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A UNIQUENESS RESULT FOR AN INVERSE PROBLEM IN A SPACE-TIME FRACTIONAL DIFFUSION EQUATION

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ABSTRACT. Fractional (nonlocal) diffusion equations replace the integer-order derivatives in space and time by fractional-order derivatives. This article considers a nonlocal inverse problem and shows that the exponents of the fractional time and space derivatives are determined uniquely by the data $u(t,0)=g(t)$, $0<t<T$. The uniqueness result is a theoretical background for determining experimentally the order of many anomalous diffusion phenomena, which are important in physics and in environmental engineering.

1. Introduction

The classical diffusion equation $\partial_t u = \Delta u$ is used to describe a cloud of spreading particles at the macroscopic level. The point source solution is a Gaussian probability density that predicts the relative particle concentration. For microscopic picture, Brownian motion is employed, which describes the path of individual particles. The space-time fractional diffusion equation $\partial_t^\beta u = \Delta^{\alpha/2} u$ with $0<\beta<1$ and $0<\alpha<2$ is used to model anomalous diffusion [11]. Here, the fractional derivative in time is used to describe particle sticking and trapping phenomena and the fractional space derivative is used to model long particle jumps. These two effects combined together produce a concentration profile with a sharper peak, and heavier tails. The fractional-time derivative considered here is the Caputo fractional derivative of order $0<\beta<1$ and is defined as

$$\frac{\partial^\beta f(t)}{\partial t^\beta} := \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial f(r)}{\partial r} \frac{dr}{(t-r)^\beta}, \tag{1.1}$$

where Γ is the Gamma function. This is intended to properly handle initial values [2, 3, 5], since its Laplace transform $s^\beta \tilde{f}(s) - s^{\beta-1} f(0)$ incorporates the initial value in the same way the first derivative does. Here, $\tilde{f}(s)$ is the usual Laplace transform. It is well-known that the Caputo derivative has a continuous spectrum [3, 12], with eigenfunctions given in terms of the Mittag-Leffler function

$$E_\beta(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(1+\beta k)}.$$

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In fact, it is easy to see that, $f(t) = E_\beta(-\lambda t^\beta)$ solves the eigenvalue equation

$$\frac{\partial^\beta f(t)}{\partial t^\beta} = -\lambda f(t),$$

for any $\lambda > 0$. This is easily verified by differentiating term-by-term and using the fact that t^p has Caputo derivative $t^{p-\beta}\frac{\Gamma(p+1)}{\Gamma(p+1-\beta)}$ for $p > 0$ and $0 < \beta \le 1$. For slow diffusion we take $0 < \beta < 1$, which is related to parameter specification of the large-time behaviour of the waiting-time distribution function, see [12] and references therein.

For $0 < \alpha < 2$, $\Delta^{\alpha/2} f$ denotes the fractional Laplacian, defined for

$$f \in \mathrm{Dom}(\Delta^{\alpha/2}) = \big\{ f \in L^2(\mathbb{R}^d; dx) : \int_{\mathbb{R}^d} |\xi|^\alpha |\widehat{f}(\xi)|^2 \, d\xi < \infty \big\}$$

as the function with Fourier transform

$$\widehat{\Delta^{\alpha/2} f}(\xi) = -|\xi|^\alpha \widehat{f}(\xi), \tag{1.2}$$

where $\widehat{f}(\xi)$ denotes the Fourier transform of $f(x)$. For sufficiently regular functions (for example, C^2 functions with bounded second derivatives), the fractional Laplacian can be defined pointwise by

$$\Delta^{\alpha/2} f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \big(f(x+y) - f(x) - \nabla f(x) \cdot y 1_{\{|y| \le 1\}} \big) \frac{c_{d,\alpha}}{|y|^{d+\alpha}} \, dy, \tag{1.3}$$

where $c_{d,\alpha} > 0$ is given by

$$c_{d,\alpha} \int_{y \in \mathbb{R}^d} \frac{1 - \cos(y_1)}{|y|^{d+\alpha}} \, dy = 1.$$

