Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 241, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF PIECEWISE CONTINUOUS MILD SOLUTIONS FOR IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH ITERATED DEVIATING ARGUMENTS

PRADEEP KUMAR, DWIJENDRA N. PANDEY, DHIRENDRA BAHUGUNA

Abstract. The objective of this article is to prove the existence of piecewise continuous mild solutions to impulsive functional differential equation with iterated deviating arguments in a Banach space. The results are obtained by using the theory of analytic semigroups and fixed point theorems.

1. Introduction

In the theory of differential equations with deviating arguments, we study differential equations involving variables (arguments) as well as unknown functions and its derivative; generally speaking, under different values of the variables (arguments). It is very important and significant branch of nonlinear analysis with numerous applications to physics, mechanics, control theory, biology, ecology, economics, theory of nuclear reactors, engineering, natural sciences, and many other areas of science and technology. A comprehensive coverage can be found in the book by El'sgol'ts and Norkin [7]. For more details on recent works in this direction, we refer to [8, 9, 10, 11, 12] and the references cited therein.

Impulsive effects are common phenomena due to short-term perturbations whose duration is negligible in comparison with the total duration of the original process. The governing equations of such phenomena may be modelled as impulsive differential equations. In recent years, there has been a growing interest in the study of impulsive differential equations as these equations approach the simulation processes in the control theory, physics, chemistry, population dynamics, biotechnology, economics and so on. The investigation of existence and uniqueness of mild solutions for impulsive differential equations have been discussed by many authors (see [4, 6, 19, 5, 3, 23, 17, 16, 24, 26, 13, 14, 25, 1, 2] and references cited therein).

However, due to theoretical and practical difficulties, the study of impulsive differential equations with deviating arguments has been developed rather slowly. Recently, the study of impulsive differential equations with deviating arguments has studied by some authors. Guobing et al. [26] established the existence solution of periodic boundary value problems for a class of impulsive neutral differential

2000 *Mathematics Subject Classification.* 34K45, 34A60, 35R12, 45J05.
Key words and phrases. Impulsive functional differential equation; deviating argument; analytic semigroup; fixed point theorem.

Submitted May 20, 2013. Published October 31, 2013.

equations with multi-deviation arguments. Jankowski [14] discussed the existence of solutions for second-order impulsive differential equations with deviating arguments (see also [16, 24, 13, 25, 1, 2] and the references cited therein).

Liu [15] studied the existence of mild solutions of the following impulsive evolution equation by using semigroup theory

$$\begin{gathered} u'(t) = Au(t) + f(t, u(t)), \quad 0 < t < T_0,\ t \neq t_i, \\ \Delta u(t_i) = I_i(u(t_i)) = u(t_i^+) - u(t_i^-), \quad i = 1, 2, \dots,\ 0 < t_1 < t_2 < \dots < T_0, \\ u(0) = u_0, \end{gathered} \tag{1.1}$$

where A is the infinitesimal generator of a C_0-semigroup, $u(t_i^+), u(t_i^-)$ represent the right and left hand limits respectively at $t = t_i$, I_i's are some operators and f is a suitable function.

Chang et al. [4] investigated the existence of mild solutions for the following impulsive problem

$$\begin{aligned} \frac{d}{dt}(u(t) - F(t, u(h_1(t)))) \in A\Big[u(t) + \int_0^t f(t-s)u(s)ds\Big] \\ + G(t, u(h_2(t)))), \quad t \in [0, a],\ t \neq t_k, \end{aligned} \tag{1.2}$$

$$\Delta u|_{t=t_k} = I_k(u(t_k^-)), \quad k = 1, \dots, m, \tag{1.3}$$

$$u(0) + g(u) = u_0, \tag{1.4}$$

where A is the infinitesimal generator of a compact analytic resolvent operator $R(t)$, $t > 0$ in a Banach space H, $u_0 \in H$, $0 = t_0 < t_1 < \dots t_m < t_{m+1} = a$, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $u(t_k^+), u(t_k^-)$ represent the right and left hand limits respectively at $t = t_k$. $h_i : [0, a] \to [0, a]$, $i = 1, 2$, $f(t)$, $t \in [0, a]$ is a bounded linear operator, and F, G, g are suitable functions.

With strong motivation from the above work, in this article, we consider the following impulsive functional differential equations with iterated deviating arguments in a Banach space $(H, \|\cdot\|)$:

$$\begin{gathered} \frac{d}{dt}u(t) + Au(t) = f(t, u(t), u[w_1(t, u(t))]), \quad t \in I = [0, T_0],\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)),\ k = 1, 2, \dots, m, \\ u(0) = u_0, \end{gathered} \tag{1.5}$$

where

$$w_1(t, u(t)) = h_1\big(t, u(h_2(t, \dots, u(h_m(t, u(t))) \dots))\big)$$

and $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators, $\{S(t), t \geq 0\}$ on H. Functions f and h_i are suitably defined and satisfying certain conditions to be stated later. $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T_0$, $I_k \in C(H, H)(k = 1, 2, \dots, m)$, are bounded functions and $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $u(t_k^-)$ and $u(t_k^+)$ represent the left and right limits of $u(t)$ at $t = t_k$, respectively.

