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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR MISCIBLE LIQUIDS MODEL IN POROUS MEDIA

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Abstract. In this article, we study the existence and uniqueness of solutions for miscible liquids model in porous media. The model describing the phenomenon is a system of equations coupling hydrodynamic equations with concentration equation taking into account the Korteweg stress. We assume that the fluid is incompressible and its motion is described by the Darcy law. We prove the existence and uniqueness of global solutions for the initial boundary value problem.

1. Introduction

Two liquids are miscible if the molecules of the one liquid can mix freely with the molecules of the other liquid. There is no sharp interface between miscible liquids, but rather a transition zone. An example of such phenomenon is the mixing of water and glycerin [8, 13]. It is possible that two liquids are not completely miscible. They may mix until the concentration reaches a certain saturation value. This saturation may be affected by the temperature and pressure of the system [1].

There exists a transient capillary phenomena between two miscible liquids. Theoretical and experimental studies of such phenomena are reviewed in [8]. The change of concentration gradients near the transition zone causes capillary forces between the two miscible liquids [6]. So we need to take into account some additional terms in the equation of motion due to the concentration inhomogeneities called Korteweg stress [3, 5].

The existence and uniqueness of the solutions of miscible liquids model with fully incompressible Navier-Stokes equations are studied in [7]. In this paper, we are interested in studying porous media. The study of miscible liquids in porous media is motivated by enhanced oil recovery, hydrology, frontal polymerization, groundwater pollution and filtration [4, 9, 12, 14, 15].

The paper is organized as follows. The next section introduces the model, while section 3 deals with the existence of the model solutions and we establish the uniqueness of solutions in section 4.

1.1. **The model.** The model describing the interaction between two miscible liquids in porous media is given by the following system of equations in the bounded

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open domain $\Omega \in \mathbb{R}^2$ with Lipschitz continuous boundary [2, 11]:

$$\frac{\partial C}{\partial t} + u \cdot \nabla C = d\Delta C, \tag{1.1}$$

$$\frac{\partial u}{\partial t} + \frac{\mu}{K} u = -\nabla p + \nabla \cdot T(C), \tag{1.2}$$

$$\operatorname{div}(u) = 0, \tag{1.3}$$

with the boundary conditions

$$\frac{\partial C}{\partial n} = 0, \quad u \cdot n = 0, \quad \text{on } \Gamma, \tag{1.4}$$

and the initial conditions

$$C(x, 0) = C_0(x), \quad u(x, 0) = u_0(x), \quad x \in \Omega. \tag{1.5}$$

Here u is the velocity, p is the pressure, C is the concentration, d is the coefficient of mass diffusion, μ is the viscosity, K is the permeability of the medium, Γ is the boundary of Ω, n is the unit outward normal vector to Γ, the additional stress tensor term is

$$T_{11} = k\big(\frac{\partial C}{\partial x_2}\big)^2, \quad T_{12} = T_{21} = -k\frac{\partial C}{\partial x_1}\frac{\partial C}{\partial x_2}, \quad T_{22} = k\big(\frac{\partial C}{\partial x_1}\big)^2, \tag{1.6}$$

where k is a nonnegative constant. The gradient, divergence and Laplace operators can be defined as follows

$$\nabla v = \big(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}\big), \quad \operatorname{div} \overrightarrow{\mathbf{v}} = \sum_{i=1}^{2} \frac{\partial \mathbf{v}_i}{\partial x_i}, \quad \Delta v = \sum_{i=1}^{2} \frac{\partial^2 v}{\partial^2 x_i},$$

the divergence of the additional tensor term will be in the form

$$\nabla \cdot T(C) = \begin{pmatrix} \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} \\ \\ \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} \end{pmatrix}. \tag{1.7}$$

1.2. **The problem in variational form.** We specify now the functional framework in which we carry out our analysis of the problem. The velocity space S_u is

$$S_u = \{u \in H(\operatorname{div}; \Omega); \operatorname{div}(u) = 0, u \cdot n = 0 \text{ on } \Gamma\}.$$

The concentration space S_C is

$$S_C = \{C \in H^2(\Omega); \frac{\partial C}{\partial n} = 0 \text{ on } \Gamma\}.$$

By the Green formula, the variational form of problem (1.1)-(1.5) is: For each B and v, find C and u such that

$$\big(\frac{\partial C}{\partial t}, B\big) + d(\nabla C, \nabla B) + (u \cdot \nabla C, B) = 0, \tag{1.8}$$

$$\big(\frac{\partial u}{\partial t}, v\big) + \mu_p(u, v) - (div\ T(C), v) = 0. \tag{1.9}$$

Here $\mu_p = \mu/K$. We will assume that $d > 0$ and $\mu_p > 0$.

