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EXISTENCE OF INFINITELY MANY PERIODIC SOLUTIONS FOR SECOND-ORDER HAMILTONIAN SYSTEMS

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Abstract. By using the variant of the fountain theorem, we study the existence of infinitely many periodic solutions for a class of superquadratic nonautonomous second-order Hamiltonian systems.

1. Introduction

Consider the second-order Hamiltonian system

$$\begin{gathered}\ddot{u}(t) - U(t)u(t) + \nabla_u W(t,u) = 0, \quad \forall t \in \mathbb{R}, \\ u(0) = u(T), \quad \dot{u}(0) = \dot{u}(T), \quad T > 0,\end{gathered} \tag{1.1}$$

where $U(\cdot)$ is a continuous T-periodic symmetric positive definite matrix and $W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is T-periodic in its first variable. Moreover, we assume that $W(t,x)$ is continuous in t for each $x \in \mathbb{R}^N$, continuously differentiable in x for each $t \in [0,T]$ and $\nabla W(t,x)$ denotes its gradient with respect to the x variable.

Inspired by the monographs [5, 6], the existence and multiplicity of periodic solutions for Hamiltonian systems have been investigated in many papers (see [2, 3, 4, 7, 8, 9, 10, 12] and the references therein) via the variational methods. In 2008, He and Wu [4] studied the existence of nontrivial T-periodic solutions for system (1.1) by a mountain pass theorem and a local link theorem. In 2010, Zhang and Tang [10] obtained some new results of T-periodic solutions for system (1.1) under weaker assumptions, which generalized the corresponding results in [4]. In [9], Zhang and Liu considered the second-order Hamiltonian system

$$\begin{gathered}\ddot{u}(t) + \nabla_u V(t,u) = 0, \quad \forall t \in \mathbb{R}, \\ u(0) = u(T), \quad \dot{u}(0) = \dot{u}(T), \quad T > 0,\end{gathered} \tag{1.2}$$

where $V \in C^1(\mathbb{R} \times \mathbb{R}^N)$ is T-periodic in t and has the form

$$V(t,u) = \frac{1}{2}\langle U(t)u, u\rangle + W(t,u).$$

Here and in the sequel, $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the standard inner product and norm in $\mathbb{R}^N$ respectively. They obtained infinitely many periodic solutions of (1.2) by

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using the variant of the fountain theorem under superquadratic assumptions (see [9, Theorem 1.2]).

Now motivated by the above papers [9, 10], we will use the following conditions to obtain the existence of infinitely many periodic solutions of system (1.1).

(S1) There exist constants $d_1 > 0$ and $\alpha > 1$ such that

$$|\nabla_u W(t,u)| \le d_1(1+|u|^\alpha), \quad \forall t \in [0,T],\ u \in \mathbb{R}^N;$$

(S2) $W(t,u) \ge 0$ for all $(t,u) \in [0,T] \times \mathbb{R}^N$, and

$$\liminf_{|u|\to\infty} \frac{W(t,u)}{|u|^2} = \infty, \quad \forall t \in [0,T];$$

(S3) There exist constants $\mu > 2$, $0 < \beta < 2$, $L > 0$ and a function $a(t) \in L^1(0,T;R^+)$ such that

$$\mu W(t,u) \le \langle \nabla_u W(t,u), u\rangle + a(t)|u|^\beta, \quad \forall |u| \ge L,\ u \in \mathbb{R}^N,\ t \in [0,T].$$

Then our main result is the following theorem.

Theorem 1.1. *Assume that* (S1)–(S3) *hold and that* $W(t,u)$ *is even in* u*. Then* (1.1) *possesses infinitely many solutions.*

Note that by (S1), we can obtain that there exists a constant $d_2 > 0$ such that

$$|W(t,u)| \le d_1(|u| + |u|^{\alpha+1}) + d_2, \quad \forall (t,u) \in [0,T] \times \mathbb{R}^N. \tag{1.3}$$

As is known, the so-called global Ambrosetti-Rabinowitz condition (AR-condition for short) was introduced by Ambrosetti and Rabinowitz in [1] and is wildly used in the study of the superquadratic case of Hamiltonian systems: there is a constant μ such that

$$0 < \mu W(t,u) \le \langle \nabla_u W(t,u), u\rangle, \quad \forall (t,u) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\}. \tag{1.4}$$

When we take $a(t) \equiv 0$, the condition (S3) reduces to (1.4). So the condition (S3) is weaker than AR-condition.