The paper is organized as follows. In Section 2, we provide some basic definitions, notation, lemmas and proposition which are used throughout the paper. In Sections 3 and 4, we prove the existence and uniqueness results concerning the piecewise continuous mild solutions. In the last section, we give an example to demonstrate the application of the main results.

2. Preliminaries and assumptions

In this section, we will introduce some basic definitions, notation, lemmas and proposition which are used throughout this paper.

It is assumed that $-A$ generates an analytic semigroup of bounded operators, denoted by $S(t)$, $t \geq 0$. It is known that there exist constants $\tilde{M} \geq 1$ and $\omega \geq 0$ such that

$$\|S(t)\| \leq \tilde{M} e^{\omega t}, \quad t \geq 0.$$

If necessary, we may assume without loss of generality that $\|S(t)\|$ is uniformly bounded by M; i.e., $\|S(t)\| \leq M$ for $t \geq 0$, and that $0 \in \rho(-A)$; i.e., $-A$ is invertible. In this case, it is possible to define the fractional power A^α for $0 \leq \alpha \leq 1$ as closed linear operator with domain $D(A^\alpha) \subseteq H$. Furthermore, $D(A^\alpha)$ is dense in H and the expression

$$\|x\|_\alpha = \|A^\alpha x\|,$$

defines a norm on $D(A^\alpha)$. Henceforth, we denote the space $D(A^\alpha)$ by H_α endowed with the norm $\|\cdot\|_\alpha$. Also, for each $\alpha > 0$, we define $H_{-\alpha} = (H_\alpha)^*$, the dual space of H_α, is a $\|x\|_{-\alpha} = \|A^{-\alpha} x\|$. For more details, we refer to the book by Pazy [18].

Lemma 2.1 ([18, pp. 72,74,195-196]). *Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, $t \geq 0$ with $\|S(t)\| \leq M$ for $t \geq 0$ and $0 \in \rho(-A)$. Then we have the following:*

- (i) *H_α is a Banach space for $0 \leq \alpha \leq 1$;*
- (ii) *For any $0 < \delta \leq \alpha$ implies $D(A^\alpha) \subset D(A^\delta)$, the embedding $H_\alpha \hookrightarrow H_\delta$ is continuous;*
- (iii) *The operator $A^\alpha S(t)$ is bounded for every $t > 0$ and*
$$\|A^\alpha S(t)\| \leq C_\alpha t^{-\alpha}.$$
- (iv) *For $\alpha \leq 0$, A^α is bounded.*

We define the space

$$X = \mathcal{PC}(H_\alpha) = \Big\{ u : [0, T_0] \to H_\alpha : u \in C((t_k, t_{k+1}], H_\alpha), k = 0, 1, \dots, m, \\ u(t_k^-), u(t_k^+) \text{ exist and } u(t_k^-) = u(t_k) \Big\},$$

is a Banach space endowed with the supremum norm

$$\|u\|_{\mathcal{PC}} := \sup_{t \in I} \|u(t)\|_\alpha.$$

We shall use the following conditions on f and h_i in its arguments:

- (H1) Let $W \subset \mathrm{Dom}(f)$ be an open subset of $\mathbb{R}_+ \times H_\alpha \times H_{\alpha-1}$, where $0 \leq \alpha < 1$. For each $(t, u, v) \in W$, there is a neighborhood $V_1 \subset W$ of (t, u, v), such that the nonlinear map $f : \mathbb{R}_+ \times H_\alpha \times H_{\alpha-1} \to H$ satisfies the condition
$$\|f(t, u, v) - f(s, u_1, v_1)\| \leq L_f \{ |t - s|^{\theta_1} + \|u - u_1\|_\alpha + \|v - v_1\|_{\alpha-1} \}$$
for all $(t, u, v), (s, u_1, v_1) \in V_1$, where $L_f = L_f(t, u, v, V_1) > 0$ and $0 < \theta_1 \leq 1$ are constants.
- (H2) Let $U_{h_i} \subset \mathrm{Dom}(h_i)$ be open subsets of $\mathbb{R}_+ \times H_{\alpha-1}$, where $0 \leq \alpha < 1$. For each $(t, u) \in U_{h_i}$, there is a neighborhood $V_{h_i} \subset U_{h_i}$ of (t, u), such that

$h_i(0,\cdot) = 0$ for each $i = 1, 2, \dots, m$, $h_i : \mathbb{R}_+ \times H_{\alpha-1} \to \mathbb{R}_+$ satisfies the condition

$$|h_i(t,u) - h_i(s,v)| \le L_{h_i}\{\|u-v\|_{\alpha-1} + |t-s|^{\theta_2}\} \tag{2.1}$$

for all $(t,u),(s,v) \in V_{h_i}$, $L_{h_i} = L_{h_i}(t,u,V_{h_i}) > 0$ and $0 < \theta_2 \le 1$ are constants.

(H3) The functions $I_k : H_\alpha \to H_\alpha$ are continuous and there exists Υ_k such that $\|I_k(u)\|_\alpha \le \Upsilon_k$, $k = 1, 2, \dots, m$.

(H4) There exist continuous nondecreasing $d_k : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|I_k(u) - I_k(v)\|_\alpha \le d_k\|u-v\|_\alpha, k = 1, 2, \dots, m$.

