$. Given any sequence of (v_{1n}, v_{2n}) there is a further subsequence (call it again (v_{1n}, v_{2n})) such that

$$|v_{1n}(x)| \leqslant h_1(x), |v_{2n}(x)| \leqslant h_2(x)$$

for some $h_1 \in L^{q_1'}(\Omega), h_2 \in L^{p_1'}(\Omega)$. It follows from (2.5) and (2.6) that

$$|f(x, v_{1n}(x), v_{2n}(x))| \leqslant c_1 |h_1(x)|^{q_1-1} + c_2 |h_2(x)|^{(q_1-1)\frac{p_1}{q_1}} + b_1(x),$$
$$|g(x, v_{1n}(x), v_{2n}(x))| \leqslant c_3 |h_1(x)|^{(p_1-1)\frac{q_1}{p_1}} + c_4 |h_2(x)|^{p_1-1} + b_2(x).$$

Since $f(x, v_{1n}(x), v_{2n}(x))$ converges a.e. to $f(x, v_1(x), v_2(x))$, $g(x, v_{1n}(x), v_{2n}(x))$ converges a.e. to $g(x, v_1(x), v_2(x))$, the result follows from the Lebesgue Dominated Convergence Theorem and a standard result on metric spaces. □

3. Existence of solutions for (1.1) using a Leray-Schauder technique

We start we the statement of the Leray-Schauder fixed point theorem.

Theorem 3.1. *Let T be a continuous and compact mapping of a Banach space X into itself, such that the set*

$$\{x \in X : x = \lambda T x \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded. Then T has a fixed point.

Since $X_1 \to L^{q_1}(\Omega)$ and $X_2 \to L^{p_1}(\Omega)$ are compact, the diagram

$$X_1 \times X_2 \xrightarrow{i} L^{q_1}(\Omega) \times L^{p_1}(\Omega) \xrightarrow{N_{f,g}} L^{q_1'}(\Omega) \times L^{p_1'}(\Omega) \xrightarrow{i^*} X_1^* \times X_2^*$$

show that $N_{f,g}$ (by which we mean $i^* N_{f,g} i$) is compact.

By Proposition 2.5, the operator $J_{q,p} : X_1 \times X_2 \to X_1^* \times X_2^*$ is bijective with its inverse $J_{q,p}^{-1}(u_1^*, u_2^*) = (J_q^{-1} u_1^*, J_p^{-1} u_2^*)$ bounded, continuous and monotone.

Consequently (1.1) can be equivalently written

$$(u_1, u_2) = J_{q,p}^{-1} N_{f,g}(u_1, u_2),$$

with $J_{q,p}^{-1} N_{f,g} : X_1 \times X_2 \to X_1 \times X_2$ a compact operator.

We define the operator $T = J_{q,p}^{-1} N_{f,g} = (T_1, T_2)$, where

$$T_1(u_1, u_2) = J_q^{-1} N_f(u_1, u_2), \quad T_2(u_1, u_2) = J_p^{-1} N_g(u_1, u_2) \tag{3.1}$$

and we shall prove that the compact operator T has at least one fixed point using the Leray-Schauder fixed point theorem.

For this it is sufficient to prove that the set

$$S = \left\{(u_1, u_2) \in X_1 \times X_2 : (u_1, u_2) = \alpha T(u_1, u_2) \text{ for some } \alpha \in [0, 1]\right\}$$

is bounded in $X_1 \times X_2$.