2. Preliminaries

In this section, we will establish the variational setting for our problem and give a variant fountain theorem. Let $E = H_T^1$ be the usual Sobolev space with the inner product

$$\langle u, v\rangle_E = \int_0^T \langle u(t), v(t)\rangle dt + \int_0^T \langle \dot u(t), \dot v(t)\rangle dt.$$

We define a functional Φ on E by

$$\Phi(u) = \frac{1}{2}\Big(\int_0^T |\dot u|^2 dt + \int_0^T \langle U(t)u, u\rangle dt\Big) - \Psi(u), \tag{2.1}$$

where $\Psi(u) = \int_0^T W(t,u(t))dt$. Then Φ and Ψ are continuously differentiable and

$$\langle \Phi'(u), v\rangle = \int_0^T \langle \dot u, \dot v\rangle dt + \int_0^T \langle U(t)u, v\rangle dt - \int_0^T \langle \nabla_u W(t,u), v\rangle dt.$$

Define a selfadjoint linear operator $\mathcal{B} : L^2([0,T], \mathbb{R}^N) \to L^2([0,T], \mathbb{R}^N)$ by

$$\langle \mathcal{B}u, v\rangle_{L^2} = \int_0^T \langle \dot u, \dot v\rangle dt + \int_0^T \langle U(t)u, v\rangle dt$$

with domain $D(\mathcal{B}) = E$. Then $\mathcal{B}$ has a sequence of eigenvalues

$$\lambda_{-m} \le \lambda_{-m+1} \le \cdots \le \lambda_{-1} < 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \ldots$$

such that $\lambda_k \to +\infty$ as $k \to +\infty$. Note that 0 may be not a eigenvalue. Let e_j be the eigenvector of $\mathcal{B}$ corresponding to λ_j (esp. e_0 is an eigenvector corresponding to the eigenvalue 0.), then $\{e_{-m}, e_{-m+1}, \ldots, e_{-1}, e_0, e_1, \ldots\}$ forms an orthogonal basis in L^2. Set

$$E^0 = \ker \mathcal{B} = \operatorname{span}\{e_0\},$$
$$E^- = \operatorname{span}\{e_j : j = -m, -m+1, \ldots, -1\},$$
$$E^+ = \operatorname{span}\{e_j : j = 1, 2, \ldots, \}.$$

Then E possess an orthogonal decomposition $E = E^- \oplus E^0 \oplus E^+$. For $u \in E$, we have

$$u = u^- + u^0 + u^+ \in E^- \oplus E^0 \oplus E^+.$$

We can define on E a new inner product and the associated norm by

$$\langle u, v\rangle_0 = \langle \mathcal{B}u^+, v^+\rangle_{L^2} - \langle \mathcal{B}u^-, v^-\rangle_{L^2} + \langle u^0, v^0\rangle_{L^2},$$
$$\|u\| = \langle u, u\rangle_0^{1/2}.$$

Therefore, Φ can be written as

$$\Phi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \Psi(u). \tag{2.2}$$

Direct computations show that

$$\begin{gathered} \langle \Psi'(u), v\rangle = \int_0^T \langle \nabla_u W(t,u), v\rangle dt \\ \langle \Phi'(u), v\rangle = \langle u^+, v^+\rangle_0 - \langle u^-, v^-\rangle_0 - \langle \Psi'(u), v\rangle \end{gathered} \tag{2.3}$$

for all $u, v \in E$ with $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+$ respectively. It is known that $\Psi' : E \to E$ is compact.

