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# GLOBAL SOLVABILITY FOR INVOLUTIVE SYSTEMS ON THE TORUS

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Abstract. In this article, we consider a class of involutive systems of $n$ smooth vector fields on the torus of dimension $n+1$. We prove that the global solvability of this class is related to an algebraic condition involving Liouville forms and the connectedness of all sublevel and superlevel sets of the primitive of a certain 1-form associated with the system.

## 1. Introduction

In this article we study the global solvability of a system of vector fields on $\mathbb{T}^{n+1} \simeq (\mathbb{R}/2\pi\mathbb{Z})^{n+1}$ given by

$$L_j = \frac{\partial}{\partial t_j} + (a_j + ib_j)(t)\frac{\partial}{\partial x}, \quad j = 1, \dots, n, \tag{1.1}$$

where $(t_1, \dots, t_n, x) = (t, x)$ denotes the coordinates on $\mathbb{T}^{n+1}$, $a_j, b_j \in C^\infty(\mathbb{T}^n; \mathbb{R})$ and for each $j$ we consider $a_j$ or $b_j$ identically zero.

We assume that the system (1.1) is involutive (see [1, 12]) or equivalently that the 1-form $c(t) = \sum_{j=1}^n (a_j + ib_j)(t)dt_j \in \wedge^1 C^\infty(\mathbb{T}^n_t)$ is closed.

When the 1-form $c(t)$ is exact the problem was treated by Cardoso and Hounie in [9]. Here, we will consider that only the imaginary part of $c(t)$ is exact, that is, the real 1-form $b(t) = \sum_{j=1}^n b_j(t)dt_j$ is exact.

The system (1.1) gives rise to a complex of differential operators $\mathbb{L}$ which at the first level acts in the following way

$$\mathbb{L}u = d_t u + c(t) \wedge \frac{\partial}{\partial x}u, \quad u \in C^\infty(\mathbb{T}^{n+1}) \quad \text{or } \mathcal{D}'(\mathbb{T}^{n+1})), \tag{1.2}$$

where $d_t$ denotes the exterior differential on the torus $\mathbb{T}^n_t$. Our aim is to carry out a study of the global solvability at the first level of this complex. In other words we study the global solvability of the equation $\mathbb{L}u = f$ where $u \in \mathcal{D}'(\mathbb{T}^{n+1})$ and $f \in C^\infty(\mathbb{T}^n_t \times \mathbb{T}^1_x; \wedge^{1,0})$.

Note that if the equation $\mathbb{L}u = f$ has a solution $u$ then $f$ must be of the form

$$f = \sum_{j=1}^n f_j(t, x)dt_j.$$

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The local solvability of this complex of operators was studied by Treves in his seminal work [11].

When each function $b_j \equiv 0$, the global solvability was treated by Bergamasco and Petronilho in [8]. In this case the system is globally solvable if and only if the real 1-form $a(t) = \sum_{j=1}^{n} a_j(t)dt_j$ is either non-Liouville or rational (see definition in [2]).

When $c(t)$ is exact the problem was solved by Cardoso and Hounie in [9]. In this case the 1-form $c(t)$ has a global primitive $C$ defined on $\mathbb{T}^n$ and global solvability is equivalent to the connectedness of all sublevels and superlevels of the real function $Im(C)$.

We are interested in global solvability when at least one of the functions $b_j \not\equiv 0$ and $c(t)$ is not exact. Moreover, we suppose that $Im(c)$ is exact and for each $j$, $a_j \equiv 0$ or $b_j \equiv 0$.

We prove that system (1.1) is globally solvable if and only if the real 1-form $a(t)$ is either non-Liouville or rational and any primitive of the 1-form $b(t)$ has only connected sublevels and superlevels on $\mathbb{T}^n$ (see Theorem 2.2).

The articles [3, 4, 5, 6, 7, 10] deal with similar questions.

## 2. Preliminaries and statement of the main result

There are natural compatibility conditions on the 1-form $f$ for the existence of a solution $u$ to the equation $\mathbb{L}u = f$. We now move on to describing them.