New concept of solutions. Here, we prove a new concept of solutions [23, 15] for the problem

$$\begin{gathered} u'(t) + Au(t) = r(t), \quad t \in [0,T_0],\ t \ne t_k, \\ u(0) = u_0 \\ u(t_k) = I_k(u(t_k^-)), \quad k = 1, 2 \dots, m, \end{gathered} \tag{2.2}$$

where $r \in H$. Let

$$\begin{gathered} v'(t) + Av(t) = r(t), \quad t \in [0,T_0], \\ v(0) = v_0, \end{gathered} \tag{2.3}$$

and

$$\begin{gathered} w'(t) + Aw(t) = 0, \quad t \in [0,T_0],\ t \ne t_k, \\ w(0) = 0, \\ w(t_k) = I_k(u(t_k^-)), \quad k = 1, 2, \dots, m, \end{gathered} \tag{2.4}$$

be the decomposition of $u(.) = v(.) + w(.)$, where v is the continuous mild solution of (2.3) and w is the piecewise continuous mild solution of (2.4).

By a mild solution for (2.3), we mean a continuous function $v : [0,T_0] \to H$ satisfying the integral equation

$$v(t) = S(t)v_0 + \int_0^t S(t-s)r(s)ds, \quad t \in [0,T_0]. \tag{2.5}$$

and by a piecewise continuous mild solution for (2.4), we mean a function $w \in \mathcal{PC}([0,T_0], D(A))$ satisfying the integral equation equivalent to the system (2.4) ([23, see Eq. (3.4)])

$$w(t) = \begin{cases} -\int_0^t Aw(s)ds, & t \in [0,t_1], \\ I_1(u(t_1^-)) - \int_0^t Aw(s)ds, & t \in (t_1,t_2], \\ \sum_{i=1}^k I_i(u(t_i^-)) - \int_0^t Aw(s)ds, & t \in (t_k,t_{k+1}], \\ k = 1, 2, \dots, m. \end{cases} \tag{2.6}$$

The above equation can be expressed as

$$w(t) = \sum_{i=1}^k \chi_i(t) I_i(w(t_i^-)) - \int_0^t Aw(s)ds, \tag{2.7}$$

for $t \in [0,T_0]$, where

$$\chi_i(t) = \begin{cases} 0, & \text{for } t \in [0,t_1], \\ 1, & \text{for } t \in [t_k,t_{k+1}]. \end{cases} \tag{2.8}$$

Taking Laplace transform of (2.7), we obtain

$$w(p) = \sum_{i=1}^{k} \frac{e^{-t_i p}}{p} I_i - \frac{Aw(p)}{p},$$

this gives

$$w(p) = \sum_{i=1}^{k} e^{-t_i p}(pI + A)^{-1} I_i. \tag{2.9}$$

We also note that $(pI + A)^{-1} = \int_0^\infty e^{-pt} S(t)dt$. Thus we can obtain the mild solution for (2.4),

$$w(t) = \sum_{i=1}^{k} \chi_i(t) S(t - t_i) I_i(w(t_i^-)). \tag{2.10}$$

Hence, the mild solution for the problem (2.2) is

$$u(t) = S(t)u_0 + \sum_{i=1}^{k} \chi_i(t) S(t - t_i) I_i(u(t_i^-)) + \int_0^t S(t - s) r(s) ds. \tag{2.11}$$

We can rewrite equation (2.11) as

$$u(t) = \begin{cases} S(t)u_0 + \int_0^t S(t-s)r(s)ds, & t \in [0, t_1], \\ S(t)u_0 + S(t - t_1) I_1(u(t_1^-)) + \int_0^t S(t-s)r(s)ds, & t \in (t_1, t_2], \\ S(t)u_0 + \sum_{i=1}^{k} S(t - t_i) I_i(u(t_i^-)) + \int_0^t S(t-s)r(s)ds, & t \in (t_k, t_{k+1}], \\ k = 1, 2, \dots, m. \end{cases} \tag{2.12}$$

3. Local existence of mild solutions

In this section, we prove the existence and uniqueness results concerning piecewise continuous-mild solutions for system (1.5). For $0 \le \alpha < 1$, we define

$$X_1 = \{u \in X : \|u(t) - u(s)\|_{\alpha-1} \le L|t - s|, \forall t, s \in (t_k, t_{k+1}], k = 0, 1, \dots, m\},$$

which is needful for proving contraction principle (See [8, 9]). Where L is a suitable positive constant to be specified later.

Definition 3.1. A function $u : [0, T_0] \to H$ solution of the problem (1.5),

$$u(t) = \begin{cases} S(t)u_0 + \int_0^t S(t-s) f(s, u(s), u(w_1(s, u(s))))ds, & t \in [0, t_1], \\ S(t)u_0 + S(t - t_1) I_1(u(t_1^-)) \\ + \int_0^t S(t-s) f(s, u(s), u(w_1(s, u(s))))ds, & t \in (t_1, t_2], \\ \dots \\ S(t)u_0 + \sum_{i=1}^{k} S(t - t_i) I_i(u(t_i^-)) \\ + \int_0^t S(t-s) f(s, u(s), u(w_1(s, u(s))))ds, & t \in (t_k, t_{k+1}], \\ k = 1, 2, \dots, m. \end{cases} \tag{3.1}$$

is said to be a piecewise continuous-mild solution.