Denote by $|\cdot|_p$ the usual norm of $L^p \equiv L^p([0,T], \mathbb{R}^N)$ for all $1 \le p \le \infty$, then by the Sobolev embedding theorem, there exists a $\tau_p > 0$ such that

$$|u|_p \le \tau_p \|u\|, \quad \forall u \in E. \tag{2.4}$$

Now we state an abstract critical point theorem founded in [11]. Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\oplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \oplus_{j=1}^k X_j$ and $Z_k = \overline{\oplus_{j=k}^\infty X_j}$. Consider the following C^1-functional $\Phi_\lambda : E \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) := A(u) - \lambda B(u), \quad \lambda \in [1,2].$$

Theorem 2.1 ([11, Theorem 2.1])**.** *Assume that the functional Φ_λ defined above satisfies*

- (F1) *Φ_λ maps bounded sets to bounded sets for $\lambda \in [1,2]$, and $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$;*
- (F2) *$B(u) \ge 0$ for all $u \in E$; moreover, $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$;*
- (F3) *There exist $r_k > \rho_k > 0$ such that*

$$\alpha_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} \Phi_\lambda(u) > \beta_k(\lambda) := \max_{u\in Y_k, \|u\|=r_k} \Phi_\lambda(u), \quad \forall \lambda \in [1,2].$$

Then
$$\alpha_k(\lambda) \le \zeta_k(\lambda) := \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} \Phi_\lambda(\gamma(u)), \quad \forall \lambda \in [1,2],$$
where $B_k = \{u \in Y_k : \|u\| \le r_k\}$ *and* $\Gamma_k := \{\gamma \in C(B_k, E) : \gamma \text{is odd}, \ \gamma|_{\partial B_k} = id\}$. *Moreover, for a.e.* $\lambda \in [1,2]$, *there exists a sequence* $\{u_m^k(\lambda)\}_{m=1}^\infty$ *such that*
$$\sup_m \|u_m^k(\lambda)\| < \infty, \quad \Phi'_\lambda(u_m^k(\lambda)) \to 0, \quad \Phi_\lambda(u_m^k(\lambda)) \to \zeta_k(\lambda) \quad \text{as } m \to \infty.$$

To apply this theorem to prove our main result, we define the functionals A, B and Φ_λ on our working space E by
$$A(u) = \frac{1}{2}\|u^+\|^2, \quad B(u) = \frac{1}{2}\|u^-\|^2 + \int_0^T W(t,u)dt, \tag{2.5}$$
$$\Phi_\lambda(u) = A(u) - \lambda B(u) = \frac{1}{2}\|u^+\|^2 - \lambda(\frac{1}{2}\|u^-\|^2 + \int_0^T W(t,u)dt) \tag{2.6}$$
for all $u = u^- + u^0 + u^+ \in E = E^- + E^0 + E^+$ and $\lambda \in [1,2]$. Then $\Phi_\lambda \in C^1(E,\mathbb{R})$ for all $\lambda \in [1,2]$ and
$$\langle \Phi'_\lambda(u), v\rangle = \langle u^+, v^+\rangle_0 - \lambda(\langle u^-, v^-\rangle_0 + \int_0^T \langle \nabla_u W(t,u), v\rangle dt). \tag{2.7}$$
Let $X_j = \operatorname{span}\{e_j\}$, $j = -m, -m+1, \dots, -1, 0, 1, 2, \dots$. Note that $\Phi_1 = \Phi$, where Φ is the functional defined in (2.2).

3. Proof of Theorem 1.1

We first establish the following lemmas and then give the proof of Theorem 1.1.

Lemma 3.1. *Assume that* (S1)–(S2) *hold. Then* $B(u) \ge 0$ *for all* $u \in E$. *Furthermore,* $A(u) \to \infty$ *or* $B(u) \to \infty$ *as* $\|u\| \to \infty$.