If $f \in C^\infty(\mathbb{T}_x^n \times \mathbb{T}_t^1; \wedge^{1,0})$ we consider the $x$-Fourier series

$$f(t,x) = \sum_{\xi\in\mathbb{Z}} \hat{f}(t,\xi)e^{i\xi x},$$

where $\hat{f}(t,\xi) = \sum_{j=1}^{n} \hat{f}_j(t,\xi)dt_j$ and $\hat{f}_j(t,\xi)$ denotes the Fourier transform with respect to $x$.

Since $b$ is exact there exists a function $B \in C^\infty(\mathbb{T}_t^n;\mathbb{R})$ such that $d_tB = b$. Moreover, we may write $a = a_0 + d_tA$ where $A \in C^\infty(\mathbb{T}_t^n;\mathbb{R})$ and $a_0 \in \wedge^1\mathbb{R}^n \simeq \mathbb{R}^n$. Thus, we may write $c(t) = a_0 + d_tC$ where $C(t) = A(t) + iB(t)$.

We will identify the 1-form $a_0 \in \wedge^1\mathbb{R}^n$ with the vector $a_0 := (a_{10}, \dots, a_{n0})$ in $\mathbb{R}^n$ consisting of the periods of the 1-form $a$ given by

$$a_{j0} = \frac{1}{2\pi}\int_0^{2\pi} a_j(0,\dots,\tau_j,\dots,0)d\tau_j.$$

Thus, if $f \in C^\infty(\mathbb{T}_t^n \times \mathbb{T}_x^1; \wedge^{1,0})$ and if there exists $u \in \mathcal{D}'(\mathbb{T}^{n+1})$ such that $\mathbb{L}u = f$ then, since $\mathbb{L}$ defines a differential complex, $\mathbb{L}f = 0$ or equivalently $L_jf_k = L_kf_j$, $j,k = 1,\dots,n$; also

$$\hat{f}(t,\xi)e^{i\xi(a_0\cdot t + C(t))} \text{ is exact when } \xi a_0 \in \mathbb{Z}. \tag{2.1}$$

We define now the set

$$\mathbb{E} = \big\{ f \in C^\infty(\mathbb{T}_t^n \times \mathbb{T}_x^1; \wedge^{1,0});\ \mathbb{L}f = 0 \text{ and (2.1) holds} \big\}.$$

**Definition 2.1.** The operator $\mathbb{L}$ is said to be globally solvable on $\mathbb{T}^{n+1}$ if for each $f \in \mathbb{E}$ there exists $u \in \mathcal{D}'(\mathbb{T}^{n+1})$ satisfying $\mathbb{L}u = f$.

Given $\alpha \notin \mathbb{Q}^n$ we say that $\alpha$ is *Liouville* when there exists a constant $C > 0$ such that for each $N \in \mathbb{N}$ the inequality

$$\max_{j=1,\dots,n} \Big|\alpha_j - \frac{p_j}{q}\Big| \leq \frac{C}{q^N},$$

has infinitely many solutions $(p_1, \dots, p_n, q) \in \mathbb{Z}^n \times \mathbb{N}$.

Let us consider the following two sets

$$J = \{j \in \{1, \dots, n\};\ b_j \equiv 0\}, \quad K = \{k \in \{1, \dots, n\};\ a_k \equiv 0\};$$

and we will write $J = \{j_1, \dots, j_m\}$ and $K = \{k_1, \dots, k_p\}$. Under the above notation, the main result of this work is the following theorem.

**Theorem 2.2.** *Let $B$ be a global primitive of the 1-form $b$. If $J \cup K = \{1, \dots, n\}$ then the operator $\mathbb{L}$ given in* (1.2) *is globally solvable if and only if one of the following two conditions holds:*

- (I) *$J \neq \emptyset$ and $(a_{j_1 0}, \dots, a_{j_m 0}) \notin \mathbb{Q}^m$ is non-Liouville.*
- (II) *The sublevels $\Omega_s = \{t \in \mathbb{T}^n,\ B(t) < s\}$ and superlevels $\Omega^s = \{t \in \mathbb{T}^n,\ B(t) > s\}$ are connected for every $s \in \mathbb{R}$ and $(a_{j_1 0}, \dots, a_{j_m 0}) \in \mathbb{Q}^m$ if $J \neq \emptyset$.*

Note that if $J = \emptyset$ then $K = \{1, \dots, n\}$ (since $J \cup K = \{1, \dots, n\}$ by hypothesis). In this case each $a_k \equiv 0$ and Theorem 2.2 says that $\mathbb{L}$ is globally solvable if and only if all the sublevels and superlevels of $B$ are connected in $\mathbb{T}^n$, which is according to [9].

