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# POINTWISE ESTIMATES FOR SOLUTIONS TO A SYSTEM OF NONLINEAR DAMPED WAVE EQUATIONS

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Abstract. In this article, we consider the existence of global solutions and pointwise estimates for the Cauchy problem of a nonlinear damped wave equation. We obtain the existence by using the approach introduced by Li and Chen in [7] and some estimates of the solution. The proofs of the estimates are based on a detailed analysis of the Green function of the linear damped wave equations. Also, we show the $L^p$ convergence rate of the solution.

## 1. Introduction

In this paper, we consider the nonlinear damped wave equation

$$\begin{gathered} \partial_t^2 u - \Delta u + \partial_t u = F(u), \quad t > 0, x \in \mathbb{R}^n, \\ u(0,x) = a(x), \quad \partial_t u(0,x) = b(x), \quad x \in \mathbb{R}^n, \end{gathered} \tag{1.1}$$

where $u(t,x) = (u_1(t,x), u_2(t,x), \dots, u_m(t,x)) : (0,T) \times \mathbb{R}^n \to \mathbb{R}^m$ is the unknown vector valued function and $a(x) = (a_1(x), \dots, a_m(x))$ and $b(x) = (b_1(x), \dots, b_m(x))$ are given initial data. The nonlinear smooth vector function $F : \mathbb{R}^m \to \mathbb{R}^m$, $F(u) = (F_1(u), \dots, F_m(u))$ such that

$$F_j(u) = O\Big(\prod_{k=1}^{l} u_k^{p_{j,k}}\Big), \tag{1.2}$$

with $p_{j,k} \ge 1$ or $p_{j,k} = 0$ for $j,k = 1,\dots,m$.

The first aim of this paper is to obtain the existence of classical global solutions to system (1.1). We show the existence directly by using the Banach fixed point theorem with a detailed analysis of the Green function. At the same time, we have the following decay rates of the solutions

$$\|u_j(t)\|_{L^\infty} \le C(1+t)^{-n/2}, \quad \|u_j(t)\|_{L^2} \le C(1+t)^{-n/4}, \quad j = 1,\dots,m. \tag{1.3}$$

The second aim is to get the pointwise estimate of the solutions to system (1.1). With the help of the pointwise estimates of the Green function and using the method of the Green function, we show the pointwise estimates of the solutions to system (1.1). This estimates represent a clear decaying structure of the solutions. Furthermore, we get the optimal $L^p$ decay estimates of the solutions.

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There are many authors working in this field. For the single nonlinear damped wave equation

$$
\begin{gathered}
\partial_t^2 u - \Delta u + \partial_t u = f(u), \quad t > 0, x \in \mathbb{R}^n, \\
u(0,x) = a_1(x), \quad \partial_t u(0,x) = b_1(x), \quad x \in \mathbb{R}^n,
\end{gathered} \tag{1.4}
$$

many results have been published. For the case $f(u) = -|u|^\theta u$, Kawashima, Nakao and Ono [6] studied the decay properties of solutions to (1.4) by using the energy method combined with $L^p$-$L^q$ estimates. Ono [19] derived sharp decay rates in the subcritical case of solutions in unbounded domains in $\mathbb{R}^n$. Nakao and Ono in [12] proved the existence and decay of global solutions weak solutions for (1.4) by using the potential well method. By employing the weighted $L^2$ energy method, Nishihara and Zhao [16] obtained that the behavior of solutions to (1.4) as $t \to \infty$ is expected to be same as that for the corresponding heat equation. The global asymptotic behaviors were studied by Nishihara [14, 15] for $n = 3, 4$ and Ikehata, Nishihara and Zhao [4] for $n \geq 1$. In [9, 11], the pointwise estimates of classical solutions to (1.4) were obtained.

For the case of $f(u) = |u|^\theta u$, Ikehata, Miyaoka and Nakatake [3] obtained the global existence of weak solutions to (1.4). Furthermore, Hosono and Ogawa [1] obtained the $L^p$-$L^q$ type estimate of the difference between the solution to (1.4) and the solutions of corresponding heat and wave equations in the two-dimensional space. Meanwhile, when $2 \leq n \leq 5$, the same type estimate was studied by Narazaki in [13].

For the general case $f(u) = O(u^{\theta+1})$, Wang and Wang [25] proved the pointwise estimates of classical solutions to (2.1). There also have been a lot of investigations for those cases. For detail results, please refer to [2, 5, 8, 20, 21, 24, 28].

For the system of the nonlinear damped wave equations

$$
\begin{gathered}
\partial_t^2 u_1 - \Delta u_1 + \partial_t u_1 = |u_m|^{p_1}, \quad t > 0,\ x \in \mathbb{R}^n, \\
\partial_t^2 u_2 - \Delta u_2 + \partial_t u_2 = |u_1|^{p_2}, \quad t > 0,\ x \in \mathbb{R}^n, \\
\dots \\
\partial_t^2 u_m - \Delta u_m + \partial_t u_m = |u_{m-1}|^{p_m}, \quad t > 0, x \in \mathbb{R}^n, \\
u_j(0,x) = a_j(x), \quad \partial_t u_j(0,x) = b_j(x), \quad x \in \mathbb{R}^n,\ (1 \leq j \leq m),
\end{gathered} \tag{1.5}
$$

Sun and Wang [22] for $m = 2$ and Takeda [23] for $m \geq 2$ obtained global weak solutions to system (1.5).

For the general case (1.1), Ogawa and Takeda in [17] obtained the existence of the global solution under some conditions, which include the results of [22] and [23]. Recently, Ogawa and Takeda in [18] proved the asymptotic behavior of solutions to the problem (1.1) by using the $L^p$-$L^q$ type decomposition of the fundamental solution of the linear damped wave equations into the dissipative part and hyperbolic part.

However, there are few studies concerning the global existence and decay property of classical solutions to the Cauchy problem of the nonlinear damped wave system. In this paper, we investigate the global existence and pointwise estimates of classical solution to system (1.1). First of all, we employ the Green function of the linear damped wave equation to express the solution of system (1.1). Then, we obtain the global solution directly by using the method introduced by Li and Chen in [7]. Unlike the usual energy method, this method needn't to prove the local existence and extend the local solution to the global one in time. In this process, the decay properties of the Green function play an important role. We employ $G_1$ and $G_2$ to define the Green function of linear equation. By a detailed analysis of the Green function, we obtain the pointwise estimates of the Green function. Compared with the methods in [11, 13, 14], the method of dealing with the existence theory in this paper is more useful to show a clear decaying structure of the solution. Secondly, with the obtained pointwise estimates of the Green function, we give the pointwise estimates of the solution to (1.4) by the method of the Green function. Finally, as a corollary of the pointwise estimates, the optimal $L^p$ $(1 \le p \le \infty)$ convergence rate can be obtained easily.