Proof. Since $W(t,u) \ge 0$, by (2.5), it is obvious that $B(u) \ge 0$ for all $u \in E$. By the proof of [9, Lemma 2.6], for any finite-dimensional subspace $F \subset E$, there exists a constant $\epsilon > 0$ such that
$$m(\{t \in [0,T] : |u| \ge \epsilon\|u\|\}) \ge \epsilon, \quad \forall u \in F \setminus \{0\}, \tag{3.1}$$
where $m(\cdot)$ is the Lebesgue measure.

Now for the finite-dimensional subspace $E^- \oplus E^0 \subset E$, there exist a constant ϵ corresponding to the one in (3.1). Let
$$\Lambda_u = \{t \in [0,T] : |u| \ge \epsilon\|u\|\}, \quad \forall u \in E^- \oplus E^0 \setminus \{0\}.$$
Then $m(\Lambda_u) \ge \epsilon$. By (S2), there exist positive constants d_3 and R_1 such that
$$W(t,u) \ge d_3|u|^2, \quad \forall t \in [0,T] \text{ and } |u| \ge R_1. \tag{3.2}$$
Note that
$$|u(t)| \ge R_1, \quad \forall t \in \Lambda_u \tag{3.3}$$
for any $u \in E^- \oplus E^0$ with $\|u\| \ge R_1/\epsilon$. Combining (3.2) and (3.3), for any $u \in E^- \oplus E^0$ with $\|u\| \ge R_1/\epsilon$, we have
$$\begin{aligned} B(u) &= \frac{1}{2}\|u^-\|^2 + \int_0^T W(t,u)dt \\ &\ge \int_{\Lambda_u} W(t,u)dt \ge \int_{\Lambda_u} d_3|u|^2 dt \end{aligned}$$

$$\geq d_3\epsilon^2\|u\|^2 \cdot m(\Lambda_u) \geq d_3\epsilon^3\|u\|^2 .$$

This implies $B(u) \to \infty$ as $\|u\| \to \infty$ on $E^- \oplus E^0$. Combining this with $E = E^- \oplus E^0 \oplus E^+$ and (2.5), we have

$$A(u) \to \infty \text{ or } B(u) \to \infty \quad \text{as } \|u\| \to \infty.$$

The proof is complete. □

Lemma 3.2. *Let* (S1)–(S3) *be satisfied. Then there exist a positive integer k_1 and two sequences $r_k > \rho_k \to \infty$ as $k \to \infty$ such that*

$$\alpha_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} \Phi_\lambda(u) > 0, \quad \forall k \geq k_1, \tag{3.4}$$

and

$$\beta_k(\lambda) := \max_{u\in Y_k, \|u\|=r_k} \Phi_\lambda(u) < 0, \quad \forall k \in \mathbb{N}, \tag{3.5}$$

where $Y_k = \oplus_{j=-m}^{k} X_j = \operatorname{span}\{e_{-m}, e_{-m+1}, \dots, e_k\}$ and

$$Z_k = \overline{\oplus_{j=k}^{\infty} X_j} = \overline{\operatorname{span}\{e_k, e_{k+1}, \dots\}}$$

for all $k \in \{-m, -m+1, \dots, 1, 2, \dots\}$.

Proof. Step 1. First we prove (3.4). By (1.3) and (2.6), for all $u \in E^+$ we have

$$\begin{aligned} \Phi_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - 2\int_0^T W(t,u)dt \\ &\geq \frac{1}{2}\|u\|^2 - 2d_1(|u|_1 + |u|_{\alpha+1}^{\alpha+1}) - 2d_2T, \quad \forall \lambda \in [1,2]. \end{aligned} \tag{3.6}$$

where d_1, d_2 are the constants in (1.3). Let

$$\iota_{\alpha+1}(k) = \sup_{u\in Z_k, \|u\|=1} |u|_{\alpha+1}, \quad \forall k \in \mathbb{N}. \tag{3.7}$$