Similar formulations have been observed before, see [3] and references therein. We note that the fractional Laplacian used here is usually called the Riesz fractional derivative (see [15] for details) in the fractional calculus community. Following [3], we observe that if f is bounded and continuous on $\mathbb{R}^d$ and f is C^2 in an open set D, then $\Delta^{\alpha/2} f$ exists pointwise and is continuous in D. Moreover, if f is C^1-function on $[0, \infty)$ satisfying $|f'(t)| \le C t^{\gamma - 1}$ for some $\gamma > 0$, then by (1.1), the Caputo derivative $\frac{\partial^\beta f(t)}{\partial t^\beta}$ of f exists for all $t > 0$ and the derivative is continuous in $t > 0$. The reader is referred to Kilbas et al. [7] and Podlubny [12] for properties of the Caputo derivative.

Important references documenting the recent interest on inverse problems with fractional derivatives are [4, 6, 9, 10, 13, 14, 16, 17, 18]. A substantial difference between our study and those in the latter references is that we consider fractional derivatives both in the time and space variable whereas those study only consider fractional derivatives in the time variable. On the other hand, our work shares with those studies the use of eigenfunction expansion of weak solutions to the initial/boundary value problem.

The main purpose of this article is to establish the determination of the unique exponents β and α in the fractional time and space derivatives by means of the observation data $u(t, 0) = g(t)$, $0 < t < T$. This uniqueness result may lead to the identification of anomalous diffusions. For the sake of simplicity and for the technical reasons on eigenvalue bounds, throughout this paper we consider $d = 1$ and $\alpha \in (1/2, 2)$.

This article is organized as follows: In the next section we provide a review of main properties of the direct problem and introduce the inverse problem. Section 3 includes both the statement and the proof of the main result of this paper.

2. Analysis of the direct problem and formulation of the inverse problem

First we consider the direct problem

$$\begin{gathered}\frac{\partial^\beta}{\partial t^\beta}u(t,x)=\Delta^{\alpha/2}u(t,x),\quad -1<x<1,\ 0<t<T,\\ u(t,-1)=u(t,1)=0,\quad 0<t<T,\\ u(0,x)=f(x),\quad -1<x<1.\end{gathered}\tag{2.1}$$

Here $T>0$ is a final time and f is a given function.

Definition 2.1 ([3]). A function $u(t,x)$ is said to be a weak solution of (2.1) if the following conditions hold:

$$\begin{gathered}u(t,\cdot)\in W_0^{\alpha/2,2}(D)\quad \text{for each } t>0,\\ \lim_{t\downarrow 0}u(x,t)=f(x)\quad \text{a.e.},\\ \frac{\partial^\beta}{\partial t^\beta}u(t,x)=\Delta^{\alpha/2}u(t,x)\quad \text{in the distrubitional sense.}\end{gathered}\tag{2.2}$$

Here $W_0^{\alpha/2,2}(D)$ is the $\sqrt{\epsilon_1}$-completion of the space $C_c^\infty(D)$ of smooth functions with compact support in D where $\epsilon_1(u,u)=\epsilon(u,u)+\int_{\mathbb{R}}u(x)^2dx$ and $\epsilon(u,v)=\varepsilon^D(u,v)$ for $u,v\in W_0^{\alpha/2,2}(D)$. We note that such a definition can be given for any bounded domain D in $\mathbb{R}^d$, replacing $(-1,1)$. The last condition of (2.2) is equivalent to the following (which comes from multiplying the equation by the test function $\phi(x)\psi(t)$, integration by parts and symmetry, for details see [3]): for each $\psi\in C_c^1(0,\infty)$ and $\phi\in C_c^2(D)$,