2. Existence of global solutions

To prove the existence of global solutions, we need the following lemmas.

Lemma 2.1. *The concentration C is bounded in the $L^\infty(0,t;L^2)$ space.*

Proof. Choosing C as test function in (1.8), we will have:

$$\frac{1}{2}\frac{\partial}{\partial t}(C,C) + d(\nabla C,\nabla C) + (u\cdot\nabla C, C) = 0.$$

Since $u \in S_u$ the last term vanishes. The second term is positive, so by integrating over time:

$$\|C(t=s)\|_{L^2}^2 \le \|C_0\|_{L^2}^2,$$

from the inequality, it follows that C is bounded in $L^\infty(0,t;L^2)$. □

Lemma 2.2. *The concentration C is bounded in $L^\infty(0,t;H^1)$ and the velocity u is bounded in $L^\infty(0,t;L^2)$.*

Proof. By choosing $-k\Delta C$ as test function in (1.8), we have

$$(\frac{\partial C}{\partial t}, -k\Delta C) + (u\cdot\nabla C, -k\Delta C) = d(\Delta C, -k\Delta C);$$

therefore,

$$\frac{k}{2}\frac{\partial}{\partial t}(\nabla C,\nabla C) + dk(\Delta C,\Delta C) - k(u\cdot\nabla C,\Delta C) = 0,$$

then

$$\frac{1}{2}\frac{\partial}{\partial t}(\nabla C,\nabla C) + d(\Delta C,\Delta C) - (u\cdot\nabla C,\Delta C) = 0. \tag{2.1}$$

Also, by choosing u as test function in (1.9),

$$\frac{1}{2}\frac{\partial}{\partial t}(u,u) + \mu_p(u,u) - (\nabla\cdot T(C), u) = 0. \tag{2.2}$$

To have an explicit expression of $\nabla\cdot T(C)$, we calculate its first component:

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} = 2k\frac{\partial C}{\partial x_2}\frac{\partial^2 C}{\partial x_1\partial x_2} - k\frac{\partial^2 C}{\partial x_1\partial x_2}\frac{\partial C}{\partial x_2} - k\frac{\partial C}{\partial x_1}\frac{\partial^2 C}{\partial x_2^2}, \tag{2.3}$$

hence

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} = k\frac{\partial C}{\partial x_1}\frac{\partial^2 C}{\partial x_1\partial x_2} + k\frac{\partial C}{\partial x_1}\frac{\partial^2 C}{\partial x_1^2} - k\frac{\partial C}{\partial x_1}\Delta C,$$

then

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} = \frac{k}{2}\frac{\partial}{\partial x_1}(\nabla C)^2 - k\frac{\partial C}{\partial x_1}\Delta C.$$

Following the same steps for the second component, we have

$$\nabla\cdot T = \frac{k}{2}\nabla(\nabla C)^2 - k\Delta C\nabla C.$$

Replacing this last equality in (2.2) and since $u \in S_u$, we have

$$\frac{1}{2}\frac{\partial}{\partial t}(u,u) + \mu_p(u,u) - k(\Delta C\nabla C, u) = 0. \tag{2.4}$$

Adding (2.1) and (2.4), and with the fact $u \in S_u$ and $C \in S_c$, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\big((u,u) + (\nabla C,\nabla C)\big) + \mu_p(u,u) + dk(\Delta C,\Delta C) = 0.$$

Since the second and the third terms are positive, by integrating over time, we have

$$k\|C(t=s)\|_{H^1}^2 + \|u(t=s)\|_{L^2}^2 \le k\|C(t=0)\|_{H^1}^2 + \|u(t=0)\|_{L^2}^2.$$

We conclude that C is bounded in $L^\infty(0,t;H^1)$ and u is bounded in $L^\infty(0,t;L^2)$. □

Now, we look for the estimates over the time derivatives of u and C. To this end we will need the following Lemmas.

Lemma 2.3. *The derivative* $\frac{\partial C}{\partial t}$ *of the concentration is bounded in* $L^2(0,t;L^2)$.

Proof. From (1.8) and by the triangular inequality, we have

$$\|\frac{\partial C}{\partial t}\|_{L^2} \le d\|\Delta C\|_{L^2} + \|u\cdot\nabla C\|_{L^2}.$$

Using Hölder inequality, we obtain

$$\|\frac{\partial C}{\partial t}\|_{L^2} \le d\|\Delta C\|_{L^2} + \|u\|_{L^4}\|\nabla C\|_{L^4},$$

and by the Gagliardo-Nirenberg inequality, it follows that there exists $N>0$ such that

$$\|\frac{\partial C}{\partial t}\|_{L^2} \le d\|\Delta C\|_{L^2} + N\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\nabla C\|_{L^2}^{1/2}\|\nabla C\|_{H^1}^{1/2}.$$

We conclude that $\frac{\partial C}{\partial t}$ is bounded in $L^2(0,t;L^2)$. □

Lemma 2.4. *The time derivative of the velocity* $\frac{\partial u}{\partial t}$ *is bounded in* $L^2(0,t;L^2)$.