For a fixed $R > 0$, we define

$$\mathcal{W} = \{u \in X \cap X_1 : u(0) = u_0, \|u - u_0\|_{\mathcal{PC}} \le R\}.$$

Clearly, $\mathcal{W}$ is a closed and bounded subset of X_1 and is a Banach space. We define a map $\mathcal{G} : \mathcal{W} \to \mathcal{W}$ by

$$(\mathcal{G}u)(t) = \begin{cases} S(t)u_0 + \int_0^t S(t-s)f(s,u(s),u(w_1(s,u(s))))ds, & t \in [0,t_1], \\ S(t)u_0 + S(t-t_1)I_1(u(t_1^-)) & \\ + \int_0^t S(t-s)f(s,u(s),u(w_1(s,u(s))))ds, & t \in (t_1,t_2], \\ \dots & \\ S(t)u_0 + \sum_{i=1}^k S(t-t_i)I_i(u(t_i^-)) & \\ + \int_0^t S(t-s)f(s,u(s),u(w_1(s,u(s))))ds, & t \in (t_k,t_{k+1}], \\ k = 1,2,\dots,m. & \end{cases} \tag{3.2}$$

Theorem 3.2. *Let $0 \le \alpha < 1$, $u_0 \in H_\alpha$ and the assumptions (H1)–(H4) hold. Then problem* (1.5) *has a mild solution provided that*

$$C_\alpha N \frac{T_0^{1-\alpha}}{1-\alpha} + M \sum_{i=1}^k \Upsilon_i \le \frac{R}{2}, \tag{3.3}$$

$$C_\alpha L_f(2 + L\ L_h)\frac{T_0^{1-\alpha}}{1-\alpha} + M\sum_{i=1}^m d_i < 1, \tag{3.4}$$

where N and L_h are positive constants to be specified later.

Proof. Clearly, $\mathcal{G} : X \to X$. We begin with showing that $\mathcal{G}u \in X_1$ for each $u \in X_1$. Let $u \in X_1$, then for each $\tau_1, \tau_2 \in [0,t_1]$, $\tau_1 < \tau_2$ and $0 \le \alpha < 1$, we have

$$\begin{aligned} &\|(\mathcal{G}u)(\tau_2) - (\mathcal{G}u)(\tau_1)\|_{\alpha-1} \\ &\le \|[S(\tau_2) - S(\tau_1)]u_0\|_{\alpha-1} \\ &\quad + \int_0^{\tau_1} \|A^{\alpha-1}[S(\tau_2-s) - S(\tau_1-s)]\|\,\|f(s,u(s),u(w_1(s,u(s))))\|ds \\ &\quad + \int_{\tau_1}^{\tau_2} \|A^{\alpha-1}S(\tau_2-s)\|\,\|f(s,u(s),u(w_1(s,u(s))))\|ds. \end{aligned} \tag{3.5}$$

Since $f(t,u(t),u(w_1(u(t),t)))$ is continuous, by assumptions (H1), (H2), there exists a constant N, such that

$$\|f(t,u(t),u(w_1(t,u(t))))\| \le N, \quad u \in X,\ t \in [0,T_0]. \tag{3.6}$$

For the first term on the right-hand side of (3.5), we have

$$\begin{aligned} \|A^{\alpha-1}[S(\tau_2) - S(\tau_1)]u_0\| &\le \int_{\tau_1}^{\tau_2} \|A^{\alpha-1}S'(s)u_0\|ds \\ &= \int_{\tau_1}^{\tau_2} \|A^\alpha S(s)u_0\|ds \\ &\le M\|u_0\|_\alpha(\tau_2 - \tau_1). \end{aligned} \tag{3.7}$$

For the second term on the right-hand side of (3.5), we have

$$\begin{aligned} \|(S(\tau_2-s) - S(\tau_1-s))\|_{\alpha-1} &\le \int_0^{\tau_2-\tau_1} \|A^{\alpha-1}S'(l)S(\tau_1-s)\|dl \\ &= \int_0^{\tau_2-\tau_1} \|S(l)A^\alpha S(\tau_1-s)\|dl \\ &\le MC_\alpha(\tau_2-\tau_1)(\tau_1-s)^{-\alpha}. \end{aligned} \tag{3.8}$$

Then using (3.8), we obtain the following bound for the second term on the right-hand side of (3.5),

$$\begin{aligned}&\int_0^{\tau_1} \|(S(\tau_2-s)-S(\tau_1-s))A^{\alpha-1}\|\,\|f(s,u(s),u(w_1(s,u(s))))\|ds\\&\le NMC_\alpha\frac{T_0^{1-\alpha}}{1-\alpha}(\tau_2-\tau_1).\end{aligned}\tag{3.9}$$

The third term on the right-hand side of (3.5) is estimated as

$$\int_{\tau_1}^{\tau_2} \|S(\tau_2-s)A^{\alpha-1}\|\|f(s,u(s),u(w_1(s,u(s))))\|ds \le \|A^{\alpha-1}\|MN(\tau_2-\tau_1).\tag{3.10}$$

Thus from inequalities (3.7), (3.9) and (3.10), we see that

$$\|(\mathcal{G}u)(\tau_2)-(\mathcal{G}u)(\tau_1)\|_{\alpha-1} \le M\{\|u_0\|_\alpha + N\ C_\alpha\frac{T_0^{1-\alpha}}{1-\alpha}+N\|A^{\alpha-1}\|\}(\tau_2-\tau_1).\tag{3.11}$$