$$\int_{\mathbb{R}}\Big(\int_0^\infty u(t,x)\frac{\partial^\beta\psi(t)}{\partial t^\beta}\,dt\Big)\phi(x)\,dx=\int_0^\infty\varepsilon^D(u(t,\cdot),\phi)\psi(t)\,dt,$$

where $D=(-1,1)$, $C_c^1(0,\infty)$ is the space of compactly supported C^1 (the class of continuously differentiable functions) functions, $C_c^2(D)$ is the space of compactly supported C^2 (the class of twice continuously differentiable functions) functions,

$$\varepsilon^D(u,v)=\frac{c_\alpha}{2}\iint_{\mathbb{R}\times\mathbb{R}}\frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{1+\alpha}}\,dx\,dy,$$

for $u,v\in\mathcal{F}$, $c_\alpha>0$ is a constant, $\varepsilon^D(u,v)$ comes from variational formulation and symmetry (see [3] for details),

$$\mathcal{F}:=W^{\frac{\alpha}{2},2}(\mathbb{R}):=\Big\{u\in L^2(\mathbb{R};dx):\iint_{\mathbb{R}\times\mathbb{R}}\frac{(u(x)-u(y))^2}{|x-y|^{1+\alpha}}\,dx\,dy<\infty\Big\}.$$

Following [3], we obtain a useful formula for the weak solution of (2.1):

$$\begin{aligned} u(t,x) &= \int_0^\infty \mathbb{E}_x[f(X_s); s < \tau_D] f_t(s)\, ds \\ &= \int_0^\infty \Big(\sum_{n=1}^\infty e^{-s\lambda_n} \langle f, \psi_n\rangle \psi_n(x) \Big) f_t(s)\, ds \\ &= \sum_{n=1}^\infty E_\beta(-\lambda_n t^\beta) \langle f, \psi_n\rangle \psi_n(x), \end{aligned} \tag{2.3}$$

where the last equality comes from a conditioning argument (see [3] for more details), $\mathbb{E}_x$ is the expected value with respect to x, $\{\lambda_n\}_{n\ge1}$ is a sequence of positive numbers $0 < \lambda_1 \le \lambda_2 \le \dots$, $\{\psi_n\}_{n\ge1}$ is an orthonormal basis for $L^2(D; dx)$ and for any $f \in L^2(D; dx)$ has the representation

$$f(x) = \sum_{n=1}^\infty \langle f, \psi_n\rangle \psi_n(x).$$

The following lemma indicates an important property of the Mittag-Leffler function, see [7, 12] for more details.

Lemma 2.2. *For each $\alpha < 2$ and $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$ there exists a constant $C_0 > 0$ such that*

$$|E_\beta(z)| \le \frac{C_0}{1+|z|}, \quad \mu \le |arg(z)| \le \pi. \tag{2.4}$$

We recall the following result from [8] about the asymptotic behavior of the eigenvalues of the fractional Laplacian in an interval, which is used for further estimates and proofs. We note that in the higher dimensional case, the estimates are not so explicit, though we believe that a similar uniqueness result holds.

Theorem 2.3. *The eigenvalues of the spectral problem for the one-dimensional fractional Laplace operator, i.e. $(-\Delta)^{\alpha/2}u(x) = \lambda u(x)$, in the interval $D \subset \mathbb{R}$ satisfy the asymptotic equality*

$$\lambda_n = \Big(\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}\Big)^\alpha + O\big(\frac{1}{n}\big). \tag{2.5}$$

Next, we show that the series on the right-hand side of (2.3) is uniformly convergent in $x \in [-1, 1]$ and $t \in (0, T]$. For this purpose, we use the following inequalities for the eigenvalues and corresponding eigenvectors of the one-dimensional fractional Laplacian:

$$\begin{gathered} C_1 n^\alpha \le \lambda_n \le C_2 n^\alpha, \quad n \ge 1, \\ |\langle f, \psi_n\rangle| \le \sqrt{M} \lambda_n^{-k}, \\ |\psi_n(x)| \le C_3 \lambda_n^{1/2\alpha}, \end{gathered} \tag{2.6}$$