When $J = \{1, \dots, n\}$ we have that $b = 0$, hence any primitive of $b$ has only connected subleves and superlevels on $\mathbb{T}^n$. In this case Theorem 2.2 says that $\mathbb{L}$ is globally solvable if and only if either $a_0 \notin \mathbb{Q}^n$ is non-Liouville or $a_0 \in \mathbb{Q}^n$, which was proved in [8]. Thus, in order to prove Theorem 2.2 it suffices to consider the following situation $\emptyset \neq J \neq \{1, \dots, n\}$.

**Remark 2.3.** As in [8], the differential operator $\mathbb{L}$ is globally solvable if and only if the differential operator

$$d_t + (a_0 + ib(t)) \wedge \frac{\partial}{\partial x} \tag{2.2}$$

is globally solvable.

Indeed, consider the automorphism

$$S : \mathcal{D}'(\mathbb{T}^{n+1}) \longrightarrow \mathcal{D}'(\mathbb{T}^{n+1})$$
$$\sum_{\xi \in \mathbb{Z}} \hat{u}(t, \xi) e^{i\xi x} \longmapsto \sum_{\xi \in \mathbb{Z}} \hat{u}(t, \xi) e^{i\xi A(t)} e^{i\xi x},$$

where $A$ is the previous smooth real valued function satisfying $d_t A = a(t) - a_0$. Observe that following relation holds:

$$S\mathbb{L}S^{-1} = d_t + (a_0 + ib(t)) \wedge \frac{\partial}{\partial x},$$

which ensures the above statement.

Therefore, it is sufficient to prove Theorem 2.2 for the operator (2.2). For the rest of this article, we will denote by $\mathbb{L}$ the operator (2.2); that is,

$$\mathbb{L} = d_t + (a_0 + ib(t)) \wedge \frac{\partial}{\partial x} \tag{2.3}$$

and by $\mathbb{E}$ the corresponding space of compatibility conditions. The new operator $\mathbb{L}$ is associated with the vector fields

$$L_j = \frac{\partial}{\partial t_j} + (a_{j0} + ib_j(t))\frac{\partial}{\partial x}, \quad j = 1, \dots, n. \tag{2.4}$$

## 3. Sufficiency part of Theorem 2.2

First assume that $(a_{j_1 0}, \dots, a_{j_m 0}) \notin \mathbb{Q}^m$ is non-Liouville where

$$J = \{j_1, \dots, j_m\} := \{j \in \{1, \dots, n\},\ b_j \equiv 0\}.$$

Then, there exist a constant $C > 0$ and an integer $N > 1$ such that

$$\max_{j \in J} |\xi a_{j0} - \kappa_j| \geq \frac{C}{|\xi|^{N-1}}, \quad \forall (\kappa, \xi) \in \mathbb{Z}^m \times \mathbb{N}. \tag{3.1}$$

Consider the set $I$ where $I \cup J = \{1, \dots, n\}$ and $I \cap J = \emptyset$. Remember that $\emptyset \neq J \neq \{1, \dots, n\}$ then $I \neq \emptyset$ and $b_\ell \not\equiv 0$ if $\ell \in I$.

We denote by $t_J$ the variables $t_{j_1}, \dots, t_{j_m}$ and by $t_I$ the other variables on $\mathbb{T}^n_t$. Let $f(t,x) = \sum_{j=1}^n f_j(t,x) dt_j \in \mathbb{E}$. Consider the $(t_J, x)$-Fourier series as follows

$$u(t,x) = \sum_{(\kappa,\xi) \in \mathbb{Z}^m \times \mathbb{Z}} \hat{u}(t_I, \kappa, \xi) e^{i(\kappa \cdot t_J + \xi x)} \tag{3.2}$$

and for each $j = 1, \dots, n$,

$$f_j(t,x) = \sum_{(\kappa,\xi) \in \mathbb{Z}^m \times \mathbb{Z}} \hat{f}_j(t_I, \kappa, \xi) e^{i(\kappa \cdot t_J + \xi x)}, \tag{3.3}$$

where $\kappa = (\kappa_{j_1}, \dots, \kappa_{j_m}) \in \mathbb{Z}^m$ and $\hat{u}(t_I, \kappa, \xi)$ and $\hat{f}_j(t_I, \kappa, \xi)$ denote the Fourier transform with respect to variables $(t_{j_1}, \dots, t_{j_m}, x)$.