Proof. To prove this lemma, it is sufficient to remark that $\nabla\cdot T(C)$ is sum of expressions of the form $\lambda D_i(D_jCD_lC)$. Where $D_i=\frac{\partial}{\partial x_i}$, $i=1,2$, and λ depends on i,j and l (see for example (2.3)). Using the problem in its variational form and using the same technics as for the previous lemmas, it follows that $\frac{\partial u}{\partial t}$ is bounded in $L^2(0,t;L^2)$. □

Now, we can give our main result as follows:

Theorem 2.5. *The problem* (1.1)-(1.5) *admits a global solution.*

Proof. A priori error estimates over concentration and speed (see all the previous Lemmas) allow us to deduce that our finite dimensional solution is global in time. In addition, applying some classical compactness theorems (see for example [10, 16]), it follows the existence of our continuous problem. □

3. Uniqueness of solutions

To prove the uniqueness, we assume that (1.1)-(1.5) has two different solutions (C_1,u_1) and (C_2,u_2). From (1.1), we have:

$$\frac{\partial}{\partial t}(C_1-C_2) - d\Delta(C_1-C_2) + u_1\nabla C_1 - u_2\nabla C_2 = 0, \tag{3.1}$$

and from (1.2), we have also:

$$\begin{aligned}&\frac{\partial}{\partial t}(u_1-u_2) + \mu_p(u_1-u_2) + \nabla(p_1-p_2)\\ &= \frac{k}{2}\nabla\left((\nabla C_1)^2-(\nabla C_2)^2\right) - k(\Delta C_1\nabla C_1 - \Delta C_2\nabla C_2).\end{aligned} \tag{3.2}$$

Multiplying (3.1) by $-k\Delta(C_1-C_2)$ and integrating, we obtain

$$
\begin{aligned}
&(\frac{\partial}{\partial t}(C_1-C_2),-k\Delta(C_1-C_2))+dk(\Delta(C_1-C_2),\Delta(C_1-C_2))\\
&+(u_1\nabla C_1,-k\Delta(C_1-C_2))+(u_2\nabla C_2,k\Delta(C_1-C_2))=0.
\end{aligned}
$$

Similarly, multiplying (3.2) by u_1-u_2 and integrating, we have

$$
\begin{aligned}
&(\frac{\partial}{\partial t}(u_1-u_2),u_1-u_2)+\mu_p(u_1-u_2,u_1-u_2)\\
&=\frac{k}{2}(\nabla\left((\nabla C_1)^2-(\nabla C_2)^2\right),u_1-u_2)-k(\Delta C_1\nabla C_1-\Delta C_2\nabla C_2,u_1-u_2).
\end{aligned}
$$

Adding the two last equalities, using the Green formula and the fact that $u_i\in S_u$, it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}(\|u_1-u_2\|_{L^2}^2+k\|\nabla C_1-\nabla C_2\|_{L^2}^2)+\mu_p\|u_1-u_2\|_{L^2}^2+kd\|\Delta(C_1-C_2)\|_{L^2}^2\\
&=k(u_1\nabla(C_1-C_2),\Delta(C_1-C_2))+k((u_1-u_2)\nabla C_2,\Delta(C_1-C_2))\\
&\quad-k(\Delta C_1\nabla(C_1-C_2),u_1-u_2)+k(-\Delta C_1\nabla C_2+\Delta C_2\nabla C_2,u_1-u_2);
\end{aligned}
$$

therefore,

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}(\|u_1-u_2\|_{L^2}^2+k\|\nabla C_1-\nabla C_2\|_{L^2}^2)+\mu_p\|u_1-u_2\|_{L^2}^2+kd\|\Delta(C_1-C_2)\|_{L^2}^2\\
&=k(u_1\nabla(C_1-C_2),\Delta(C_1-C_2))-k(\Delta C_1\nabla(C_1-C_2),u_1-u_2).
\end{aligned}
\tag{3.3}
$$

Now we look for estimates of the right-hand terms. We put $C=C_1-C_2$ and $u=u_1-u_2$, by Hölder inequality, it follows that

$$
\begin{aligned}
|(\Delta C_1\nabla C,u)|&\le\|\Delta C_1\|_{L^2}\|\nabla C\cdot u\|_{L^2}\\
&\le\|\Delta C_1\|_{L^2}\|\nabla C\|_{L^4}\|u\|_{(L^4)^2}.
\end{aligned}
$$