where C_i, $i = 1, 2, 3$ are positive constants, and

$$M := \sum_{n=1}^\infty \lambda_n^{2k} \langle f, \psi_n\rangle^2 < \infty,$$

for some k that satisfies the following inequality (see [1, 3] for details):

$$k > -1 + \frac{7}{2\alpha}. \tag{2.7}$$

Then by (2.4) and (2.6) we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} \max_{0\le x\le l} |E_\beta(-\lambda_n t^\beta)\langle f,\psi_n\rangle \psi_n(x)| &\le \sqrt{M} C_0 C_3 \sum_{n=1}^{\infty} \frac{1}{1+|\lambda_n t^\beta|} \lambda_n^{-k} \lambda_n^{1/2\alpha} \\
&\le C_\star \sum_{n=1}^{\infty} \frac{1}{1+|\lambda_n t^\beta|} n^{\alpha(\frac{1}{2\alpha}-k)} < \infty,
\end{aligned}
\tag{2.8}
$$

where $C_\star = \sqrt{M} C_0 C_2^2 C_3$ is a positive constant. Now using (2.5), we see that the series on the last line of (2.8) is uniformly convergent if $k > -1 + \frac{3}{2\alpha}$, which is already guaranteed by the condition (2.7).

The inverse problem consists of determining the unknown orders β and α of the time and space derivatives in the space-time fractional diffusion problem (2.1) from the measured output data (also called additional condition)

$$
u(t,0) = g(t), \quad 0 < t < T. \tag{2.9}
$$

For technical reasons in the proof of determining the exponents β and α uniquely, we will need a specific class of the initial functions $f(x)$ satisfying

$$
\langle f(x), \psi_n(x)\rangle > 0 \quad (\text{or } \langle f(x), \psi_n(x)\rangle < 0) \quad \text{for all } n \ge 1, \tag{2.10}
$$

where $\{\psi_n(x)\}_{n\ge 1}$ is the orthonormal basis for $L^2(D;dx)$ considered above. In this article, we assume that $g(t) \not\equiv 0$. Next section is devoted to the statement and the proof of the uniqueness result for the inverse problem.

3. Statement and the proof of the main result

The main result of the paper, whose proof is also included in this section, reads as follows.

Theorem 3.1. *Let u be the weak solution of* (2.1) *and let v be the weak solution of the following equation with the same initial and boundary conditions:*

$$
\frac{\partial^\gamma}{\partial t^\gamma} u(t,x) = \Delta^{\eta/2} u(t,x), \quad -1 < x < 1,\ 0 < t < T. \tag{3.1}
$$

If $u(t,0) = v(t,0)$, $0 < t < T$ and (2.10) *holds, then $\beta = \gamma$ and $\alpha = \eta$.*

Proof. Using the explicit formula (2.3), the weak solutions $u(t,x)$ and $v(t,x)$ can be written as follows:

$$
u(t,x) = \sum_{n=1}^{\infty} E_\beta(-\lambda_n t^\beta)\langle f,\psi_n\rangle \psi_n(x), \tag{3.2}
$$

$$
v(t,x) = \sum_{n=1}^{\infty} E_\gamma(-\mu_n t^\gamma)\langle f,\varphi_n\rangle \varphi_n(x). \tag{3.3}
$$

Here ψ_n and φ_n are the eigenfunctions corresponding to λ_n and μ_n which are the eigenvalues of the equations $(-\Delta)^{\alpha/2}u(x) = \lambda u(x)$ and $(-\Delta)^{\eta/2}v(x) = \lambda v(x)$, respectively satisfying $\psi_n(0) = 1$ and $\varphi_n(0) = 1$. Consequently, assuming that $u(t,0) = v(t,0)$ we have

$$
\sum_{n=1}^{\infty} E_\beta(-\lambda_n t^\beta)\langle f,\psi_n\rangle = \sum_{n=1}^{\infty} E_\gamma(-\mu_n t^\gamma)\langle f,\varphi_n\rangle. \tag{3.4}
$$