Substituting the formal series (3.2) and (3.3) in the equations $L_j u = f_j$, $j \in J$, we have for each $(\kappa, \xi) \neq (0,0)$

$$i(\kappa_j + \xi a_{j0}) \hat{u}(t_I, \kappa, \xi) = \hat{f}_j(t_I, \kappa, \xi), \quad j \in J.$$

Also, from the compatibility conditions $L_j f_\ell = L_\ell f_j$, for all $j, \ell \in J$, we obtain the equations

$$(\kappa_j + \xi a_{j0}) \hat{f}_\ell(t_I, \kappa, \xi) = (\kappa_\ell + \xi a_{\ell 0}) \hat{f}_j(t_I, \kappa, \xi), \ \ j, \ell \in J.$$

By the preceding equations we have

$$\hat{u}(t_I, \kappa, \xi) = \frac{1}{i(\kappa_M + \xi a_{M0})} \hat{f}_M(t_I, \kappa, \xi), \quad (\kappa, \xi) \neq (0,0), \tag{3.4}$$

where $M \in J$, $M = M(\xi)$ is such that

$$|\kappa_M + \xi a_{M0}| = \max_{j \in J} |\kappa_j + \xi a_{j0}| \neq 0.$$

If $(\kappa, \xi) = (0,0)$, since $\hat{f}(t_I, 0, 0)$ is exact, there exists $v \in C^\infty(\mathbb{T}^{n-m}_{t_I})$ such that $dv = \hat{f}(\cdot, 0, 0)$. Thus, we choose $\hat{u}(t_I, 0, 0) = v(t_I)$.

Given $\alpha \in \mathbb{Z}^{n-m}_+$ we obtain from (3.1) and (3.4) the inequality

$$|\partial^\alpha \hat{u}(t_I, \kappa, \xi)| \leq \frac{1}{C} |\xi|^{N-1} |\partial^\alpha \hat{f}_M(t_I, \kappa, \xi)|.$$

Since each $f_j$ is a smooth function we conclude that

$$u(t,x) = \sum_{(\kappa,\xi)\in\mathbb{Z}^m\times\mathbb{Z}} \hat{u}(t_I,\kappa,\xi)e^{i(\kappa\cdot t_J+\xi x)} \in C^\infty(\mathbb{T}^{n+1}).$$

By construction $u$ is a solution of

$$L_j u = f_j, \quad j \in J.$$

Now, we will prove that $u$ is also a solution to the equations

$$L_\ell u = f_\ell, \quad \ell \in I.$$

Let $\ell \in I$. Given $(\kappa,\xi) \neq (0,0)$ by the compatibility condition $L_M f_\ell = L_\ell f_M$ we have

$$i(\kappa_M + \xi a_{M0})\hat{f}_\ell(t_I,\kappa,\xi) = \frac{\partial}{\partial t_\ell}\hat{f}_M(t_I,\kappa,\xi) - \xi b_\ell(t)\hat{f}_M(t_I,\kappa,\xi). \tag{3.5}$$

Therefore, (3.4) and (3.5) imply

$$\begin{aligned}
&\frac{\partial}{\partial t_\ell}\hat{u}(t_I,\kappa,\xi) - \xi b_\ell(t)\hat{u}(t_I,\kappa,\xi)\\
&= \frac{1}{i(\kappa_M+\xi a_{M0})}\frac{\partial}{\partial t_\ell}\hat{f}_M(t_I,\kappa,\xi) - \xi b_\ell(t)\frac{1}{i(\kappa_M+\xi a_{M0})}\hat{f}_M(t_I,\kappa,\xi)\\
&= \frac{1}{i(\kappa_M+\xi a_{M0})}\Big(\frac{\partial}{\partial t_\ell}\hat{f}_M(t_I,\kappa,\xi) - \xi b_\ell(t)\hat{f}_M(t_I,\kappa,\xi)\Big)\\
&= \hat{f}_\ell(t_I,\kappa,\xi).
\end{aligned}$$

If $(\kappa,\xi) = (0,0)$ then $\frac{\partial}{\partial t_\ell}\hat{u}(t_I,0,0) = \hat{f}_\ell(t_I,0,0)$.