By (3.1), (1.2) and (1.3) for $(u_1, u_2) \in X_1 \times X_2$ we have

$$\begin{aligned}
&\|T_1(u_1, u_2)\|_{X_1}^q \\
&= \langle J_q(T_1(u_1, u_2)), T_1(u_1, u_2)\rangle \\
&= \langle N_f(u_1, u_2), T_1(u_1, u_2)\rangle \\
&= \int_\Omega f(x, u_1(x), u_2(x)) T_1(u_1(x), u_2(x)) dx
\end{aligned}$$

$$\le \int_\Omega \Big(c_1|u_1(x)|^{q_1-1} + c_2|u_2(x)|^{(q_1-1)\frac{p_1}{q_1}} + |b_1(x)|\Big)|T_1(u_1(x),u_2(x))|dx,$$

and similarly

$$\begin{aligned}
&\|T_2(u_1,u_2)\|_{X_2}^p \\
&= \langle J_p(T_2(u_1,u_2)), T_2(u_1,u_2)\rangle \\
&= \langle N_g(u_1,u_2), T_2(u_1,u_2)\rangle \\
&= \int_\Omega g(x,u_1(x),u_2(x))T_2(u_1(x),u_2(x))dx \\
&\le \int_\Omega \Big(c_3|u_1(x)|^{(p_1-1)\frac{q_1}{p_1}} + c_4|u_2(x)|^{p_1-1} + |b_2(x)|\Big)|T_2(u_1(x),u_2(x))|dx.
\end{aligned}$$

If $(u_1,u_2) \in S$, that is $(u_1,u_2) = \alpha T(u_1,u_2) = (T_1(u_1,u_2), T_2(u_1,u_2))$ with $\alpha \in [0,1]$, we have

$$\begin{aligned}
&\|T_1(u_1,u_2)\|_{X_1}^q \\
&\le \int_\Omega \Big(c_1\alpha^{q_1-1}|T_1(u_1(x),u_2(x))|^{q_1-1} \\
&\quad + c_2\alpha^{(q_1-1)\frac{p_1}{q_1}}|T_2(u_1(x),u_2(x))|^{(q_1-1)\frac{p_1}{q_1}} + |b_1(x)|\Big)|T_1(u_1(x),u_2(x))|dx \\
&\le c_1\alpha^{q_1-1}\|T_1(u_1,u_2)\|_{L^{q_1}(\Omega)}^{q_1} + c_2\alpha^{(q_1-1)\frac{p_1}{q_1}}\|T_2(u_1,u_2)\|_{L^{p_1}(\Omega)}^{(q_1-1)\frac{p_1}{q_1}}\|T_1(u_1,u_2)\|_{L^{q_1}(\Omega)} \\
&\quad + \|b_1\|_{L^{q_1'}(\Omega)}\|T_1(u_1,u_2)\|_{L^{q_1}(\Omega)} \\
&\le c_1k_1^{q_1}\|T_1(u_1,u_2)\|_{X_1}^{q_1} + c_2k_1k_2^{(q_1-1)\frac{p_1}{q_1}}\|T_2(u_1,u_2)\|_{X_2}^{(q_1-1)\frac{p_1}{q_1}}\|T_1(u_1,u_2)\|_{X_1} \\
&\quad + k_1\|b_1\|_{L^{q_1'}(\Omega)}\|T_1(u_1,u_2)\|_{X_1},
\end{aligned}$$

where $k_1, k_2 > 0$ are coming from the compact embeddings $X_1 \to L^{q_1}(\Omega)$ and $X_2 \to L^{p_1}(\Omega)$, respectively.

In the same way we obtain

$$\begin{aligned}
\|T_2(u_1,u_2)\|_{X_2}^p \le{}& c_3k_1^{(p_1-1)\frac{q_1}{p_1}}k_2\|T_1(u_1,u_2)\|_{X_1}^{(p_1-1)\frac{q_1}{p_1}}\|T_2(u_1,u_2)\|_{X_2} \\
&+ c_4k_2^{p_1}\|T_2(u_1,u_2)\|_{X_2}^{p_1} + k_2\|b_2\|_{L^{p_1'}(\Omega)}\|T_2(u_1,u_2)\|_{X_2}.
\end{aligned}$$