Now we derive an asymptotic equality for the left-hand side of (3.4) using the following well known asymptotic property of the Mittag-Leffler function [7, 12]:

$$E_\beta(-t) = \frac{1}{t\Gamma(1-\beta)} + O(|t|^{-2}). \tag{3.5}$$

By (2.5) and (3.5), there exists a constant $C_4 > 0$ such that

$$\Big|E_\beta(-\lambda_n t^\beta) - \frac{1}{\Gamma(1-\beta)}\frac{1}{\lambda_n t^\beta}\Big| \le \frac{C_4}{t^{2\beta}}. \tag{3.6}$$

Now we use the asymptotic behaviour of (3.6) on the left-hand side of (3.4). For this purpose, we add and subtract the term $\frac{1}{\Gamma(1-\beta)\lambda_n t^\beta}$ to right hand side of (3.2). Then, we obtain

$$\begin{aligned} &\sum_{n=1}^\infty E_\beta(-\lambda_n t^\beta)\langle f,\psi_n\rangle \\ &= \sum_{n=1}^\infty \langle f,\psi_n\rangle\Big[\frac{1}{\Gamma(1-\beta)\lambda_n t^\beta} + \big\{E_\beta(-\lambda_n t^\beta) - \frac{1}{\Gamma(1-\beta)\lambda_n t^\beta}\big\}\Big]. \end{aligned} \tag{3.7}$$

Finally, we get the following asymptotic equality for the left hand side of (3.4) by using (3.6) in (3.7)

$$\sum_{n=1}^\infty E_\beta(-\lambda_n t^\beta)\langle f,\psi_n\rangle = \sum_{n=1}^\infty \langle f,\psi_n\rangle \frac{1}{\Gamma(1-\beta)}\frac{1}{\lambda_n t^\beta} + O\big(|\frac{1}{t^{2\beta}}|\big). \tag{3.8}$$

Arguing similarly for $\sum_{n=1}^\infty E_\gamma(-\mu_n t^\gamma)\langle f,\varphi_n\rangle$, we have

$$\sum_{n=1}^\infty E_\gamma(-\mu_n t^\gamma)\langle f,\varphi_n\rangle = \sum_{n=1}^\infty \langle f,\varphi_n\rangle \frac{1}{\Gamma(1-\gamma)}\frac{1}{\mu_n t^\gamma} + O\big(|\frac{1}{t^{2\gamma}}|\big). \tag{3.9}$$

Therefore, from (3.4), (3.8) and (3.9), we have, as $t \to \infty$,

$$\begin{aligned} &\sum_{n=1}^\infty \langle f,\psi_n\rangle \frac{1}{\Gamma(1-\beta)}\frac{1}{\lambda_n t^\beta} + O\big(|\frac{1}{t^{2\beta}}|\big) \\ &= \sum_{n=1}^\infty \langle f,\varphi_n\rangle \frac{1}{\Gamma(1-\gamma)}\frac{1}{\mu_n t^\gamma} + O\big(|\frac{1}{t^{2\gamma}}|\big). \end{aligned} \tag{3.10}$$

To complete the proof, for the moment, we suppose that $\beta > \gamma$. Then multiplying (3.10) by t^γ yields that

$$\begin{aligned} &-\frac{t^\gamma}{t^\beta}\sum_{n=1}^\infty \langle f,\psi_n\rangle \frac{1}{\Gamma(1-\beta)}\frac{1}{\lambda_n} + O\big(|\frac{t^\gamma}{t^{2\beta}}|\big) \\ &+ \sum_{n=1}^\infty \langle f,\varphi_n\rangle \frac{1}{\Gamma(1-\gamma)}\frac{1}{\mu_n} + O\big(|\frac{1}{t^{\gamma}}|\big) = 0. \end{aligned} \tag{3.11}$$

Letting $t \to \infty$ in (3.11), we deduce that

$$\sum_{n=1}^\infty \langle f,\varphi_n\rangle \frac{1}{\Gamma(1-\gamma)}\frac{1}{\mu_n} = 0. \tag{3.12}$$

By (2.10), this is a contradiction since the left-hand side of (3.12) can not be zero. Similarly, the assumption $\gamma > \beta$ leads to a contradiction. Therefore, we conclude that $\gamma = \beta$, and this completes the first part of the proof.