We have thus proved that condition (I) implies global solvability.

Suppose now that the condition (II) holds. Let $q_J$ be the smallest positive integer such that $q_J(a_{j_1 0},\dots,a_{j_m 0}) \in \mathbb{Z}^m$.

We denote by $\mathcal{A} := q_J\mathbb{Z}$ and $\mathcal{B} := \mathbb{Z}\backslash\mathcal{A}$ and define

$$\mathcal{D}'_{\mathcal{A}}(\mathbb{T}^{n+1}) := \big\{u \in \mathcal{D}'(\mathbb{T}^{n+1}); \quad u(t,x) = \sum_{\xi\in\mathcal{A}} \hat{u}(t,\xi)e^{i\xi x}\big\}.$$

Let $\mathbb{L}_{\mathcal{A}}$ be the operator $\mathbb{L}$ acting on $\mathcal{D}'_{\mathcal{A}}(\mathbb{T}^{n+1})$. Similarly, we define $\mathcal{D}'_{\mathcal{B}}(\mathbb{T}^{n+1})$ and $\mathbb{L}_{\mathcal{B}}$.

Then $\mathbb{L}$ is globally solvable if and only if $\mathbb{L}_{\mathcal{A}}$ and $\mathbb{L}_{\mathcal{B}}$ are globally solvable (see [3]).

**Lemma 3.1.** *The operator $\mathbb{L}_{\mathcal{A}}$ is globally solvable.*

*Proof.* Since $q_J a_0 \in \mathbb{Z}^n$, we define

$$\begin{gathered}
T : \mathcal{D}'_{\mathcal{A}}(\mathbb{T}^{n+1}) \longrightarrow \mathcal{D}'_{\mathcal{A}}(\mathbb{T}^{n+1})\\
\sum_{\xi\in\mathcal{A}} \hat{u}(t,\xi)e^{i\xi x} \longmapsto \sum_{\xi\in\mathcal{A}} \hat{u}(t,\xi)e^{-i\xi a_0\cdot t}e^{i\xi x}.
\end{gathered}$$

Note that $T$ is an automorphism of $\mathcal{D}'_{\mathcal{A}}(\mathbb{T}^{n+1})$ (and of $C^\infty_{\mathcal{A}}(\mathbb{T}^{n+1})$). Furthermore the following relation holds:

$$T^{-1}\mathbb{L}_{\mathcal{A}}T = \mathbb{L}_{0,\mathcal{A}}, \tag{3.6}$$

where $\mathbb{L}_0 := d_t + ib(t) \wedge \frac{\partial}{\partial x}$.

Let $B$ be a global primitive of $b$ on $\mathbb{T}^n$. Since all the sublevels and superlevels of $B$ are connected in $\mathbb{T}^n$, by work [8] we have $\mathbb{L}_0$ globally solvable, hence $\mathbb{L}_{0,\mathcal{A}}$ is globally solvable. Since $T$ is an automorphism, from equality (3.6) we obtain that $\mathbb{L}_{\mathcal{A}}$ is globally solvable. □

If $q_J = 1$ then $\mathcal{A} = \mathbb{Z}$ and the proof is complete. Otherwise we have:

**Lemma 3.2.** *The operator $\mathbb{L}_{\mathcal{B}}$ is globally solvable.*

*Proof.* Let $(\kappa, \xi) \in \mathbb{Z}^m \times \mathcal{B}$. Since $q_J$ is defined as the smallest natural such that $q_J(a_{j_1 0}, \dots, a_{j_m 0}) \in \mathbb{Z}^m$, there exists $\ell \in J$ such that

$$\big|a_{\ell 0} - \frac{\kappa_\ell}{\xi}\big| \geq \frac{C}{|\xi|},$$

where $C = 1/q_J$. Therefore

$$\max_{j \in J} \big|a_{j0} - \frac{\kappa_j}{\xi}\big| \geq \big|a_{\ell 0} - \frac{\kappa_\ell}{\xi}\big| \geq \frac{C}{|\xi|}, \quad (\kappa, \xi) \in \mathbb{Z}^m \times \mathcal{B}.$$

Note that if the denominators $\xi \in \mathcal{B}$ then $(a_{j_1 0}, \dots, a_{j_m 0})$ behaves as non-Liouville. Thus, the rest of the proof is analogous to the case where $(a_{j_1 0}, \dots, a_{j_m 0})$ is non-Liouville. □

## 4. Necessity part of Theorem 2.2

Assume first that $(a_{j_1 0}, \dots, a_{j_m 0}) \in \mathbb{Q}^m$ and the global primitive $B : \mathbb{T}^n \to \mathbb{R}$ of $b$ has a disconnected sublevel or superlevel on $\mathbb{T}^n$.