Consequently, for each $(u_1,u_2) \in S$ it hold

$$\begin{aligned}
&\|T_1(u_1,u_2)\|_{X_1}^q - c_5\|T_1(u_1,u_2)\|_{X_1}^{q_1} \\
&- c_6\|T_2(u_1,u_2)\|_{X_2}^{(q_1-1)\frac{p_1}{q_1}}\|T_1(u_1,u_2)\|_{X_1} - c_7\|T_1(u_1,u_2)\|_{X_1} \le 0
\end{aligned}$$

and

$$\begin{aligned}
&\|T_2(u_1,u_2)\|_{X_2}^p - c_8\|T_1(u_1,u_2)\|_{X_1}^{(p_1-1)\frac{q_1}{p_1}}\|T_2(u_1,u_2)\|_{X_2} \\
&- c_9\|T_2(u_1,u_2)\|_{X_2}^{p_1} - c_{10}\|T_2(u_1,u_2)\|_{X_2} \le 0,
\end{aligned}$$

with $c_5,\dots,c_{10}$ positive constants.

Lemma 3.2. *Let $q > p > 1$, $1 < p_1 < p$, $1 < q_1 < q$ and $a, b > 0$ such that*

$$\begin{gathered}
a^q \le c_5a^{q_1} + c_6ab^{(q_1-1)\frac{p_1}{q_1}} + c_7a, \\
b^p \le c_8a^{(p_1-1)\frac{q_1}{p_1}}b + c_9b^{p_1} + c_{10}b,
\end{gathered}$$

where $c_5, \dots, c_{10} > 0$ positve constants. Then there is the constant $K > 0$ be such that $a + b \leq K$.

Proof. We consider the following cases:

(1) If $a \leq 1, b \leq 1$ then $a + b \leq 2$.
(2) If $a \leq 1, b > 1$ we have $b^p \leq c_8 b + c_9 b^{p_1} + c_{10} b$ and since $p > p_1 > 1$, there is a constant $K_1 > 0$ such that $b \leq K_1$. Consequently $a + b \leq 1 + K_1$.
(3) If $a > 1, b \leq 1$ we have $a^q \leq c_5 a^{q_1} + c_6 a + c_7 a$ and since $q > q_1 > 1$, there is a constant $K_2 > 0$ such that $a \leq K_2$. Consequently $a + b \leq 1 + K_2$.
(4) We consider $a > 1, b > 1$. Let us remark that

$$\max\left((q_1 - 1)\frac{p_1}{q_1}, (p_1 - 1)\frac{q_1}{p_1}\right) \leq \max(p_1 - 1, q_1 - 1).$$

If $a \geq b$ we have $a^q \leq c_5 a^{q_1} + c_6 a b^{\max(p_1 - 1, q_1 - 1)} + c_7 a \leq c_5 a^{q_1} + c_6 a^{\max(p_1, q_1)} + c_7 a$, and since $q > q_1, q > \max(p_1, q_1) > 1$, there is a constant $K_3 > 0$ such that $a \leq K_3$ and so $a + b \leq 2K_3$. If $a \leq b$ we reasoning similarly. $\square$

Now, by Lemma 3.2, there exists a constant $K > 0$ such that $\|T(u_1, u_2)\|_{X_1 \times X_2} = \|T_1(u_1, u_2)\|_{X_1} + \|T_2(u_1, u_2)\|_{X_2} \leq K$ for $(u_1, u_2) \in S$ and then

$$\|(u_1, u_2)\|_{X_1 \times X_2} = \alpha \|T(u_1, u_2)\| \leq \alpha K \leq K, \quad \text{for } (u_1, u_2) \in S,$$

that is S is bounded. We have obtained the following result.

Theorem 3.3. *Assume that X_1, X_2 are locally uniformly convex, $J_q : X_1 \to X_1^*$, $J_p : X_2 \to X_2^*$ and the Carathèodory functions f and g satisfy (1.2) and (1.3), respectively with $q_1 \in (1, q)$ and $p_1 \in (1, p)$. Then the operator $T = J_{q,p}^{-1} N_{f,g}$ has one fixed point in $X_1 \times X_2$ or equivalently problem (1.1) has a solution. Moreover, the set of solutions of problem (1.1) is bounded in $X_1 \times X_2$.*