For $\tau_1,\tau_2\in(t_1,t_2]$, $\tau_1<\tau_2$ and $0\le\alpha<1$, we have

$$\begin{aligned}&\|(\mathcal{G}u)(\tau_2)-(\mathcal{G}u)(\tau_1)\|_{\alpha-1}\\&\le \|[S(\tau_2-S(\tau_1)]u_0\|_{\alpha-1}+\|A^{\alpha-1}[S(\tau_2-t_1)-S(\tau_1-t_1)]I_1(u(t_1^-))\|\\&\quad+\int_0^{\tau_1}\|A^{\alpha-1}[S(\tau_2-s)-S(\tau_1-s)]\|\,\|f(s,u(s),u(w_1(s,u(s))))\|ds\\&\quad+\int_{\tau_1}^{\tau_2}\|A^{\alpha-1}S(\tau_2-s)\|\,\|f(s,u(s),u(w_1(s,u(s))))\|ds.\end{aligned}\tag{3.12}$$

The second term on the right-hand side of (3.12) is estimated as

$$\begin{aligned}&\|A^{\alpha-1}[S(\tau_2-t_1)-S(\tau_1-t_1)]I_1(u(t_1^-))\|\\&\le\int_{\tau_1}^{\tau_2}\|A^{\alpha-1}S'(l-t_1)I_1(u(t_1^-))\|dl\\&=\int_{\tau_1}^{\tau_2}\|A^{\alpha}S(l-t_1)I_1(u(t_1^-))\|dl\\&\le M\|I_1(u(t_1^-))\|_\alpha(\tau_2-\tau_1).\end{aligned}\tag{3.13}$$

Thus, from inequalities (3.7), (3.9), (3.10) and (3.13), we obtain

$$\begin{aligned}&\|(\mathcal{G}u)(\tau_2)-(\mathcal{G}u)(\tau_1)\|_{\alpha-1}\\&\le M\Big\{\|u_0\|_\alpha+\Upsilon_1+NC_\alpha\frac{T_0^{1-\alpha}}{1-\alpha}+N\|A^{\alpha-1}\|\Big\}(\tau_2-\tau_1).\end{aligned}\tag{3.14}$$

Similarly, for $\tau_1,\tau_2\in(t_k,t_{k+1}]$, $\tau_1<\tau_2$, $k=1,2,\dots,m$ and $0\le\alpha<1$, we have

$$\begin{aligned}&\|(\mathcal{G}u)(\tau_2)-(\mathcal{G}u)(\tau_1)\|_{\alpha-1}\\&\le M\Big\{\|u_0\|_\alpha+\sum_{i=1}^k\Upsilon_i+NC_\alpha\frac{T_0^{1-\alpha}}{1-\alpha}+N\|A^{\alpha-1}\|\Big\}(\tau_2-\tau_1).\end{aligned}\tag{3.15}$$

Thus, for each $\tau_1,\tau_2\in[0,T_0]$, $\tau_1<\tau_2$ and $0\le\alpha<1$, we have

$$\|(\mathcal{G}u)(\tau_2)-(\mathcal{G}u)(\tau_1)\|_{\alpha-1}\le L(\tau_2-\tau_1),\tag{3.16}$$

where

$$L=\max\{M\|u_0\|_\alpha, M\ \sum_{i=1}^m\Upsilon_i, NMC_\alpha\frac{T_0^{1-\alpha}}{1-\alpha}, NM\|A^{1-\alpha}\|\}.$$

Therefore, $\mathcal{G}$ is piecewise Lipschitz continuous on $[0,T_0]$ and so $\mathcal{G} : X_1 \to X_1$.

Next we will show that $\mathcal{G} : \mathcal{W} \to \mathcal{W}$. Let $u \in X \cap X_1$, then for each $t \in [0,t_1]$,

$$\begin{aligned}\|(\mathcal{G}u)(t) - u_0\|_\alpha &\le \|(S(t)-I)A^\alpha u_0\| \\ &\quad + \int_0^t \|S(t-s)A^\alpha\|\|f(s,u(s),u(w_1(s,u(s))))\|ds \\ &\le \|(S(t)-I)A^\alpha u_0\| + C_\alpha N \frac{T_0^{1-\alpha}}{1-\alpha}.\end{aligned}$$

Similarly, let $u \in X \cap X_1$, then for each $t \in (t_k, t_{k+1}], k = 1, \dots, m$,

$$\begin{aligned}&\|(\mathcal{G}u)(t) - u_0\|_\alpha \\ &\le \|(S(t)-I)A^\alpha u_0\| + \int_0^t \|S(t-s)A^\alpha\|\|f(s,u(s),u(w_1(s,u(s))))\|ds \\ &\quad + \sum_{i=1}^k \|A^\alpha S(t-t_i) I_i(u(t_i^-))\| \\ &\le \|(S(t)-I)A^\alpha u_0\| + C_\alpha N \frac{T_0^{1-\alpha}}{1-\alpha} + M\sum_{i=1}^k \Upsilon_i.\end{aligned}$$

Part (iii) of Lemma 2.1, implies that

$$\|(S(t)-I)A^\alpha(u_0)\| \le \frac{R}{2}. \tag{3.17}$$

Thus, from (3.3) and (3.17), it is clear that

$$\|\mathcal{G}u - u_0\|_{\mathcal{PC}} \le R.$$

Therefore, $\mathcal{G} : \mathcal{W} \to \mathcal{W}$.