Throughout this paper, we assume that the nonlinear term $\{F_j(u)\}_{j=1}^m$ satisfies the following conditions, for $p_{j,k} \in [0, +\infty) \cup \{0\}$, $(j = 1, \dots, m;\ k = 1, \dots, m)$,

$$|\partial^{\tilde{\alpha}_j} F_j(u)| \le C_{\tilde{\alpha}_j,\delta} \sum_{\alpha_{j,1}+\dots+\alpha_{j,m}=\tilde{\alpha}_j} \prod_{k=1}^m |u_k|^{(p_{j,k}-\alpha_{j,k})_+}, \quad |u_j| \le \delta,\ 0 \le \tilde{\alpha}_j \le \tilde{p}_j, \tag{1.6}$$

$$|\partial^{\tilde{\alpha}_j} F_j(u)| \le C_{\tilde{\alpha}_j,\delta}, \quad |u_j| \le \delta,\ \tilde{p}_j \le \tilde{\alpha}_j \le l, \tag{1.7}$$

and for $|u_j| \le \delta,\ |v_j| \le \delta,\ \tilde{\alpha}_j \le l$,

$$\begin{aligned} &|\partial^{\tilde{\alpha}_j} F_j(u) - \partial^{\tilde{\alpha}_j} F_j(v)| \\ &\le C_{\tilde{\alpha}_j,\delta} \sum_{\alpha_{j,1}+\dots+\alpha_{j,m}=\tilde{\alpha}_j} \sum_{l=1}^m \Big\{ \prod_{k=1}^{s-1} |u_k|^{(p_{j,k}-\alpha_{j,k})_+} \prod_{k=s+1}^m |v_k|^{(p_{j,k}-\alpha_{j,k})_+} \\ &\quad \times \Big( |u_l|^{(p_{j,s}-\alpha_{j,s}-1)_+} + |v_l|^{(p_{j,s}-\alpha_{j,s}-1)_+} \Big) |u_s - v_s| \Big\}, \end{aligned} \tag{1.8}$$

where

$$\tilde{p}_j = \sum_{k=1}^m p_{j,k}, \quad \tilde{\alpha}_j = \sum_{k=1}^m \alpha_{j,k}$$

with $\alpha_{j,k} \ge 0$ and $(a)_+ = \max\{a, 0\}$.

Our main results are the following two theorems.

**Theorem 1.1.** *Assume that $\tilde{p}_j \ge 2$, $\tilde{p}_j > 1 + \frac{2}{n}$, $l \ge n + 1$ and the initial data $\{(a_j, b_j)|_{j=1}^m \subset (H^{l+1}(\mathbb{R}^n) \cap W^{l,1}(\mathbb{R}^n)) \times (H^l(\mathbb{R}^n) \cap W^{l,1}(\mathbb{R}^n))$ and*

$$N_0 := \sum_{j=1}^m \big( \|a_j\|_{H^{l+1}(\mathbb{R}^n) \cap W^{l,1}(\mathbb{R}^n)} + \|b_j\|_{H^l(\mathbb{R}^n) \cap W^{l,1}(\mathbb{R}^n)} \big), \tag{1.9}$$

*is sufficiently small and the nonlinear coupling $F(u)$ satisfies the assumptions* (1.6), (1.7) *and* (1.8). *Then there exists a unique global classical solution $\{u_j(t)\}_{j=1}^m$ of system* (1.1).

*Moreover, for $j = 1, 2, \dots, m$, we have the decay estimates*

$$\|u_j\|_{W^{l-n-1,\infty}(\mathbb{R}^n)} \le C(1+t)^{-n/2}, \quad \text{and} \quad \|u_j\|_{H^l} \le C(1+t)^{-n/4}. \tag{1.10}$$

For the solution in the above theorem, we have the following pointwise estimates.

**Theorem 1.2.** *Under the assumptions of Theorem 1.1, if for any multi-index $\alpha$, $|\alpha| < l$, there exist some constant $r > n$ and a small positive constant $\varepsilon_0$, such that*

$$|\partial_x^\alpha a_j| + |\partial_x^\alpha b_j| \le \varepsilon_0 (1 + |x|^2)^{-r}, \tag{1.11}$$

*then for $|\alpha| < l - n$ the solution to (1.1) satisfies:*

$$|D_x^\alpha u_j(t)| \le C(1+t)^{-\frac{n+|\alpha|}{2}} B_{\frac{n}{2}}(|x|,t), \tag{1.12}$$

*where $B_N(|x|,t) = \big(1 + \frac{|x|^2}{1+t}\big)^{-N}$. And for $p \in [1,\infty]$, we have*

$$\|D_x^\alpha u_j(t)\|_{L^p(\mathbb{R}^n)} \le C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}. \tag{1.13}$$

**Notation.** Various positive constants will be denoted by $C$. $W^{m,p}(\mathbb{R}^n)$, with $m \in \mathbb{Z}_+$ and $p \in [1,\infty]$, denotes the usual Sobolev space with the norm

$$\|f\|_{W^{m,p}(\mathbb{R}^n)} := \sum_{k=0}^{m} \|\partial_x^k f\|_{L^p(\mathbb{R}^n)}.$$

In particular, we use $W^{m,2}(\mathbb{R}^n) = H^m(\mathbb{R}^n)$, $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^n)}$, $\|\cdot\|_{m,p} = \|\cdot\|_{W^{m,p}(\mathbb{R}^n)}$ and $\|\cdot\|_m = \|\cdot\|_{H^m(\mathbb{R}^n)}$.

The rest of this article is organized as follows. In the next section, we show the pointwise estimates of the Green function. Then the existence of global solutions is proved in Section 3. Furthermore, the pointwise estimates of solutions for nonlinear equations are obtained in Section 4.

## 2. Green Function

To obtain the global existence and pointwise estimates of the solutions, we should first derive representation formulas of the solutions through the Green function. The single linearized equation of (1.1) is

$$\begin{gathered} \partial_t^2 u_j - \Delta u_j + \partial_t u_j = 0, \\ u_j|_{t=0} = a_j, \quad u_{jt}|_{t=0} = b_j. \end{gathered} \tag{2.1}$$

Then, the Green function of (2.1) can be defined as follows:

$$\begin{gathered} \partial_t^2 G_1 - \Delta G_1 + \partial_t G_1 = 0, \\ G_1|_{t=0} = \delta(x), \quad G_{1t}|_{t=0} = 0, \end{gathered}$$

and

$$\begin{gathered} \partial_t^2 G_2 - \Delta G_2 + \partial_t G_2 = 0, \\ G_2|_{t=0} = 0, \quad G_{2t}|_{t=0} = \delta(x). \end{gathered}$$

Now, we show the formulas of the solutions in the following theorem. The proof of the theorem is similar to that of [27, Theorem 2.5]. We show the proof here for the convenience of the readers.