Then

$$\iota_{\alpha+1}(k) \to 0 \quad \text{as } k \to \infty \tag{3.8}$$

since E is compactly embedded into $L^{\alpha+1}$. Note that

$$Z_k \subset E^+, \quad \forall k \geq 1. \tag{3.9}$$

Combining (2.4), (3.6), (3.7) and (3.9), for $k \geq 1$, we have

$$\Phi_\lambda(u) \geq \frac{1}{2}\|u\|^2 - 2d_1\tau_1\|u\| - 2d_2T - 2d_1\iota_{\alpha+1}^{\alpha+1}(k)\|u\|^{\alpha+1}, \tag{3.10}$$

for all $(\lambda, u) \in [1,2] \times Z_k$, where τ_1 is the constant given in (2.4). By (3.8), there exists a positive integer $k_1 \geq 1$ such that

$$\rho_k := (16d_1\iota_{\alpha+1}^{\alpha+1}(k))^{1/(1-\alpha)} > \max\{16d_1\tau_1 + 1, 16d_2T\}, \quad \forall k \geq k_1 \tag{3.11}$$

since $\alpha > 1$. Clearly,

$$\rho_k \to \infty \quad \text{as } k \to \infty. \tag{3.12}$$

Combining (3.10) and (3.11), direct computation shows

$$\alpha_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} \Phi_\lambda(u) \geq \rho_k^2/4 > 0, \quad \forall k \geq k_1.$$

Step 2. We prove (3.5). Note that for any $k \in \{-m, -m+1, \dots, 1, 2, \dots\}$, Y_k is of finite dimension, so we can choose $M_1 > 0$ sufficiently large such that

$$\|u\| \le M_1 \Big(\int_0^T |u|^2\Big)^{1/2}, \quad \forall u \in Y_k. \tag{3.13}$$

By (S2) and (1.3), for the former M_1, there exists a $M_2 > 0$ such that

$$W(t,u) \ge M_1^2 |u|^2 - M_2, \quad \forall (t,u) \in [0,T] \times \mathbb{R}^N. \tag{3.14}$$

Consequently, by (3.13) and (3.14), we have

$$\begin{aligned}
\Phi_\lambda(u) &\le \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_0^T W(t,u)dt \\
&\le \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - M_1^2 \int_0^T |u|^2 dt + M_2 T \\
&\le \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - M_1^2 \Big(\frac{1}{M_1^2}\|u^+\|^2 + \frac{1}{M_1^2}\|u^0\|^2\Big) + M_2 T \\
&\le -\frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \|u^0\|^2 + M_2 T \\
&\le -\frac{1}{2}\|u\|^2 + M_2 T
\end{aligned} \tag{3.15}$$

for all $u = u^- + u^0 + u^+ \in Y_k$. Now for any $k \in \{-m, -m+1, \dots, 1, 2, \dots\}$, if we choose

$$r_k > \max\{\rho_k, \sqrt{2M_2T}\}$$

then (3.15) implies

$$\beta_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u) < 0, \quad \forall k \in \mathbb{N}.$$

The proof is complete. □

Now we prove our main result.

Proof of Theorem 1.1. In view of (1.3), (2.4) and (2.6), Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. By the evenness of $W(t,u)$ in u, it holds that $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$. Therefore condition (F1) of Theorem 2.1 holds. Lemma 3.1 shows that condition (F2) holds, whereas Lemma 3.2 implies that condition (F3) holds for all $k \ge k_1$, where k_1 is given in Lemma 3.2. Thus, by Theorem 2.1, for each $k \ge k_1$ and a.e. $\lambda \in [1,2]$, there exists a sequence $\{u_m^k(\lambda)\}_{m=1}^\infty \subset E$ such that

$$\sup_m \|u_m^k(\lambda)\| < \infty, \ \Phi_\lambda'(u_m^k(\lambda)) \to 0 \ \textit{and} \ \Phi_\lambda(u_m^k(\lambda)) \to \zeta_k(\lambda) \tag{3.16}$$

as $m \to \infty$, where

$$\zeta_k(\lambda) := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \Phi_\lambda(\gamma(u)), \quad \forall \lambda \in [1,2]$$

with $B_k = \{u \in Y_k : \|u\| \le r_k\}$ and $\Gamma_k := \{\gamma \in C(B_k, E) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = \mathrm{id}\}$.