By Lemma 3.1 we have that $\mathbb{L}_{\mathcal{A}}$ is globally solvable if and only if $\mathbb{L}_{0,\mathcal{A}}$ is globally solvable, where $\mathcal{A} = q_J\mathbb{Z}$ and $\mathbb{L}_0 = d_t + ib(t) \wedge \frac{\partial}{\partial x}$. Since $B$ has a disconnected sublevel or superlevel, we have $\mathbb{L}_{0,\mathcal{A}}$ not globally solvable by [9]. Therefore $\mathbb{L}$ is not globally solvable.

Suppose now that $(a_{j_1 0}, \dots, a_{j_m 0}) \notin \mathbb{Q}^m$ is Liouville. Therefore, by work [8] the involutive system $\mathbb{L}_J$ generated by the vector fields

$$L_j = \frac{\partial}{\partial t_j} + a_{j0}\frac{\partial}{\partial x}, \quad j \in J = \{j_1, \dots, j_m\}, \tag{4.1}$$

is not globally solvable on $\mathbb{T}^{m+1}$.

As in the sufficiency part, we will consider the set $I$ such that $J \cup I = \{1, \dots, n\}$ and $J \cap I = \emptyset$.

Consider the space of compatibility conditions $\mathbb{E}_J$ associated to $\mathbb{L}_J$. Since (4.1) is not globally solvable on $\mathbb{T}^{m+1}$ there exists $g(t_J, x) = \sum_{j \in J} g_j(t_J, x) dt_j \in \mathbb{E}_J$ such that

$$\mathbb{L}_J v = g$$

has no solution $v \in \mathcal{D}'(\mathbb{T}^{m+1})$.

Now, we define smooth functions $f_1, \dots, f_n$ on $\mathbb{T}^{n+1}$ such that $f = \sum_{j=1}^n f_j dt_j \in \mathbb{E}$ and $\mathbb{L}u = f$ has no solution $u \in \mathcal{D}'(\mathbb{T}^{n+1})$.

Let $B$ be a primitive of the 1-form $b$. Thus, we have $\frac{\partial}{\partial t_j} B = b_j$. Since for each $j \in J$ the function $b_j \equiv 0$ then $B$ depends only on the variables $t_I$; that is, $B = B(t_I)$.

For $\ell \in I$ we choose $f_\ell \equiv 0$ and for $j \in J$ we define

$$f_j(t, x) := \sum_{\xi \in \mathbb{Z}} \hat{f}_j(t, \xi) e^{i\xi x},$$

where

$$\hat{f}_j(t,\xi) := \begin{cases} \hat{g}_j(t_J,\xi)e^{\xi(B(t_I)-M)} & \text{if } \xi \geq 0 \\ \hat{g}_j(t_J,\xi)e^{\xi(B(t_I)-\mu)} & \text{if } \xi < 0, \end{cases}$$

where $M$ and $\mu$ are, respectively, the maximum and minimum of $B$ over $\mathbb{T}^n$.

Given $\alpha \in \mathbb{Z}_+^n$, for each $j \in J$ we obtain

$$\partial^\alpha \hat{f}_j(t,\xi) = [\partial^{\alpha_J} g_j(t_J,\xi)]\xi^{|\alpha_I|}[\partial^{\alpha_I} B(t_I)]e^{\xi(B(t_I)-M)}, \quad \xi \geq 0,$$

and

$$\partial^\alpha \hat{f}_j(t,\xi) = [\partial^{\alpha_J} g_j(t_J,\xi)]\xi^{|\alpha_I|}[\partial^{\alpha_I} B(t_I)]e^{\xi(B(t_I)-\mu)}, \quad \xi < 0,$$

where $|\alpha_I| := \sum_{i\in I}\alpha_i$. Since the derivatives of $B$ are bounded on $\mathbb{T}^n$ then, there exists a constant $C_\alpha > 0$ such that $|\partial^{\alpha_I} B(t_I)| \leq C_\alpha$ for all $t_I \in \mathbb{T}^{n-m}_{t_I}$. Therefore,

$$|\partial^\alpha \hat{f}_j(t,\xi)| \leq C_\alpha|\xi|^{|\alpha_I|}|\partial^{\alpha_J} g_j(t_J,\xi)|, \quad \xi \in \mathbb{Z}.$$