**Theorem 2.1.** *The solution of (2.1) is*

$$u_j(x,t) = G_1 * a_j + G_2 * b_j + \int_0^t G_2(t-s) * F_j(u)(s)\mathrm{d}s. \tag{2.2}$$

*Proof.* It is obvious that

$$u_j(x,0) = G_1(x,0) * a_j + G_2(x,0) * b_j = \delta(x) * a_j = a_j. \tag{2.3}$$

$$\begin{aligned} u_{jt}(x,t) = {} & G_{1t}(x,0) * a_j + G_{2t}(x,0) * b_j + G_2(t-t) * F_j(u)(t) \\ & + \int_0^t G_{2t}(t-s) * F_j(u)(s)\mathrm{d}s. \end{aligned} \tag{2.4}$$

Then, we have

$$u_{jt}(x,0) = \delta(x) * b_j = b_j, \tag{2.5}$$

$$\Delta u_j = \Delta G_1 * a_j + \Delta G_2 * b_j + \int_0^t \Delta G_2(t-s) * F_j(u)(s)\mathrm{d}s, \tag{2.6}$$

$$\partial_t u_j = \partial_t G_1 * a_j + \partial_t G_2 * b_j + G_2(t-t) * F_j(u)(t) + \int_0^t \partial_t G_2(t-s) * F_j(u)(s)\mathrm{d}s, \tag{2.7}$$

$$\begin{aligned} \partial_t^2 u_j &= \partial_t^2 G_1 * a_j + \partial_t^2 G_2 * b_j + G_{2t}(t-t) * F_j(u)(t) \\ &\quad + \int_0^t \partial_t^2 G_2(t-s) * F_j(u)(s)\mathrm{d}s \\ &= \partial_t^2 G_1 * a_j + \partial_t^2 G_2 * b_j + F_j(u)(t) + \int_0^t \partial_t^2 G_2(t-s) * F_j(u)(s)\mathrm{d}s. \end{aligned} \tag{2.8}$$

Then, by the definition of the Green function, we obtain

$$\begin{aligned} &\partial_t^2 u_j - \Delta u_j + \partial_t u_j \\ &= (\partial_t^2 G_1 - \Delta G_1 + \partial_t G_1) * a_j + (\partial_t^2 G_2 - \Delta G_2 + \partial_t G_2) * b_j \\ &\quad + F_j(u)(t) + \int_0^t (\partial_t^2 G_2 - \Delta G_2 + \partial_t G_2)(t-s) * F_j(u)(s)\mathrm{d}s \\ &= F_j(u)(t). \end{aligned} \tag{2.9}$$

The proof is complete. □

By the Fourier transform and a direct calculation, we have

$$\hat{G}_1(\xi,t) = -\frac{\tau_-}{\tau_+ - \tau_-}\mathrm{e}^{\tau_+ t} + \frac{\tau_+}{\tau_+ - \tau_-}\mathrm{e}^{\tau_- t}, \tag{2.10}$$

$$\hat{G}_2(\xi,t) = \frac{1}{\tau_+ - \tau_-}\mathrm{e}^{\tau_+ t} - \frac{1}{\tau_+ - \tau_-}\mathrm{e}^{\tau_- t}, \tag{2.11}$$

where $\hat{G}_i(\xi,t)$ $(i=1,2)$ are the Fourier transform corresponding to $G_i(x,t)$ and

$$\tau_\pm = \frac{-1 \pm \sqrt{1-4|\xi|^2}}{2}. \tag{2.12}$$

In what follows, we show the asymptotic behavior of $u$ by using the pointwise estimates of $G_1$, $G_2$. First, we divide $|\xi|$ into three cases: $|\xi|$ is small, bounded and big enough. Here, we set

$$\chi_1(\xi) = \begin{cases} 1, & \text{if } \xi < \epsilon, \\ 0, & \text{if } \xi > 2\epsilon, \end{cases} \quad \text{and} \quad \chi_3(\xi) = \begin{cases} 1, & \text{if } \xi > R, \\ 0, & \text{if } \xi < R-1, \end{cases} \tag{2.13}$$

where $\epsilon$ is small enough, $R$ is large enough, and $\chi_1, \chi_3$ are smooth functions. We denote $\chi_2(\xi) = 1 - \chi_1(\xi) - \chi_3(\xi)$. Then, we set $\hat{G}_{i,j}(\xi,t) = \chi_j(\xi)G_i(\xi,t)$ where $i = 1,2; j = 1,2,3$.

To deal with the coupling of Green function, we need the following two lemmas. The following lemma corresponds to [26, Lemma 3.2]. We omit its proof here.

**Lemma 2.2.** *Assume that* $\operatorname{supp}\hat{f} \subset \{\xi : |\xi| > R\}$ *with* $R$ *large enough,* $|\partial_\xi^\beta \hat{f}| \le C|\xi|^{-|\beta|-1}$*, then there exist distributions* $f_1(x), f_2(x)$*, such that* $f(x) = f_1(x) +$

*$f_2(x)$. And $|\partial_x^\alpha f_1(x)| \le C(1+|x|^2)^{-N}$ for positive number $2N > n+|\alpha|$, $\|f_2\|_{L^1} \le C$, $\operatorname{supp} f_2(x) \subset \{x, |x| < 2\varepsilon\}$ with $\varepsilon$ being sufficiently small.*

The proof of the following lemma can be founded in [10]. We omit it here.

**Lemma 2.3.** *If the functions $H(x,t)$ and $S(x,t)$ satisfy*

$$
\begin{aligned}
|\partial_x^\alpha H(x,t)| &\le C(1+t)^{-(n+|\alpha|)/2} B_N(|x|,t),\\
|\partial_x^\alpha S(x,t)| &\le C(1+t)^{-(2n+|\alpha|)/2} B_n(|x|,t),
\end{aligned}
$$

*then*

$$
\Big|\partial_x^\alpha \int_0^t (H(t-\tau) * S(\tau))\mathrm{d}\tau\Big| \le C(1+t)^{-(n+|\alpha|)/2} B_{\frac{n}{2}}(|x|,t).
$$

Next we estimate $G_{i,j}(x,t)$, $(i=1,2; j=1,2,3)$ which are the inverse Fourier transform corresponding to $\hat{G}_{i,j}(\xi,t)$. First of all, for $G_{i,j}$, $(i=1,2; j=1,2)$, we can use the following results from [11].

**Proposition 2.4.** *For any positive number $N$, we have*

$$
|\partial_x^\alpha G_{i,1}| \le C(1+t)^{-\frac{n+|\alpha|}{2}} B_N(|x|,t), \quad i=1,2.
$$

**Proposition 2.5.** *There exists a positive number $c_0$, such that*

$$
|\partial_x^\alpha G_{i,2}| \le C\mathrm{e}^{-c_0 t} B_N(|x|,t), \quad i=1,2.
$$

For $G_{i,3}$, $(i=1,2)$, we show a subtle analysis as follows. When $|\xi|$ is large enough, using the Taylor expansion, we have

$$
\sqrt{1-4|\xi|^2} = |\xi|\sqrt{|\xi|^{-2}-4} = 2\sqrt{-1}|\xi| - \frac{\sqrt{-1}}{4}|\xi|^{-1} + O(|\xi|^{-3}), \tag{2.14}
$$

$$
\frac{1}{\sqrt{1-4|\xi|^2}} = |\xi|^{-1}\frac{1}{\sqrt{|\xi|^{-2}-4}} = -\frac{\sqrt{-1}}{2}|\xi|^{-1} + \frac{\sqrt{-1}}{16}|\xi|^{-3} + O(|\xi|^{-5}), \tag{2.15}
$$

By using the Taylor expansion, we have

$$
\begin{aligned}
\mathrm{e}^{\tau_\pm t} = \mathrm{e}^{(-\frac{1}{2}\pm\sqrt{-1}|\xi|)t}\Big(1 &+ \big(\sum_{j=1}^{k-1}(\pm a_j)|\xi|^{1-2j}\big)t + \dots\\
&+ \frac{1}{k!}\big(\sum_{j=1}^{k-1}(\pm a_j)|\xi|^{1-2j}\big)^k t^k + R^\pm(\xi,t)\Big),
\end{aligned} \tag{2.16}
$$

where $R^\pm(\xi,t) \le (1+t)^{k+1}(1+|\xi|)^{1-2k}$.