Also, by Gagliardo-Nirenberg inequality, it follows that

$$
|(\Delta C_1\nabla C,u)|\le N_1\|\Delta C_1\|_{L^2}\|\nabla C\|_{L^2}^{1/2}\|\Delta C\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}.
$$

By applying Young inequality, we obtain

$$
|(\Delta C_1\nabla C,u)|\le\frac{N_1}{4}\|\Delta C\|_{L^2}^2+\frac{3N_1}{4}\|\Delta C_1\|_{L^2}^{4/3}\|\nabla C\|_{L^2}^{2/3}\|u\|_{L^2}^{2/3}\|\nabla u\|_{L^2}^{2/3}.
$$

Using that same technics, we obtain the following inequalities: first,

$$
\begin{aligned}
|(u_1\nabla C,\Delta C)|&\le\|\Delta C\|_{L^2}\|\nabla C.u_1\|_{L^2}\\
&\le\|\Delta C\|_{L^2}\|\nabla C\|_{L^4}\|u_1\|_{(L^4)^2};
\end{aligned}
$$

therefore,

$$
|(u_1\nabla C,\Delta C)|\le N_2\|\Delta C\|_{L^2}^{3/2}\|\nabla C\|_{L^2}^{1/2}\|u_1\|_{L^2}^{1/2}\|\nabla u_1\|_{L^2}^{1/2}.
$$

Finally,

$$
|(u_1\nabla C,\Delta C)|\le\frac{3N_2}{4}\|\Delta C\|_{L^2}^2+\frac{N_2}{4}\|\nabla C\|_{L^2}^2\|u_1\|_{L^2}^2\|\nabla u_1\|_{L^2}^2.
$$

From (3.3) and assuming that $N_1+3N_2\le 4d$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}(\|u\|_{L^2}^2+k\|\nabla C\|_{L^2}^2)\\
&\le\frac{3N_1k}{2}\|\Delta C_1\|_{L^2}^{4/3}\|\nabla C\|_{L^2}^{2/3}\|u\|_{L^2}^{2/3}\|\nabla u\|_{L^2}^{2/3}+\frac{N_2k}{2}\|\nabla C\|_{L^2}^2\|u_1\|_{L^2}^2\|\nabla u_1\|_{L^2}^2
\end{aligned}
$$

$$\le (\|u\|_{L^2}^2 + k\|\nabla C\|_{L^2}^2)\Big(\frac{N_2}{2}\|\nabla C\|_{L^2}^2\|u_1\|_{L^2}^2\|\nabla u_1\|_{L^2}^2 + \frac{3N_1k}{2}\|\Delta C_1\|_{L^2}^{4/3}\|\nabla C\|_{L^2}^{2/3}\|u\|_{L^2}^{-4/3}\|\nabla u\|_{L^2}^{2/3}\Big).$$

If we denote

$$\phi(t) = \|\nabla C\|_{L^2}^2\|u_1\|_{L^2}^2\|\nabla u_1\|_{L^2}^2 + \|\Delta C_1\|_{L^2}^{4/3}\|\nabla C\|_{L^2}^{2/3}\|u\|_{L^2}^{-4/3}\|\nabla u\|_{L^2}^{2/3},$$
$$M = \max(\frac{N_2}{2}, \frac{3N_1k}{2}),$$

we have

$$\frac{d}{dt}(exp(M\int_0^t \phi(s)ds)(\|u\|_{L^2}^2 + k\|\nabla C\|_{L^2}^2)) \le 0quad\forall t \ge 0\,.$$

We deduce:

$$\exp(M\int_0^t \phi(s)ds)(\|u\|_{L^2}^2 + k\|\nabla C\|_{L^2}^2) \le \|u(0)\|_{L^2}^2 + k\|\nabla C(0)\|_{L^2}^2.$$

Since $u(0) = C(0) = 0$, we conclude the uniqueness of the solution.

Now, we can state our second theorem as follows.

Theorem 3.1. *Problem* (1.1)-(1.5) *admits a unique solution.*

Concluding remarks. The interaction between two miscible liquids is modelled by a system of equations, coupling hydrodynamic equations and the species conservation equation. Due to the interfacial interaction between the two liquids, we have taken into account two additional terms, called Korteweg stress tensor. We have chosen the appropriate functional framework for our variational problem. We have established a priori estimates over the concentration and speed which allow us to establish the existence of the solution. Furthermore, we have also proved the uniqueness of solution.

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