Now, we prove the second part of the theorem, i.e. $\alpha = \eta$. For this purpose we show $\lambda_n = \mu_n$, $n = 1, 2, 3, \dots$. By (3.4) and $\gamma = \beta$ we have

$$\sum_{n=1}^{\infty} E_\beta(-\lambda_n t^\beta)\langle f, \psi_n\rangle = \sum_{n=1}^{\infty} E_\beta(-\mu_n t^\beta)\langle f, \varphi_n\rangle. \tag{3.13}$$

We take the Laplace transform of $E_\beta(-\lambda_n t^\beta)$ as follows:

$$\int_0^\infty e^{-zt} E_\beta(-\lambda_n t^\beta)\, dt = \frac{z^{\beta-1}}{z^\beta + \lambda_n}, \quad \operatorname{Re} z > 0. \tag{3.14}$$

Moreover, if we take the Laplace transform of the Mittag-Leffler function term by term, we obtain

$$\int_0^\infty e^{-zt} E_\beta(-\lambda_n t^\beta)\, dt = \frac{z^{\beta-1}}{z^\beta + \lambda_n}, \quad \operatorname{Re} z > \lambda_n^{1/\beta}. \tag{3.15}$$

Then by (2.4), we conclude that $\sup_{t\ge 0} |E_\beta(-\lambda_n t^\beta)| < \infty$, and this implies that $\int_0^\infty e^{-zt} E_\beta(-\lambda_n t^\beta)\, dt$ is analytic in z for $\operatorname{Re} z > 0$. Thus the analytic continuation yields (3.14) for $\operatorname{Re} z > 0$. By using (2.4), (2.5), (2.6) and Lebesgue's convergence theorem, we get that $e^{-t \operatorname{Re} z} t^{-\beta}$ is integrable for $t \in (0, \infty)$ with fixed z satisfying $\operatorname{Re} z > 0$ and

$$\begin{aligned} \Big| e^{-t \operatorname{Re} z} \sum_{n=1}^{\infty} \langle f, \psi_n\rangle E_\beta(-\lambda_n t^\beta)\Big| &\le C_0 e^{-t \operatorname{Re} z}\Big(\sum_{n=2}^{\infty} \langle f, \psi_n\rangle \frac{1}{|\lambda_n|}\frac{1}{t^\beta}\Big) \\ &\le \frac{C_0'}{t^\beta} e^{-t \operatorname{Re} z} \sum_{n=1}^{\infty} n^{-\alpha(k+1)}, \end{aligned}$$

where $C_0' = \sqrt{M} C_0 C_2$. Here we note that the series $\sum_{n=1}^{\infty} n^{-\alpha(k+1)}$ is convergent by (2.7). Then, for $\operatorname{Re} z > 0$ we obtain

$$\int_0^\infty e^{-zt} \sum_{n=1}^{\infty} \langle f, \psi_n\rangle E_\beta(-\lambda_n t^\beta)\, dt = \sum_{n=1}^{\infty} \langle f, \psi_n\rangle \frac{z^{\beta-1}}{z^\beta + \lambda_n}. \tag{3.16}$$

Similarly,

$$\int_0^\infty e^{-zt} \sum_{n=1}^{\infty} \langle f, \varphi_n\rangle E_\beta(-\mu_n t^\beta)\, dt = \sum_{n=1}^{\infty} \langle f, \varphi_n\rangle \frac{z^{\beta-1}}{z^\beta + \mu_n}. \tag{3.17}$$