Then, by using (2.15) and (2.16), we have

$$
\frac{1}{\tau_+-\tau_-}\mathrm{e}^{\tau_+ t} = \mathrm{e}^{(-\frac{1}{2}+\sqrt{-1}|\xi|)t}\Big(\sum_{j=1}^{2k-2} p_j^+(t)|\xi|^{-j} + p_{2k-1}^+(t)O(|\xi|^{1-2k})\Big), \tag{2.17}
$$

$$
\frac{1}{\tau_+-\tau_-}\mathrm{e}^{\tau_- t} = \mathrm{e}^{(-\frac{1}{2}-\sqrt{-1}|\xi|)t}\Big(\sum_{j=1}^{2k-2} p_j^-(t)|\xi|^{-j} + p_{2k-1}^-(t)O(|\xi|^{1-2k})\Big), \tag{2.18}
$$

$$
\frac{\tau_+}{\tau_+-\tau_-}\mathrm{e}^{\tau_- t} = \mathrm{e}^{(-\frac{1}{2}-\sqrt{-1}|\xi|)t}\Big(\sum_{j=0}^{2k-2} q_j^-(t)|\xi|^{-j} + q_{2k-1}^-(t)O(|\xi|^{1-2k})\Big), \tag{2.19}
$$

$$\frac{\tau_-}{\tau_+-\tau_-}\mathrm{e}^{\tau_+ t}=\mathrm{e}^{(-\frac12+\sqrt{-1}|\xi|)t}\Big(\sum_{j=0}^{2k-2}q_j^+(t)|\xi|^{-j}+q_{2k-1}^+(t)O(|\xi|^{1-2k})\Big). \tag{2.20}$$

Here, $p_j^\pm$, $q_j^\pm$ are polynomials in $t$ with degree no more then $j$.

Since $|\xi|>R$, it is observed that

$$|\hat G_{1,3}(\xi,t)|+|\hat G_{2,3}(\xi,t)|\le C\mathrm{e}^{-t/4}. \tag{2.21}$$

Now, we take

$$\begin{aligned}\hat F_{1\alpha}&=-\chi_3(\xi)\mathrm{e}^{(-\frac12+\sqrt{-1}|\xi|)t}\sum_{j=0}^{|\alpha|+n+1}q_j^+(t)|\xi|^{-j}\\&\quad+\chi_3(\xi)\mathrm{e}^{(-\frac12-\sqrt{-1}|\xi|)t}\sum_{j=0}^{|\alpha|+n+1}q_j^-(t)|\xi|^{-j},\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}\hat F_{2\alpha}&=\chi_3(\xi)\mathrm{e}^{(-\frac12+\sqrt{-1}|\xi|)t}\sum_{j=1}^{|\alpha|+n+1}p_j^+(t)|\xi|^{-j}\\&\quad-\chi_3(\xi)\mathrm{e}^{(-\frac12-\sqrt{-1}|\xi|)t}\sum_{j=1}^{|\alpha|+n+1}p_j^-(t)|\xi|^{-j}.\end{aligned} \tag{2.23}$$

Then, for the high frequency part, we have the following result.

**Proposition 2.6.** *There exists a positive number $c_1$, such that*

$$|\partial_x^\alpha(G_{i,3}-F_{i\alpha})|\le C\mathrm{e}^{-c_1t}B_N(|x|,t),\quad i=1,2. \tag{2.24}$$

*Proof.* It is obvious that

$$\begin{aligned}|x^\beta(\partial_x^\alpha(G_{i,3}-F_{i\alpha}))|&\le\int|\partial_\xi^\beta(\xi^\alpha(\hat G_{i,3}-\hat F_{i\alpha}))|\mathrm{d}\xi\\&\le C\mathrm{e}^{-c_1t}\int|\xi|^{|\alpha|-|\beta|-(|\alpha|+n+1)-1}\mathrm{d}\xi\\&\le C\mathrm{e}^{-c_1t}.\end{aligned} \tag{2.25}$$

Take $|\beta|=0$ or $|\beta|=2N$, we obtain the the statement of Proposition 2.6. □

## 3. Global classical solutions

The solution can be constructed in the complete metric space

$$X=\{u(t)=(u_1(t),u_2(t),\dots,u_m(t))|\|u\|_X\le E\}, \tag{3.1}$$

where $E$ is a positive constant and

$$\|u\|_X=\sup_{t\ge0}\sum_{j=1}^m(1+t)^{\frac n2}\|u_j(\cdot,t)\|_{W^{l-n-1,\infty}(\mathbb{R}^n)}+\sup_{t\ge0}\sum_{j=1}^m(1+t)^{\frac n4}\|u_j(\cdot,t)\|_{H^l(\mathbb{R}^n)}.$$

Then $(X,\|\cdot\|_X)$ is a Banach space. Let

$$T[u](t):=(T_1[u](t),T_2[u](t),\dots,T_m[u](t)), \tag{3.2}$$

where

$$T_j[u](t)=G_1*a_j+G_2*b_j+\int_0^tG_2(t-s)*F_j(u(s))(x)\mathrm{d}s,\quad(1\le j\le m). \tag{3.3}$$

In the following lemma, we show that $T$ is a map from $X$ to itself.

**Lemma 3.1.** *If $E$ and $N_0$ are sufficiently small with $N_0 \ll E$, then $T$ is a map from $X$ to $X$.*

*Proof.* Firstly, we note that

$$\begin{aligned}
\|T_j[u](t)\|_{l-n-1,\infty} &\le \|(G_1 - F_{1\alpha})(t) * a_j\|_{l-n-1,\infty} + \|(G_2 - F_{2\alpha})(t) * b_j\|_{l-n-1,\infty} \\
&\quad + \|F_{1\alpha}(t) * a_j\|_{l-n-1,\infty} + \|F_{2\alpha}(t) * b_j\|_{l-n-1,\infty} \\
&\quad + \int_0^t \|(G_2 - F_{2\alpha})(t-\tau) * F_j(u)(\tau)\|_{l-n-1,\infty} \mathrm{d}\tau \\
&\quad + \int_0^t \|F_{2\alpha}(t-\tau) * F_j(u)(\tau)\|_{l-n-1,\infty} \mathrm{d}\tau \\
&:= \sum_{i=1}^6 I_i.
\end{aligned}$$

For $I_1$, it follows from the Young inequality and Propositions 2.4–2.6 that

$$I_1 \le \|(G_1 - F_{1\alpha})(t)\|_{L^\infty} \|a_j\|_{l-n-1,1} \le C(1+t)^{-n/2} \|a_j\|_{l-n-1,1}. \tag{3.4}$$

Similarly to the estimates of $I_1$, we obtain

$$I_2 \le C(1+t)^{-n/2} \|b_j\|_{l-n-1,1}. \tag{3.5}$$

By noticing $|\alpha| \le l-n-1$ and the definition of $F_{1\alpha}$, for some positive number $c_2$, we have

$$\begin{aligned}
|\partial_x^\alpha F_{1\alpha} * a_j| &\le \int |\xi^\alpha \hat{F}_{1\alpha} \hat{a}_j| \mathrm{d}\xi \\
&\le C\|a_j\|_{l,1} \mathrm{e}^{-c_2 t} \int \chi_3(\xi) |\xi|^{|\alpha|-l} \mathrm{d}\xi \\
&\le C\|a_j\|_{l,1} \mathrm{e}^{-c_2 t}.
\end{aligned} \tag{3.6}$$