4. Examples

4.1. **Dirichlet problem for systems with (q, p)-Laplacian.** If $X_1 \times X_2 = W_0^{1,q}(\Omega) \times W_0^{1,p}(\Omega)$, then $J_{q,p} = (-\Delta_q, -\Delta_p)$ and the solutions set of equation $J_{q,p}(u_1, u_2) = N_{f,g}(u_1, u_2)$ coincides with the solutions set of the Dirichlet problem

$$\begin{gathered} -\Delta_q u_1 = f(x, u_1, u_2) \quad \text{in } \Omega, \\ -\Delta_p u_2 = g(x, u_1, u_2) \quad \text{in } \Omega, \\ u_1 = u_2 = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.1}$$

4.2. **Neumann problem for systems with (q, p)-Laplacian.** We consider $X_1 \times X_2 = W^{1,q}(\Omega) \times W^{1,p}$, endowed with the norm

$$\|(u_1, u_2)\| = \|u_1\|_{1,q} + \|u_2\|_{1,p}$$

where

$$\begin{gathered} \|u_1\|_{1,q}^q = \|u_1\|_{0,q}^q + \||\nabla u_1|\|_{0,q}^q \quad \text{for all } u_1 \in W^{1,q}(\Omega), \\ \|u_2\|_{1,p}^p = \|u_2\|_{0,p}^p + \||\nabla u_2|\|_{0,p}^p \quad \text{for all } u_2 \in W^{1,p}(\Omega), \end{gathered}$$

which are equivalent with the standard norms on the spaces $W^{1,q}(\Omega)$, $W^{1,p}(\Omega)$ respectively (see [3]).

In this case, the duality mappings J_q, J_p on $\big(W^{1,q}(\Omega), \|\cdot\|_{1,q}\big)$, $\big(W^{1,p}(\Omega), \|\cdot\|_{1,p}\big)$, respectively, corresponding to the gauge functions $\varphi_1(t) = t^{q-1}$ and $\varphi_2(t) = t^{p-1}$ are defined by

$$\begin{gathered} J_q : \big(W^{1,q}(\Omega), \|\cdot\|_{1,q}\big) \to \big(W^{1,q}(\Omega), \|\cdot\|_{1,q}\big)^* \\ J_q u_1 = -\Delta_q u_1 + |u_1|^{q-2}u_1 \text{ for all } u_1 \in W^{1,q}(\Omega) \end{gathered} \tag{4.2}$$

$$\begin{gathered} J_p : \big(W^{1,p}(\Omega), \|\cdot\|_{1,p}\big) \to \big(W^{1,p}(\Omega), \|\cdot\|_{1,p}\big)^* \\ J_p u_2 = -\Delta_p u_2 + |u_2|^{p-2}u_2 \quad \text{ for all } u_2 \in W^{1,p}(\Omega) \end{gathered} \tag{4.3}$$

(see [5]).

By a weak solution of the Neumann problem

$$\begin{gathered} -\Delta_q u_1 + |u_1|^{q-2}u_1 = f(x, u_1, u_2) \quad \text{in } \Omega, \\ -\Delta_p u_2 + |u_2|^{p-2}u_2 = g(x, u_1, u_2) \quad \text{in } \Omega, \\ |\nabla u_1|^{q-2}\frac{\partial u_1}{\partial n} = 0 \quad \text{on } \partial\Omega, \\ |\nabla u_2|^{p-2}\frac{\partial u_2}{\partial n} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4.4}$$

we mean an element $(u_1, u_2) \in W^{1,q}(\Omega) \times W^{1,p}(\Omega)$ which satisfies

$$\begin{aligned} &\int_\Omega |\nabla u_1(x)|^{q-2}\nabla u_1(x)\nabla v_1(x)dx + \int_\Omega |u_1(x)|^{q-2}u_1(x)v_1(x)dx \\ &+ \int_\Omega |\nabla u_2(x)|^{p-2}\nabla u_2(x)\nabla v_2(x)dx + \int_\Omega |u_2(x)|^{p-2}u_2(x)v_2(x)dx \\ &= \int_\Omega f(x, u_1(x), u_2(x))v_1(x) + g(x, u_1(x), u_2(x))v_2(x)dx, \end{aligned} \tag{4.5}$$

for all $(v_1, v_2) \in W^{1,q}(\Omega) \times W^{1,p}(\Omega)$.