Finally, we will claim that $\mathcal{G}$ is a contraction map. If $t \in [0,t_1]$ and $u, v \in \mathcal{W}$, then

$$\begin{aligned}&\|(\mathcal{G}u)(t) - (\mathcal{G}v)(t)\|_\alpha \\ &\le \int_0^t \|S(t-s)A^\alpha\| \; \|f(s,u(s),u(w_1(s,u(s)))) - f(s,v(s),v(w(s,v(s))))\|ds.\end{aligned} \tag{3.18}$$

Also, we note that

$$\begin{aligned}&\|f(t,u(t),u(w_1(t,u(t)))) - f(t,v(t),v(w_1(t,v(t))))\| \\ &\le L_f\{\|u(t)-v(t)\|_\alpha + \|u(w_1(t,u(t))) - v(w_1(t,v(t)))\|_{\alpha-1}\} \\ &\le L_f\Big[\|u(t)-v(t)\|_\alpha + \|A^{-1}\| \; \|u(w_1(t,v(t))) - v(w_1(t,v(t)))\|_\alpha \\ &\quad + \|u(w_1(t,u(t))) - u(w_1(t,v(t)))\|_{\alpha-1}\Big] \\ &\le L_f\big\{2\|u-v\|_{\mathcal{PC}} + L|w_1(t,u(t)) - w_1(t,v(t))|\big\}.\end{aligned} \tag{3.19}$$

Now, let

$$w_i(t,u(t)) = h_i(t, u(h_{i+1}(t, \dots u(t, h_m(t,u(t))) \dots))), \; i = 1, \dots, m,$$

with $w_{m+1}(t,u(t)) = t$. For more details, we refer to [21, p. 2183].

Hence, we have

$$|w_1(t,u(t)) - w_1(t,v(t))| = |h_1(t,u(w_2(t,u(t)))) - h_1(t,v(w_2(t,v(t))))$$

$$
\begin{aligned}
&\le L_{h_1}\|u(w_2(t,u(t))) - u(w_2(t,u(t)))\|_{\alpha-1}\\
&\le L_{h_1}\Big\{\|u(w_2(t,u(t))) - u(w_2(t,v(t)))\|_{\alpha-1}\\
&\quad + \|u(w_2(t,v(t))) - v(w_2(t,v(t)))\|_{\alpha-1}\Big\}\\
&\le L_{h_1}\Big\{L|h_2(t,u(w_3(t,u(t))) - h_2(t,v(w_3(t,v(t)))|\\
&\quad + \|A\|^{-1}\|u-v\|_{\mathcal{PC}}\Big\}\\
&\dots\\
&\le (L^{m-1}L_{h_1}\dots L_{h_m} + L^{m-2}L_{h_1}\dots L_{h_{m-1}} + \dots\\
&\quad + LL_{h_1}L_{h_2} + L_{h_1})\|A\|^{-1}\|u-v\|_{\mathcal{PC}}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\|f(t,u(t),u(w_1(t,u(t)))) - f(t,v(t),v(w_1(t,v(t))))\|\\
&\le L_f\Big(2 + LL_h\|A^{-1}\|\Big)\|u-v\|_{\mathcal{PC}}\\
&\le L_f(2+LL_h)\|u-v\|_{\mathcal{PC}}.
\end{aligned} \tag{3.20}
$$

where $L_h = (L^{m-1}L_{h_1}\dots L_{h_m} + L^{m-1}L_{h_1}\dots L_{h_{m-1}} + \dots + L_{h_1}) > 0$. We use (3.20) in (3.18), we obtain

$$
\|(\mathcal{G}u)(t) - (\mathcal{G}v)(t)\|_\alpha \le C_\alpha L_f(2+LL_h)\frac{T_0^{1-\alpha}}{1-\alpha}\|u-v\|_{\mathcal{PC}}. \tag{3.21}
$$

For $t \in (t_1, t_2]$, we have

$$
\|(\mathcal{G}u)(t) - (\mathcal{G}v)(t)\|_\alpha \le \Big[C_\alpha L_f(2+LL_h)\frac{T_0^{1-\alpha}}{1-\alpha} + Md_1\Big]\|u-v\|_{\mathcal{PC}}. \tag{3.22}
$$

For $t \in (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$, we have

$$
\|(\mathcal{G}u)(t) - (\mathcal{G}v)(t)\|_\alpha \le \Big[C_\alpha L_f(2+LL_h)\frac{T_0^{1-\alpha}}{1-\alpha} + M\sum_{i=1}^{k} d_i\Big]\|u-v\|_{\mathcal{PC}}. \tag{3.23}
$$

Thus, for each $t \in [0, T_0]$, we have

$$
\|(\mathcal{G}u)(t) - (\mathcal{G}v)(t)\|_\alpha \le \Big[C_\alpha L_f(2+LL_h)\frac{T_0^{1-\alpha}}{1-\alpha} + M\sum_{i=1}^{m} d_i\Big]\|u-v\|_{\mathcal{PC}}.
$$

Therefore, the map $\mathcal{G}$ is a contraction map, hence $\mathcal{G}$ has a unique fixed point $u \in \mathcal{W}$. That is, problem (1.5) has a unique mild solution. □

4. Further existence results

Theorem 3.2 can be proved if we omit the hypothesis (H1). In that case the proof is based on the idea of Wang et al. [23].