Similarly, we have

$$|\partial_x^\alpha F_{2\alpha} * b_j| \le C\|b_j\|_{l-1,1} \mathrm{e}^{-c_2 t}. \tag{3.7}$$

Then, we obtain

$$I_3 \le C(1+t)^{-n/2} \|a_j\|_{l,1} \quad \text{and} \quad I_4 \le C(1+t)^{-n/2} \|b_j\|_{l-1,1}. \tag{3.8}$$

To estimate $I_5$ and $I_6$, we give the estimates of $F_j(u)(t)$ as follows:

$$\begin{aligned}
|\partial_x^\alpha F_j(u)(t)| &\le |\partial_u^1 F_j(u)(t)| \sum_{i=1}^m |\partial_x^\alpha u_i| \\
&\quad + |\partial_u^2 F_j(u)(t)| \sum_{1 \le k_1, k_2 \le m; \alpha_1 + \alpha_2 = \alpha} |\partial_x^{\alpha_1} u_{k_1} \partial_x^{\alpha_2} u_{k_2}| + \dots \\
&\quad + |\partial_u^\alpha F_j(u)(t)| \sum_{1 \le k_1, \dots, k_m \le m,\ \alpha_1 + \dots + \alpha_m = \alpha} |\partial_x^{\alpha_1} u_{k_1} \dots \partial_x^{\alpha_m} u_{k_m}|,
\end{aligned} \tag{3.9}$$

where $0 \le \alpha_k \le \alpha$, $(k = 1, \dots, m)$.

Then, by using (1.6), (1.7), (3.9), the Hölder inequality ($\|fg\|_{L^1} \le \|f\|_{L^2} \|g\|_{L^2}$) and the assumption $\|u\|_X \le E$, we have

$$\|F_j(u)(t)\|_{l-n-1,1} \le C(1+t)^{-\frac{n}{2}(\tilde{p}_j - 1)} E^{\tilde{p}_j}, \tag{3.10}$$

$$\|F_j(u)(t)\|_l \le C(1+t)^{-\frac{n}{2}(\tilde p_j-1)-\frac{n}{4}} E^{\tilde p_j}. \tag{3.11}$$

Using the Young inequality, (3.10) and Proposition 2.4 and noticing $\tilde p_j > 1+\frac{2}{n}$, for $I_5$, we have

$$\begin{aligned} I_5 &\le \int_0^t \|(G_1 - F_{1\alpha})(t-\tau)\|_{L^\infty} \|F_j(u)(\tau)\|_{l-n-1,1} \mathrm{d}\tau \\ &\le C\int_0^t (1+t-\tau)^{-n/2} \|F_j(u)(\tau)\|_{l-n-1,1} \mathrm{d}\tau \\ &\le CE^{\tilde p_j} \int_0^t (1+t-\tau)^{-n/2}(1+\tau)^{-\frac{n}{2}(\tilde p_j-1)} \mathrm{d}\tau \\ &\le C(1+t)^{-n/2}E^{\tilde p_j}. \end{aligned}$$

For $I_6$, it follows from Lemma 2.2, (3.11) and the Sobolev inequality that

$$\begin{aligned} I_6 &\le C\int_0^t \mathrm{e}^{-(t-\tau)/4} \|(f_1+f_2) * F_j(u)(\tau)\|_{l-n-1,1} \mathrm{d}\tau \\ &\le C\int_0^t \mathrm{e}^{-(t-\tau)/4} (\|f_1\|_{L^1} + \|f_2\|_{L^1}) \|F_j(u)(\tau)\|_{l-n-1,\infty} \mathrm{d}\tau \\ &\le C\int_0^t \mathrm{e}^{-(t-\tau)/4} \|F_j(u)(\tau)\|_{l-n-1,\infty} \mathrm{d}\tau \\ &\le C\int_0^t \mathrm{e}^{-(t-\tau)/4} \|F_j(u)(\tau)\|_{l-[\frac{n}{2}]} \mathrm{d}\tau \\ &\le CE^{\tilde p_j} \int_0^t \mathrm{e}^{-(t-\tau)/4} (1+\tau)^{-\frac{n}{2}(\tilde p_j-1)-\frac{n}{4}} \mathrm{d}\tau \\ &\le C(1+t)^{-n/2}E^{\tilde p_j}. \end{aligned} \tag{3.12}$$

Thus, the combination of (3.4)-(3.12) gives

$$\begin{aligned} &\|T_j[u](t)\|_{l-n-1,\infty} \\ &\le C(1+t)^{-n/2}\big(\|a_j\|_{l,1} + \|b_j\|_{l-1,1} + \|a_j\|_{l-n-1,1} + \|b_j\|_{l-n-1,1} + E^{\tilde p_j}\big). \end{aligned} \tag{3.13}$$

To estimate $H^l$ norm of $T_j[u](t)$, we consider

$$\begin{aligned} \|T_j[u](t)\|_l &\le \|(G_1 - G_{1,3})(t) * a_j\|_l + \|\partial_t(G_2 - G_{2,3})(t) * b_j\|_l \\ &\quad + \|G_{1,3}(t) * a_j\|_l + \|G_{2,3}(t) * b_j\|_l \\ &\quad + \int_0^t \|(G_2 - G_{2,3})(t-\tau) * F_j(u)(\tau)\|_l \mathrm{d}\tau \\ &\quad + \int_0^t \|G_{2,3}(t-\tau) * F_j(u)(\tau)\|_l \mathrm{d}\tau \\ &:= \sum_{i=1}^6 J_i. \end{aligned}$$

By using Propositions 2.4-2.5 and the Young inequality, for $J_1$, we obtain

$$J_1 \le \|(G_1 - G_{1,3})(t)\|\|a_j\|_{l,1} \le C(1+t)^{-n/4}\|a_j\|_{l,1}. \tag{3.14}$$

For $J_3$, it follows from the Plancherel theorem and (2.21) that

$$J_3 \le \sum_{|\alpha|=0}^{l} \|G_{1,3}(t) * \partial_x^\alpha a_j\| = \sum_{|\alpha|=0}^{l} \|\hat{G}_{1,3}(t)\widehat{\partial_x^\alpha a_j}\| \le C\mathrm{e}^{-t/4}\|a_j\|_l. \tag{3.15}$$

Similar to the estimates of $J_1$ and $J_3$, we obtain

$$J_2 \le C(1+t)^{-n/4}\|b_j\|_{l,1}, \quad \text{and} \quad J_4 \le C\mathrm{e}^{-t/4}\|b_j\|_l. \tag{3.16}$$