Moreover, by the proof of Lemma 3.2, we have

$$\zeta_k(\lambda) \in [\overline{\alpha}_k, \overline{\zeta}_k], \quad \forall k \ge k_1, \tag{3.17}$$

where $\overline{\zeta}_k := \max_{u \in B_k} \Phi_1(u)$ and $\overline{\alpha}_k := \rho_k^2/4 \to \infty$ as $k \to \infty$ by (3.12).

Since the sequence $\{u_m^k(\lambda)\}_{m=1}^{\infty}$ obtained by (3.16) is bounded, it is clear that for each $k \geq k_1$, we can choose $\lambda_n \to 1$ such that the sequence $\{u_m^k(\lambda_n)\}_{m=1}^{\infty}$ has a strong convergent subsequence.

In fact, without loss of generality, we assume that

$$u_m^k(\lambda_n)^- \to u_0^k(\lambda_n)^-, \quad u_m^k(\lambda_n)^0 \to u_0^k(\lambda_n)^0, \quad u_m^k(\lambda_n)^+ \rightharpoonup u_0^k(\lambda_n)^+ \tag{3.18}$$

as $m \to \infty$ and

$$u_m^k(\lambda_n) \rightharpoonup u_0^k(\lambda_n) \quad \text{as } m \to \infty \tag{3.19}$$

for some $u_0^k(\lambda_n) = u_0^k(\lambda_n)^- + u_0^k(\lambda_n)^0 + u_0^k(\lambda_n)^+ \in E = E^- \oplus E^0 \oplus E^+$ since $\dim(E^- \oplus E^0) < \infty$. Note that

$$\Phi'_{\lambda_n}(u_m^k(\lambda_n)) = u_m^k(\lambda_n)^+ - \lambda_n(u_m^k(\lambda_n)^- + \Psi'(u_m^k(\lambda_n)), \quad \forall n \in \mathbb{N}.$$

That is,

$$u_m^k(\lambda_n)^+ = \Phi'_{\lambda_n}(u_m^k(\lambda_n)) + \lambda_n(u_m^k(\lambda_n)^- + \Psi'(u_m^k(\lambda_n)), \quad \forall m \in \mathbb{N}. \tag{3.20}$$

In view of (3.16), (3.18), (3.19) and the compactness of Ψ', the right-hand side of (3.20) converges strongly in E and hence $u_m^k(\lambda_n)^+ \to u_0^k(\lambda_n)^+$ in E. Together with (3.18), $\{u_m^k(\lambda_n)\}_{m=1}^{\infty}$ has a strong convergent subsequence in E.

Without loss of generality, we may assume that

$$\lim_{m\to\infty} u_m^k(\lambda_n) = u_n^k, \quad \forall n \in \mathbb{N} \textit{ and } k \geq k_1.$$

This together with (3.16) and (3.17) yields

$$\Phi'_{\lambda_n}(u_n^k) = 0, \quad \Phi_{\lambda_n}(u_n^k) \in [\overline{\alpha}_k, \overline{\zeta}_k], \quad \forall n \in \mathbb{N} \textit{ and } k \geq k_1. \tag{3.21}$$

Now we claim that the sequence $\{u_n^k\}_{n=1}^{\infty}$ in (3.21) is bounded in E and possesses a strong convergent subsequence with the limit $u^k \in E$ for all $k \geq k_1$. For the sake of notational simplicity, throughout the remaining proof of Theorem 1.1 we denote $u_n = u_n^k$. For $u_n \in E$, let $\overline{u}_n = \frac{1}{T}\int_o^T u_n(t)dt$, $u_n = \widetilde{u}_n + \overline{u}_n$. By (2.4), there exists a constant τ_∞ for any $u \in E$ such that

$$|u|_\infty \leq \tau_\infty \|u\|. \tag{3.22}$$

Assume by contradiction, first, we prove that $\{u_n\}$ is bounded in E. Otherwise, going to a subsequence if necessary, we can assume that $\|u_n\| \to \infty$ as $n \to \infty$. Put $v_n = \frac{u_n}{\|u_n\|}$, then v_n is bounded in E. Hence, there exists a subsequence, still denoted by v_n, such that

$$v_n \rightharpoonup v_0 \;\textit{ in } E, \quad v_n \to v_0 \;\textit{ in } C([0,T], \mathbb{R}^N).$$