Theorem 4.1. *Assume the conditions (H3)-(H4) hold, the semigroup $\{S(t), t \ge 0\}$ is compact, and $f : I \times H \times H \to H$ is continuous. For $u_0 \in H_\alpha$ there exists a constant $r > 0$ such that*

$$
M\{\|u_0\|_\alpha + \sum_{i=1}^{m}\Upsilon_i\} + C_\alpha N_f\frac{T_0^{1-\alpha}}{1-\alpha} \le r, \tag{4.1}
$$

where

$$N_f = \sup_{s\in[0,T_0],u\in\Omega} \|f(s,u(s),u(w_1(s,u(s))))\|,$$

and $u \in \Omega = \{v \in \mathcal{PC}(H_\alpha) : \|v\|_{\mathcal{PC}} \leq r\}$. *Then there exists a mild solution* $u \in \mathcal{PC}(H_\alpha)$ *of problem* (1.5).

Proof. Let us define a map $\Upsilon : \mathcal{PC}(H_\alpha) \to \mathcal{PC}(H_\alpha)$, by

$$(\Upsilon u)(t) = \begin{cases} S(t)u_0 + \int_0^t S(t-s)f(s,u(s),u(w_1(s,u(s))))ds, & t\in[0,t_1],\\ S(t)u_0 + S(t-t_1)I_1(u(t_1^-)) & \\ +\int_0^t S(t-s)f(s,u(s),u(w_1(s,u(s))))ds, & t\in(t_1,t_2],\\ \dots & \\ S(t)u_0 + \sum_{i=1}^k S(t-t_i)I_i(u(t_i^-)) & \\ +\int_0^t S(t-s)f(s,u(s),u(w_1(s,u(s))))ds, & t\in(t_k,t_{k+1}],\\ k=1,2,\dots,m. & \end{cases}$$

Step 1. First we show that Υ is continuous. It follows from the continuity of f that

$$\|f(s,u_n(s),u_n(w_1(s,u_n(s)))) - f(s,u(s),u(w_1(s,u(s))))\| \leq \epsilon, \quad \text{as } n\to\infty,$$

for $s\in[0,t]$, $t\in[0,T_0]$.

Now, for each $t\in[0,t_1]$, we have

$$\|(\Upsilon u_n)(t) - (\Upsilon u)(t)\|_\alpha \leq C_\alpha \frac{T_0^{1-\alpha}}{1-\alpha}\epsilon \to 0, \quad \text{as } n\to\infty. \tag{4.2}$$

For, $t\in(t_1,t_2]$, we have

$$\begin{aligned} &\|(\Upsilon u_n)(t) - (\Upsilon u)(t)\|_\alpha \\ &\leq M\|I_1(u_n(t_1^-)) - I_1(u(t_1^-))\|_\alpha + C_\alpha \frac{T_0^{1-\alpha}}{1-\alpha}\epsilon \to 0, \quad \text{as } n\to\infty. \end{aligned} \tag{4.3}$$

Similarly, for each $t\in(t_k,t_{k+1}]$, $k=1,2,\dots,m$,

$$\begin{aligned} &\|(\Upsilon u_n)(t) - (\Upsilon u)(t)\|_\alpha \\ &\leq M\sum_{i=1}^k \|I_i(u_n(t_i^-)) - I_i(u(t_i^-))\|_\alpha + C_\alpha \frac{T_0^{1-\alpha}}{1-\alpha}\epsilon \to 0, \quad \text{as } n\to\infty. \end{aligned} \tag{4.4}$$

We have that Υ is continuous.

Step 2. Next we show that Υ maps bounded sets into bounded sets in $\mathcal{PC}(H_\alpha)$. Let $u\in\Omega$, then for $t\in[0,t_1]$, we have

$$\|(\Upsilon u)(t)\|_\alpha \leq M\|u_0\|_\alpha + N_f C_\alpha \frac{T_0^{1-\alpha}}{1-\alpha}, \tag{4.5}$$

For, $t\in(t_1,t_2]$, we have

$$\|(\Upsilon u)(t)\|_\alpha \leq M\big\{\|u_0\|_\alpha + \Upsilon_1\big\} + N_f C_\alpha \frac{T_0^{1-\alpha}}{1-\alpha}. \tag{4.6}$$

Similarly, for each $t\in(t_k,t_{k+1}]$, $k=1,2,\dots,m$, we have

$$\|(\Upsilon u)(t)\|_\alpha \leq M\Big\{\|u_0\|_\alpha + \sum_{i=1}^k \Upsilon_i\Big\} + N_f C_\alpha \frac{T_0^{1-\alpha}}{1-\alpha}. \tag{4.7}$$

Thus, from inequality (4.1), we see that $\Upsilon : \Omega \to \Omega$.