Using Propositions 2.4-2.5, the Young inequality and noticing $\tilde{p}_j > 1 + \frac{2}{n}$, we have

$$\begin{aligned} J_5 &\le C\int_0^t \|(G_2 - G_{2,3})(t-\tau)\|\|F_j(u)(\tau)\|_l \mathrm{d}\tau \\ &\le CE^{\tilde{p}_j}\int_0^t (1+t-\tau)^{-n/4}(1+t)^{-\frac{n}{2}(\tilde{p}_j-1)-\frac{n}{4}}\mathrm{d}\tau \\ &\le C(1+t)^{-n/4}E^{\tilde{p}_j}. \end{aligned} \tag{3.17}$$

For $J_6$, by using the Plancherel theorem and (2.21), we get

$$J_6 \le C\int_0^t \mathrm{e}^{-(t-\tau)/4}\|F_j(u)(\tau)\|_l \mathrm{d}\tau \le C(1+t)^{-n/4}E^{\tilde{p}_j}. \tag{3.18}$$

Thus, we obtain

$$\|T_j[u](t)\|_l \le C(1+t)^{-n/4}\left(\|a_j\|_l + \|b_j\|_l + \|a_j\|_{l,1} + \|b_j\|_{l,1} + E^{\tilde{p}_j}\right). \tag{3.19}$$

By using (3.13), (3.19), $\tilde{p}_j > 1 + \frac{2}{n}$ and the small assumptions of $E$ and $N_0$ with $N_0 \ll E$, we get $\|T[u](t)\|_X \le E$. Thus, the proof of Lemma 3.1 is complete. □

Next, we proof that this map $T$ is a contraction mapping.

**Lemma 3.2.** *Assume $u, v \in X$ and $E > 0$ is sufficiently small, then there exists a constant $\gamma$ with $0 < \gamma < 1$, such that*

$$\|T[u] - T[v]\|_X \le \gamma\|u - v\|_X.$$

*Proof.* By the Duhamel principle and the triangle inequality, we have

$$\begin{aligned} &\|T[u] - T[v]\|_{l-n-1,\infty} \\ &\le \int_0^t \|(G_2 - F_{2\alpha})(t-\tau) * (F_j(u) - F_j(v))(\tau)\|_{l-n-1,\infty}\,\mathrm{d}\tau \\ &\quad + \int_0^t \|F_{2\alpha}(t-\tau) * (F_j(u) - F_j(v))(\tau)\|_{l-n-1,\infty}\,\mathrm{d}\tau \\ &:= H_1 + H_2. \end{aligned} \tag{3.20}$$

By a directly calculation, we have

$$
\begin{aligned}
&|\partial_x^\alpha F(u)-\partial_x^\alpha F(v)|\\
&\le \Big|\sum_{k=1}^m \partial_{u_k}^1 F(u)\partial_x^\alpha u_k-\sum_{s=1}^m \partial_{v_s}^1 F(v)\partial_x^\alpha v_s\Big|\\
&\quad +\Big|\sum_{1\le k_1,k_2\le m;\alpha_{k_1}+\alpha_{k_2}=\alpha}\partial_{u_{k_1}u_{k_2}}^2 F(u)\partial_x^{\alpha_{k_1}}u_{k_1}\partial_x^{\alpha_{k_2}}u_{k_2}\\
&\quad -\sum_{1\le s_1,s_2\le m;\alpha_{s_1}+\alpha_{s_2}=\alpha}\partial_{v_{s_1}v_{s_2}}^2 F(v)\partial_x^{\alpha_{s_1}}v_{s_1}\partial_x^{\alpha_{s_2}}v_{s_2}\Big|+\dots\\
&\quad +\Big|\sum_{1\le k_i\le m,i=1,\dots,\alpha;\alpha_{k_1}+\dots+\alpha_{k_\alpha}=\alpha}\partial_{u_{k_1}u_{k_2}\dots u_{k_\alpha}}^\alpha F(u)\partial_x^{\alpha_{k_1}}u_{k_1}\dots\partial_x^{\alpha_{k_\alpha}}u_{k_\alpha}\\
&\quad -\sum_{1\le s_i\le m,i=1,\dots,\alpha;\alpha_{s_1}+\dots+\alpha_{s_\alpha}=\alpha}\partial_{v_{s_1}v_{s_2}\dots v_{s_\alpha}}^\alpha F(u)\partial_x^{\alpha_{s_1}}v_{s_1}\dots\partial_x^{\alpha_{s_\alpha}}v_{s_\alpha}\Big|\\
&\le \sum_{i=1}^m |\partial_u^1 F(u)\partial_x^\alpha u_i-\partial_v^1 F(v)\partial_x^\alpha v_i|\\
&\quad +\sum_{1\le k,s\le m;\alpha_k+\alpha_s=\alpha}|\partial_u^2 F(u)\partial_x^{\alpha_k}u_k\partial_x^{\alpha_s}u_s-\partial_v^2 F(v)\partial_x^{\alpha_k}v_k\partial_x^{\alpha_s}v_s|+\dots\\
&\quad +\sum_{1\le k_i\le m,i=1,\dots,\alpha;\alpha_{k_1}+\dots+\alpha_{k_\alpha}=\alpha}|\partial_u^\alpha F(u)\partial_x^{\alpha_{k_1}}u_{k_1}\dots\partial_x^{\alpha_{k_\alpha}}u_{k_\alpha}\\
&\quad -\partial_v^\alpha F(v)\partial_x^{\alpha_{k_1}}v_{k_1}\dots\partial_x^{\alpha_{k_\alpha}}v_{k_\alpha}|.
\end{aligned}
\tag{3.21}
$$

Using (3.21) and the assumption (1.8), we have

$$
\begin{aligned}
&\|F_j(u)-F_j(v)\|_{l,1}\\
&\le C\sum_{s=1}^m\sum_{\tilde\alpha_j=1}^l\Big(\sum_{\alpha_{j,1}+\dots+\alpha_{j,m}=\tilde\alpha_j}\prod_{k=1}^{s-1}\|u_k\|_{L^\infty}^{p_{j,k}-\alpha_{j,k}}\\
&\quad\times\prod_{k=s+1}^m\|v_k\|_{L^\infty}^{(p_{j,k}-\alpha_{j,k})_+}\big(\|u_s\|_{L^\infty}^{(p_{j,s}-\alpha_{j,s}-1)_+}+\|v_s\|_{L^\infty}^{(p_{j,s}-\alpha_{j,s}-1)_+}\big)\\
&\quad\times\Big(\|u_s-v_s\|_2\sum_{i=1}^m(\|u_i\|_l+\|v_i\|_l)\Big)\Big)\\
&\quad +C\sum_{s=1}^m\sum_{\tilde\alpha_j=1}^l\Big(\sum_{\alpha_{j,1}+\dots+\alpha_{j,m}=\tilde\alpha_j}\prod_{k=1}^{s-1}\|v_k\|_{L^\infty}^{(p_{j,k}-\alpha_{j,k})_+}\\
&\quad\times\prod_{k=s+1}^m\|v_k\|_{L^\infty}^{(p_{j,k}-\alpha_{j,k})_+}\big(\|v_s\|_{L^\infty}^{(p_{j,s}-\alpha_{j,s}-1)_+}\|v_s\|_{L^2}\big)\Big)\\
&\quad\times\sum_{s=1}^m\sum_{\tilde\alpha_j=1}^l\Big(\sum_{\alpha_{j,1}+\dots+\alpha_{j,m}=\tilde\alpha_j}\prod_{k=1}^{s-1}\|\partial_x^{\alpha_k}u_k\|_{L^\infty}\prod_{k=s+1}^m\|\partial_x^{\alpha_k}v_k\|_{L^\infty}\|u_s-v_s\|_l\Big).
\end{aligned}
\tag{3.22}
$$