Since $g_j$ are smooth functions it is possible to conclude by the above inequality that $f_j$, $j \in J$, are smooth functions. Moreover, it is easy to check that $f = \sum_{j=1}^n f_j dt_j \in \mathbb{E}$.

Suppose that there exists $u \in \mathcal{D}'(\mathbb{T}^{n+1})$ such that $\mathbb{L}u = f$. Then, if $u(t,x) = \sum_{\xi\in\mathbb{Z}} \hat{u}(t,\xi)e^{i\xi x}$, for each $\xi \in \mathbb{Z}$ we have

$$\frac{\partial}{\partial t_j}\hat{u}(t,\xi) + i\xi a_{j0}\hat{u}(t,\xi) = \hat{f}_j(t,\xi), \quad j \in J \tag{4.2}$$

and

$$\frac{\partial}{\partial t_\ell}\hat{u}(t,\xi) - \xi b_\ell(t)\hat{u}(t,\xi) = 0, \quad \ell \in I \tag{4.3}$$

Thus, for each $\ell \in I$ we may write (4.3) as follows

$$\begin{gathered} \frac{\partial}{\partial t_\ell}\big(\hat{u}(t,\xi)e^{-\xi(B(t_I)-M)}\big) = 0, \quad \text{if } \xi \geq 0, \\ \frac{\partial}{\partial t_\ell}\big(\hat{u}(t,\xi)e^{-\xi(B(t_I)-\mu)}\big) = 0, \quad \text{if } \xi < 0. \end{gathered}$$

Therefore,

$$\begin{gathered} \hat{u}(t,\xi)e^{-\xi(B(t_I)-M)} := \varphi_\xi(t_J), \quad \xi \geq 0, \\ \hat{u}(t,\xi)e^{-\xi(B(t_I)-\mu)} := \varphi_\xi(t_J), \quad \xi < 0. \end{gathered} \tag{4.4}$$

Let $t_I{}^*$ and $t_{I*}$ such that $B(t_I{}^*) = M$ and $B(t_{I*}) = \mu$. Thus, $\varphi_\xi(t_J) = \hat{u}(t_J, t_I{}^*, \xi)$ if $\xi \geq 0$ and $\varphi_\xi(t_J) = \hat{u}(t_J, t_{I*}, \xi)$ if $\xi < 0$ for all $t_J$. Since $u \in \mathcal{D}'(\mathbb{T}^{n+1})$ we have

$$v(t_J,x) := \sum_{\xi\in\mathbb{Z}} \varphi_\xi(t_J)e^{i\xi x} \in \mathcal{D}'(\mathbb{T}^{m+1}). \tag{4.5}$$

On the other hand, by (4.2) and (4.4) we have for each $j \in J$

$$\begin{gathered} \frac{\partial}{\partial t_j}(\varphi_\xi(t_J)e^{\xi(B(t_I)-M)}) + i\xi a_{j0}(\varphi_\xi(t_J)e^{\xi(B(t_I)-M)}) = \hat{f}_j(t,\xi), \quad \xi \geq 0, \\ \frac{\partial}{\partial t_j}(\varphi_\xi(t_J)e^{\xi(B(t_I)-\mu)}) + i\xi a_{j0}(\varphi_\xi(t_J)e^{\xi(B(t_I)-\mu)}) = \hat{f}_j(t,\xi), \quad \xi < 0, \end{gathered}$$

thus

$$\frac{\partial}{\partial t_j}\varphi_\xi(t_J) + i\xi a_{j0}\varphi_\xi(t_J) = \hat{g}_j(t_J,\xi), \quad \xi \in \mathbb{Z},\ j \in J.$$

We conclude that the $v$ given by (4.5) is a solution of $\mathbb{L}_J v = g$, which is a contradiction.

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