Then, from (3.13), (3.16) and (3.17) we deduce that

$$\sum_{n=1}^{\infty} \frac{\langle f, \psi_n\rangle}{z^\beta + \lambda_n} = \sum_{n=1}^{\infty} \frac{\langle f, \varphi_n\rangle}{z^\beta + \mu_n}, \quad \operatorname{Re} z > 0, \tag{3.18}$$

or equivalently,

$$\sum_{n=1}^{\infty} \frac{\langle f, \psi_n\rangle}{\rho + \lambda_n} = \sum_{n=1}^{\infty} \frac{\langle f, \varphi_n\rangle}{\rho + \mu_n}, \quad \operatorname{Re} \rho > 0. \tag{3.19}$$

Since we can continue analytically both sides of (3.19) in ρ, this equality holds for $\rho \in \mathbb{C} \setminus \big(\{-\lambda_n\}_{n\ge 1} \cup \{-\mu_n\}_{n\ge 1}\big)$. Now we prove that $\lambda_1 = \mu_1$. For this purpose, assume that $\lambda_1 \ne \mu_1$. Without loss of generality, we can suppose $\lambda_1 < \mu_1$. Then we can find a suitable disk that contains $-\lambda_1$ but does not contain $\{-\lambda_n\}_{n\ge 2} \cup \{-\mu_n\}_{n\ge 1}$. If we integrate (3.19) in this disk, the Cauchy's integral formula yields

$$2\pi i \langle f, \psi_1\rangle = 0.$$

By (2.10), this is a contradiction since $\langle f, \psi_1 \rangle$ can not be zero. This means $\lambda_1 = \mu_1$. By repeating the same argument, we obtain $\lambda_2 = \mu_2$. Continuing inductively we finally deduce that

$$\lambda_n = \mu_n, \quad n = 1, 2, 3, \dots . \tag{3.20}$$

This means that the following equality holds for $n = 1, 2, 3, \dots,$

$$\Big(\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}\Big)^{\alpha} + O\big(\frac{1}{n}\big) = \big(\frac{n\pi}{2} - \frac{(2-\eta)\pi}{8}\big)^{\eta} + O\big(\frac{1}{n}\big). \tag{3.21}$$

To conclude $\alpha = \eta$, we prove that the function $H(\alpha) = \big(\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}\big)^{\alpha}$, $n = 1, 2, 3, \dots$ is a monotone increasing function of α. For this purpose, we need to find the derivative of the function $H(\alpha)$. By using the logarithmic differentiation we obtain

$$H'(\alpha) = H(\alpha)\Big\{ \ln\Big(\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}\Big) + \alpha \frac{\pi/8}{\big(\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}\big)} \Big\}. \tag{3.22}$$

Since the function $H(\alpha)$ and the second term in the bracket are positive for $n = 1, 2, 3, \dots$ and $\alpha \in (1/2, 2)$, we estimate the first term in the bracket. We know that the function $\ln(x)$ is positive for $x > 1$. This means we solve the following inequality with respect to n:

$$\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8} \geq 1. \tag{3.23}$$

Solving (3.23) yields

$$n \geq \frac{2}{\pi} + \frac{2-\alpha}{4} > \frac{2}{\pi} + \frac{2-1/2}{4} \approx 1.01.$$

Then we deduce that the function $H(\alpha)$ is monotone increasing function for $n = 2, 3, 4, \dots$. In addition, for $n = 1$ we need to solve the inequality

$$\ln\Big(\frac{\pi}{2} - \frac{(2-\alpha)\pi}{8}\Big) + \alpha \frac{\pi/8}{\big(\frac{\pi}{2} - \frac{(2-\alpha)\pi}{8}\big)} \geq 0. \tag{3.24}$$

Solving (3.24) yields $\alpha \geq 0.27$ which is already guaranteed by the condition $\alpha \in (1/2, 2)$. This completes the proof. $\square$

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