For $H_1$, by using the Young inequality and Propositions 2.4–2.6 and noticing the definition of $\|\cdot\|_X$ and $\tilde{p}_j > 1 + \frac{2}{n}$, we have

$$
\begin{aligned}
H_1 &\leq \int_0^t \|(G_2 - F_{2\alpha})(t-\tau)\|_{L^\infty} \|(F_j(u) - F_j(v))(\tau)\|_{l-n-1,1} \mathrm{d}\tau \\
&\leq C \int_0^t (1+t-\tau)^{-n/2} \|(F_j(u) - F_j(v))(\tau)\|_{l-n-1,1} \mathrm{d}\tau \\
&\leq C E^{\tilde{p}_j - 1} \int_0^t (1+t-\tau)^{-n/2} (1+\tau)^{-n/2(\tilde{p}_j - 2) - n/4} \|u - v\|_X \mathrm{d}\tau \\
&\leq C E^{\tilde{p}_j - 1} (1+t)^{-n/2} \|u - v\|_X .
\end{aligned}
\tag{3.23}
$$

Similarly, by using (3.21), we have

$$
\begin{aligned}
&\|F_j(u) - F_j(v)\|_l \\
&\leq C \sum_{s=1}^m \sum_{\tilde{\alpha}_j = 1}^l \Big( \sum_{\alpha_{j,1} + \dots + \alpha_{j,m} = \tilde{\alpha}_j} \prod_{k=1}^{s-1} \|u_k\|_{L^\infty}^{p_{j,k} - \alpha_{j,k}} \\
&\quad \times \prod_{k=s+1}^m \|v_k\|_{L^\infty}^{(p_{j,k} - \alpha_{j,k})_+} \big( \|u_s\|_{L^\infty}^{(p_{j,s} - \alpha_{j,s} - 1)_+} + \|v_s\|_{L^\infty}^{(p_{j,s} - \alpha_{j,s} - 1)_+} \big) \\
&\quad \times \Big( \|u_s - v_s\|_{L^\infty} \sum_{i=1}^m (\|u_i\|_l + \|v_i\|_l) \Big) \Big) \\
&\quad + C \sum_{s=1}^m \sum_{\tilde{\alpha}_j = 1}^l \sum_{\alpha_{j,1} + \dots + \alpha_{j,m} = \tilde{\alpha}_j} \prod_{k=1}^m \|v_k\|_{L^\infty}^{(p_{j,k} - \alpha_{j,k})_+} \\
&\quad \times \sum_{s=1}^m \sum_{\tilde{\alpha}_j = 1}^l \Big( \sum_{\alpha_{j,1} + \dots + \alpha_{j,m} = \tilde{\alpha}_j} \prod_{k=1}^{s-1} \|\partial_x^{\alpha_k} u_k\|_{L^\infty} \prod_{k=s+1}^m \|\partial_x^{\alpha_k} v_k\|_{L^\infty} \|u_s - v_s\|_l \Big).
\end{aligned}
\tag{3.24}
$$

For $H_2$, similar to the estimate of $I_6$, it follows from Lemma 2.2 and the Sobolev inequality that

$$
\begin{aligned}
H_2 &\leq C \int_0^t \mathrm{e}^{-(t-\tau)/4} \|(F_j(u) - F_j(v))(\tau)\|_{l-n-1,\infty} \mathrm{d}\tau \\
&\leq C \int_0^t \mathrm{e}^{-(t-\tau)/4} \|(F_j(u) - F_j(v))(\tau)\|_{l-[\frac{n}{2}]} \mathrm{d}\tau \\
&\leq C E^{\tilde{p}_j} \int_0^t \mathrm{e}^{-(t-\tau)/4} (1+\tau)^{-n/2(\tilde{p}_j - 1) - n/4} \|u - v\|_X \mathrm{d}\tau \\
&\leq C E^{\tilde{p}_j} (1+t)^{-n/2} \|u - v\|_X ,
\end{aligned}
$$

where we used $\tilde{p}_j > 1 + \frac{2}{n}$. Then, we obtain

$$
\|T[u] - T[v]\|_{l-n-1,\infty} \leq C E^{\tilde{p}_j} (1+t)^{-n/2} \|u - v\|_X . \tag{3.25}
$$

On the other hand, by using the Young inequality, the Plancherel theorem, (2.21) and Propositions 2.4-2.5, we have

$$
\|T[u] - T[v]\|_l \leq \int_0^t \|(G_2 - G_{2,3})(t-\tau) * (F_j(u) - F_j(v))(\tau)\|_l \mathrm{d}\tau
$$

$$
\begin{aligned}
&+\int_0^t \|G_{2,3}(t-\tau)(t-\tau)*(F_j(u)-F_j(v))(\tau)\|_l\,\mathrm{d}\tau \\
&\le C\int_0^t (1+t-\tau)^{-n/4}\left(\|F_j(u)-F_j(v)\|_{l,1}+\|F_j(u)-F_j(v)\|_l\right)\mathrm{d}\tau \\
&\le CE^{\tilde{p}_j}(1+t)^{-n/4}\|u-v\|_X.
\end{aligned}
$$

Then

$$
\|T[u]-T[v]\|_l \le CE^{\tilde{p}_j}(1+t)^{-n/4}\|u-v\|_X. \tag{3.26}
$$

Combining (3.25) with (3.26), we obtain $\|Tu-Tv\|_X \le CE^{\tilde{p}_j}\|u-v\|_X$. Since the smallness assumption of $E$ and $\tilde{p}_j > 1+\frac{2}{n}$, we complete the proof of Lemma 3.2. $\square$

*Proof of Theorem 1.1.* Lemmas 3.1 and 3.2 show that for sufficiently small initial data

$$
(a_j,b_j)\in (W^{l,1}(\mathbb{R}^n)\cap H^{l+1}(\mathbb{R}^n))\times(W^{l,1}(\mathbb{R}^n)\cap H^l(\mathbb{R}^n))
$$

for $j=1,2,\dots,m$, $T:X\to X$ is a contraction mapping. By the Banach fixed point theorem, there exists a fixed point $u\in X$. Here, we obtain the solution $\{u_j(t)\}_{j=1}^m$ to system (1.1) satisfies $\|u\|_X\le E$. Then, the proof is complete. $\square$

## 4. Pointwise estimates

In this section, we show the pointwise estimates of the solutions to system (1.1). First of all, we recall

$$
\begin{aligned}
&\partial_x^\alpha u_j(x,t)\\
&=\partial_x^\alpha G_1*a_j+\partial_x^\alpha G_2*b_j+\partial_x^\alpha\int_0^t G_2(t-s)*F_j(u)(s)\mathrm{d}s\\
&=(\partial_x^\alpha(G_1-F_{1\alpha})(t))*a_j+(\partial_x^\alpha(G_2-F_{2\alpha})(t))*b_j+\partial_x^\alpha F_{1\alpha}(t)*a_j+\partial_x^\alpha F_{2\alpha}(t)*b_j\\
&\quad+\int_0^t(\partial_x^\alpha(G_2-F_{2\alpha})(t-\tau))*F_j(u)(\tau)\mathrm{d}\tau+\int_0^t\partial_x^\alpha F_{2\alpha}(t-\tau)*F_j(u)(\tau)\mathrm{d}\tau.
\end{aligned} \tag{4.1}
$$