It is easy to see that $(u_1, u_2) \in W^{1,q}(\Omega) \times W^{1,p}(\Omega)$ is a solution of the problem (4.4), in the sense of (4.5) if and only if

$$J_{q,p}(u_1, u_2) = (i^* N_{f,g} i)(u_1, u_2),$$

where $J_{q,p}(u_1, u_2) = (J_q u_1, J_p u_2)$ and J_q, J_p are given by (4.2) and (4.3), $i(u_1, u_2) = (i_1 u_1, i_2 u_2)$, and $i_1 : W^{1,q}(\Omega) \to L^{q_1}(\Omega)$, $i_2 : W^{1,p}(\Omega) \to L^{p_1}(\Omega)$ are the compact embeddings of $W^{1,q}(\Omega)$ into $L^{q_1}(\Omega)$ and of $W^{1,p}(\Omega)$ into $L^{p_1}(\Omega)$, respectively. By $i^* : L^{q_1'}(\Omega) \times L^{p_1'}(\Omega) \to (W^{1,q}(\Omega), \|\cdot\|_{1,q})^* \times (W^{1,p}(\Omega), \|\cdot\|_{1,p})^*$ we denoted the dual of i.

So, we are in the functional framework described in introduction. Indeed, the spaces $\big(W^{1,q}(\Omega), \|\cdot\|_{1,q}\big)$ and $\big(W^{1,p}(\Omega), \|\cdot\|_{1,p}\big)$ are smooth reflexive Banach spaces, compactly embedded in $L^{q_1}(\Omega)$ and $L^{p_1}(\Omega)$, respectively. $J_q : \big(W^{1,q}(\Omega), \|\cdot\|_{1,q}\big) \to \big(W^{1,q}(\Omega), \|\cdot\|_{1,q}\big)^*$ and $J_p : \big(W^{1,p}(\Omega), \|\cdot\|_{1,p}\big) \to \big(W^{1,p}(\Omega), \|\cdot\|_{1,p}\big)^*$ are single valued, continuous and satisfies the (S_+) condition (see [3]). Consequently, the existence result given in section 3 becomes the existence result for the Neumann problem (4.4).

Remark 4.1. We note that using the same method it is possible to proved the existence of a solution for the Dirichlet and Neumann problems with (q, p)–pseudo-Laplacian or with (A_q, A_p)–Laplacian (see [6]).

Remark 4.2. In [5] the authors used the same method to proved the existence of a solution for the Dirichlet problem with p-Laplacian.

References

[1] F. E. Browder; *Problèmes Non-Lineaires*, Les Presses de l'Université de Montreal, 1964.
[2] I. Ciorănescu; *Duality mapping in nonlinear functional analysis*, Publishing House of the Romanian Academy, Bucharest, 1974.
[3] J. Crînganu; *Variational and topological methods for Neumann problems with p-Laplacian*, Communications on Applied Nonlinear Analysis 11 (2004), 1-38.
[4] G. Dincă, P. Jebelean, J. Mawhin; *Variational and Topological methods for Dirichlet problems with p-Laplacian*, Portugaliae Mathematica vol. 58 no. 3 (2001), 339-378.
[5] G. Dincă, P. Jebelean; *Une méthode de point fixe pour le p-Laplacien*, C.R. Acad. Sci. Paris t. 324, Serie I (1997), 165-168.
[6] G. Dincă, P. Jebelean; *Some existence results for a class of nonlinear equations involving a duality mapping*, Nonlinear Analysis 46 (2001), 347-363.

Jenică Crînganu
Department of Mathematics, University of Galaţi, Str. Domnească 47, Galaţi, Romania
E-mail address: jcringanu@ugal.ro

Daniel Paşca
Department of Mathematics and Informatics, University of Oradea, University Street 1, 410087 Oradea, Romania
E-mail address: dpasca@uoradea.ro