Then, we have

$$\overline{v}_n \to \overline{v}_0. \tag{3.23}$$

By (1.3), for all $|u| \leq L$, we have

$$W(t,u) \leq d_1(|u| + |u|^{\alpha+1}) + d_2 \leq d_1(L + L^{\alpha+1}) + d_2,$$

which together with (S3) yields

$$\mu W(t,u) \leq \langle \nabla_u W(t,u), u\rangle + a(t)|u|^\beta + \mu d_1(L + L^{\alpha+1}) + \mu d_2, \tag{3.24}$$

for all $u \in \mathbb{R}^N$ and $t \in [0,T]$. It follows from (2.6), (2.7) that

$$\begin{aligned}\mu\Phi_{\lambda_n}(u_n) - \langle \Phi'_{\lambda_n}(u_n), u_n\rangle &= (\frac{\mu}{2} - 1)(\|u_n^+\|^2 - \|u_n^-\|^2) - (\lambda_n - 1)(\frac{\mu}{2} - 1)\|u_n^-\|^2 \\ &\quad - \lambda_n \int_0^T (\mu W(t,u_n) - \langle \nabla_u W(t,u_n), u_n\rangle)dt\,.\end{aligned}$$

In the following, we denote $C_i > 0 (i = 0, 1, 2, \dots)$ for different positive constants. Comparing (2.1) with (2.2), we learn that

$$\begin{aligned}
&(\frac{\mu}{2} - 1)\|\dot{u_n}\|_{L^2}^2 \\
&= \mu\Phi_{\lambda_n}(u_n) - \langle \Phi'_{\lambda_n}(u_n), u_n\rangle - (\frac{\mu}{2} - 1)\int_0^T \langle U(t)u_n, u_n\rangle dt \\
&\quad + (\lambda_n - 1)(\frac{\mu}{2} - 1)\|u_n^-\|^2 + \lambda_n \int_0^T (\mu W(t, u_n) - \langle \nabla_u W(t, u_n), u_n\rangle)dt.
\end{aligned}$$

This together with the positive definite assumption of matrix U, (2.4), (3.21), (3.24) and $\mu > 2$ implies

$$\begin{aligned}
(\frac{\mu}{2} - 1)\|\dot{u_n}\|_{L^2}^2 &\le C_1 + (\frac{\mu}{2} - 1)(\lambda_n - 1)\|u_n^-\|^2 \\
&\quad + \lambda_n \int_0^T (a(t)|u_n|^\beta + \mu d_1(L + L^{\alpha+1}) + \mu d_2)dt \\
&\le C_2 + C_3(\lambda_n - 1)\|u_n^-\|^2 + C_4\|u_n\|^\beta.
\end{aligned} \tag{3.25}$$

Note that $0 < \beta < 2$, $\lambda_n \to 1$ and $\|u_n^-\|^2 \le \|u_n\|^2$, we have

$$\frac{\|\dot{u_n}\|_{L^2}^2}{\|u_n\|^2} \le \frac{C_5}{\|u_n\|^2} + C_6(\lambda_n - 1)\frac{\|u_n^-\|^2}{\|u_n\|^2} + C_7\frac{\|u_n\|^\beta}{\|u_n\|^2} \to 0 \quad \text{as } n \to \infty;$$

i.e., $\|\dot{v_n}\|_{L^2} \to 0$ as $n \to \infty$. Together with (3.23), we have $v_n \to \overline{v}_0$ as $n \to \infty$. Therefore, we obtain

$$v_0 = \overline{v}_0, \quad T|\overline{v}_0|^2 = \|\overline{v}_0\|^2 = 1\,.$$