From Propositions 2.4–2.5 and the assumption (1.11), by using Lemma 2.3, we have

$$
\begin{aligned}
&|(\partial_x^\alpha(G_1-F_{1\alpha})(t))*a_j+(\partial_x^\alpha(G_2-F_{2\alpha})(t))*b_j|\\
&\le C\varepsilon_0(1+t)^{-\frac{n+|\alpha|}{2}}B_r(|x|,t).
\end{aligned} \tag{4.2}
$$

For some positive number $b$, by noticing the definition of $F_{1\alpha}$, $|\alpha|<l$ and assumption (1.11) with $r>n$, we have

$$
|x^\beta\partial_x^\alpha F_{1\alpha}*a_j|\le\int|\partial_\xi^\beta\xi^\alpha\hat{F}_{1\alpha}\hat{a}_j|\mathrm{d}\xi\le C\varepsilon_0\mathrm{e}^{-bt}. \tag{4.3}
$$

Take $|\beta|=0$ or $|\beta|=n$, we obtain

$$
|\partial_x^\alpha F_{1\alpha}*a_j|\le C\varepsilon_0\mathrm{e}^{-\frac{b}{2}t}B_{\frac{n}{2}}(|x|,t). \tag{4.4}
$$

Similarly, we have

$$
|\partial_x^\alpha F_{2\alpha}*b_j|\le C\varepsilon_0\mathrm{e}^{-bt/2}B_r(|x|,t). \tag{4.5}
$$

To estimate the other parts in (4.1), we set

$$
\varphi_\alpha(x,t)=(1+t)^{\frac{n+|\alpha|}{2}}(B_{\frac{n}{2}}(|x|,t))^{-1}, \tag{4.6}
$$

and

$$M(t)=\sup_{0\le s,\tau\le t,\ |\alpha|\le l}\sum_{j=1}^{m}|\partial_x^\alpha u_j(x,\tau)|\varphi_\alpha(x,s). \tag{4.7}$$

When $|\alpha|\le l-1$, from the assumptions (1.6), (1.7) and the definition of $M$, we have

$$|\partial_x^\alpha F_j(u)(x,t)|\le M^2(t)(1+t)^{-n-\frac{|\alpha|}{2}}B_n(|x|,t). \tag{4.8}$$

When $|\alpha|=l$, from the definition of $M$, by using Theorem 1.1 and Lemma 3.1, we have

$$|\partial_x^\alpha F_j(u)(x,s)|\le M^2(t)(1+t)^{-n-\frac{|\alpha|}{2}}B_n(|x|,t)+EM(t)(1+t)^{-n-\frac{|\alpha|}{2}}B_{\frac{n}{2}}(|x|,t). \tag{4.9}$$

Set

$$R^\alpha=\Big|\int_0^t F_{2\alpha}(t-s)*\partial_x^\alpha F_j(u)(s)\mathrm{d}s\Big|.$$

From Lemma 2.2 and (4.8) and (4.9), we obtain

$$\begin{aligned} R^\alpha &\le \Big|\int_0^t \mathrm{e}^{-b(t-s)}(f_1+f_2)*\partial_x^\alpha F_j(u)(s)\mathrm{d}s\Big| \\ &\le \Big|\int_0^t \mathrm{e}^{-b(t-s)}f_1*\partial_x^\alpha F_j(u)(s)\mathrm{d}s\Big|+\Big|\int_0^t \mathrm{e}^{-b(t-s)}f_2*\partial_x^\alpha F_j(u)(s)\mathrm{d}s\Big| \\ &:= R_1^\alpha+R_2^\alpha. \end{aligned} \tag{4.10}$$

The right-hand side of the above inequality can be estimated as follows.

$$\begin{aligned} &R_2^\alpha \\ &\le \Big|\int_0^t \mathrm{e}^{-b(t-s)}\int_{\mathbb{R}^n} f_2(x-y)\partial_x^\alpha F_j(u)(y,s)|_{|y-x|<\varepsilon}\mathrm{d}y\mathrm{d}s\Big| \\ &\le \int_0^t \mathrm{e}^{-b(t-s)}\int_{\mathbb{R}^n}|f_2(x-y)|(M^2(t)+EM(t))(1+t)^{-n-\frac{|\alpha|}{2}}B_{\frac{n}{2}}(|y|,t)_{|y-x|<\varepsilon}\mathrm{d}y\mathrm{d}s \\ &\le \int_0^t \mathrm{e}^{-b(t-s)}\|f_2\|_{L^1}(M^2(t)+EM(t))(1+t)^{-n-\frac{|\alpha|}{2}}B_{\frac{n}{2}}(|x|,t)\mathrm{d}s \\ &\le C(1+t)^{-\frac{n+|\alpha|}{2}}(M^2(t)+EM(t))B_{\frac{n}{2}}(|x|,t). \end{aligned}$$

Since

$$|\partial_x^\alpha(\mathrm{e}^{-bt}f_1(x))|\le C(1+t)^{-\frac{n+|\alpha|+1}{2}}B_N(|x|,t),$$

we have

$$R_1^\alpha=\Big|\int_0^t \mathrm{e}^{-b(t-s)}f_1*F_j(u)(s)\mathrm{d}s\Big|\le C(1+t)^{-\frac{n+|\alpha|}{2}}(M^2(t)+M(t)E)B_{\frac{n}{2}}(|x|,t).$$

From the definition of $M$, we have

$$|\partial_x^\alpha F_j(u)(x,s)|\le CM^2(t)(1+s)^{-n-\frac{|\alpha|}{2}}B_n(|x|,s). \tag{4.11}$$

From Proposition 2.6 and Lemma 2.3, we obtain

$$\begin{aligned} &\Big|\int_0^t(\partial_x^\alpha(G_2(t-s)-F_{2\alpha}))*F_j(u)(s)\mathrm{d}s\Big| \\ &\le C(M^2(t)+M(t)E)(1+t)^{-\frac{n+|\alpha|}{2}}B_{\frac{n}{2}}(|x|,t). \end{aligned} \tag{4.12}$$

From the above inequalities and the definition of $M$, we obtain

$$M(t) \le C\big(M^2(t) + EM(t) + \varepsilon_0\big). \tag{4.13}$$

Since $E$ and $\varepsilon_0$ are small enough, we have $M(t) \le C$. It yields that

$$|\partial_x^\alpha u_j(t)| \le C(1+t)^{-\frac{n+|\alpha|}{2}} B_{\frac{n}{2}}(|x|, t). \tag{4.14}$$

Thus, we can easily obtain the optimal $L^p$, $1 \le p \le \infty$, convergence rate as follows.

**Corollary 4.1.** *Under the assumptions of Theorem 1.1, for $p \in [1, \infty]$, $|\alpha| \le l$, we have*

$$\|\partial_x^\alpha u_j(\cdot, t)\|_{L^p} \le C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}, \quad j = 1, \dots, m. \tag{4.15}$$

Thus, we have complete the proof of Theorem 1.2.

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