Consequently, $|u_n| \to \infty$ as $n \to \infty$ uniformly for a.e. $t \in [0.T]$. From (S2), we obtain

$$\begin{aligned}
\liminf_{|u_n|\to\infty} \frac{\int_0^T W(t, u_n)dt}{\|u_n\|^2} &\ge \frac{\int_0^T \liminf_{|u_n|\to\infty} W(t, u_n)dt}{\|u_n\|^2} \\
&= \int_0^T [\liminf_{|u_n|\to\infty} \frac{W(t, u_n)}{|u_n|^2}|v_n|^2]dt \\
&= \int_0^T [\liminf_{|u_n|\to\infty} \frac{W(t, u_n)}{|u_n|^2}|v_0|^2]dt > 0\,.
\end{aligned}$$

Hence,

$$\liminf_{|u_n|\to\infty} \frac{\int_0^T W(t, u_n)dt}{\|u_n\|^2} > 0. \tag{3.26}$$

On the other hand, from (2.4), (2.6), (2.7), (3.21)and (3.24), we have

$$\begin{aligned}
(\frac{\mu}{2} - 1)(\|u_n^+\|^2 - \lambda_n\|u_n^-\|^2) &= \mu\Phi_{\lambda_n}(u_n) - \langle \Phi'_{\lambda_n}(u_n), u_n\rangle \\
&\quad + \lambda_n \int_0^T (\mu W(t, u_n) - \langle \nabla_u W(t, u_n), u_n\rangle)dt \\
&\le C_1 + \lambda_n \int_0^T (a(t)|u_n|^\beta + \mu d_1(L + L^{\alpha+1}) + \mu d_2)dt \\
&\le C_8 + C_9\lambda_n\|u_n\|^\beta\,.
\end{aligned}$$

Note that $\mu > 2$; then we obtain

$$(\|u_n^+\|^2 - \lambda_n\|u_n^-\|^2) \leq \frac{2C_8}{\mu-2} + \frac{2C_9}{\mu-2}\lambda_n\|u_n\|^\beta. \tag{3.27}$$

By the boundedness of $\Phi_{\lambda_n}(u_n)$, and (3.27), we have

$$\begin{aligned}\frac{\Phi_{\lambda_n}(u_n)}{\|u_n\|^2} &= \frac{\frac{1}{2}(\|u_n^+\|^2 - \lambda_n\|u_n^-\|^2)}{\|u_n\|^2} - \frac{\lambda_n\int_0^T W(t,u_n)dt}{\|u_n\|^2} \\ &\leq \frac{\frac{C_8}{\mu-2}}{\|u_n\|^2} + \frac{\frac{C_9}{\mu-2}\lambda_n\|u_n\|^\beta}{\|u_n\|^2} - \frac{\lambda_n\int_0^T W(t,u_n)dt}{\|u_n\|^2},\end{aligned}$$

which together with $0 < \beta < 2$ implies

$$\lim_{|u_n|\to\infty}\inf\frac{\int_0^T W(t,u_n)dt}{\|u_n\|^2} = 0.$$

This contradicts to (3.26). Thus, $\{u_n\}$ is bounded in E.

The proof that $\{u_n\}$ has a strong convergent subsequence is the same as the preceding proof of $\{u_m^k(\lambda_n)\}_{m=1}^\infty$.

Now for each $k \geq k_1$, by (3.21), the limit u^k is just a critical point of $\Phi = \Phi_1$ with $\Phi(u^k) \in [\overline{\alpha}_k, \overline{\zeta}_k]$. Since $\overline{\alpha}_k \to \infty$ as $k \to \infty$ in (3.17), we obtain infinitely many nontrivial critical points of Φ. Therefore, system (1.1) possesses infinitely many nontrivial solutions. The proof of Theorem 1.1 is complete. □

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