Step 3. In this step, we show that Υ maps bounded sets into equicontinuous sets in $\mathcal{PC}(H_\alpha)$. Let $\tau_1, \tau_2 \in [0, t_1], \tau_1 < \tau_2$, we have

$$\begin{aligned} &\|(\Upsilon u)(\tau_2) - (\Upsilon u)(\tau_1)\|_\alpha \\ &\le M\Big\{\|u_0\|_\alpha + N_f C_\alpha \frac{T_0^{1-\alpha}}{1-\alpha} + \|A^{\alpha-1}\| N_f\Big\}(\tau_2 - \tau_1). \end{aligned} \tag{4.8}$$

Similarly, for each $\tau_1, \tau_2 \in (t_k, t_{k+1}]$, $\tau_1 < \tau_2$, $k = 1, 2, \dots, m$, we have

$$\begin{aligned} &\|(\Upsilon u)(\tau_2) - (\Upsilon u)(\tau_1)\|_\alpha \\ &\le M\Big\{\|u_0\|_\alpha + \sum_{i=1}^{k} \Upsilon_i + N_f C_\alpha \frac{T_0^{1-\alpha}}{1-\alpha} + \|A^{\alpha-1}\| N_f\Big\}(\tau_2 - \tau_1). \end{aligned} \tag{4.9}$$

As $\tau_2 \to \tau_1$ the right-hand side of the above inequality tends to zero. So, $\Upsilon(\Omega)$ is equicontinuous.

Step 4. Υ maps Ω into a compact set in H_α. For this purpose, we decompose Υ as $\Upsilon = \Upsilon_1 + \Upsilon_2$, where

$$(\Upsilon_1 u)(t) = S(t)u_0 + (\mathcal{T}_1 u)(t), \quad t \in I \backslash \{t_1, \dots, t_m\},$$

where $(\mathcal{T}_1 u)(t) = \int_0^t S(t-s) f(s, u(s), u(w_1(s, u(s)))) ds$. and

$$(\Upsilon_2 u)(t) = \begin{cases} 0, & t \in [0, t_1], \\ \sum_{i=1}^{k} S(t - t_i) I_i(u(t_i^-)), & t \in (t_k, t_{k+1}],\ k = 1, 2, \dots, m. \end{cases}$$

Υ_2 is a constant map and hence compact.

Finally, we need to prove that for $0 \le t \le T_0$, $(\Upsilon_1 u)(t)$ is relatively compact in Ω. For each $t \in [0, T_0]$, the set $\{S(t)u_0\}$ is precompact in H_α since $\{S(t), t \ge 0\}$ is compact.

For $t \in (0, T_0]$, and $\epsilon > 0$ sufficiently small, we define

$$(\mathcal{T}_1{}^\epsilon u)(t) = S(\epsilon) \int_0^{t-\epsilon} S(t - \epsilon - s) f(s, u(s), u(w_1(s, u(s)))) ds, \quad u \in \Omega.$$

The set $\{(\mathcal{T}_1{}^\epsilon u)(t) : u \in \Omega\}$ is precompact in H_α since $S(\epsilon)$ is compact. Moreover, for any $u \in \Omega$, we have

$$\begin{aligned} \|(\mathcal{T}_1 u)(t) - (\mathcal{T}_1^\epsilon u)(t)\|_\alpha &\le \int_{t-\epsilon}^{t} \|A^\alpha S(t-s)\| \|f(s, u(s), u(w_1(s, u(s))))\| ds \\ &\le (C_\alpha N_f) \epsilon^{1-\alpha}. \end{aligned}$$

Therefore, $\{(\mathcal{T}_1^\epsilon u)(t) : u \in \Omega\}$ is arbitrarily close to the set $\{(\mathcal{T}_1 u)(t) : u \in \Omega\}, t > 0$. Hence the set $\{(\Upsilon_1 u)(t) : u \in \Omega\}$ is precompact in H_α.

Thus, Υ_1 is a compact operator by Arzela-Ascoli theorem, and hence Υ is a compact operator. Then Schauder fixed point theorem ensures that Υ has a fixed point, which gives rise to a piecewise continuous-mild solution. $\square$

5. Applications

We consider the following semi-linear heat equation with a deviating argument (See also [8]).

$$\begin{gathered}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \tilde{H}(x, u(x,t)) + G(t, x, u(x,t)), \\
x \in (0,1), \quad t \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1), \\
\Delta u|_{t=\frac{1}{2}} = \frac{2u(\frac{1}{2})^-}{2 + u(\frac{1}{2})^-} \\
u(0,t) = u(1,t) = 0, \\
u(x,0) = u_0(x), \quad x \in (0,1),
\end{gathered} \tag{5.1}$$

where

$$\tilde{H}(x, u(x,t)) = \int_0^x K(x,y) u(y, P(t)) dy,$$
$$P(t) = g_1(t) \Big| u\Big(x, g_2(t)|u(x, \dots g_m(t)|u(x,t)|)|\Big)\Big|,$$

and the function $G : \mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is measurable in x, locally Hölder continuous in t, locally Lipschitz continuous in u, uniformly in x. Assume that $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ are locally Hölder continuous in t with $g_i(0) = 0$ for each $i = 1, 2, \dots, m$, and $K \in C^1([0,1] \times [0,1]; \mathbb{R})$. Let

$$X = L^2((0,1); \mathbb{R}), \quad Au = \frac{d^2 u}{dx^2}, \quad D(A) = H^2(0,1) \cap H_0^1(0,1),$$
$$X_{1/2} = D((A)^{1/2}) = H_0^1(0,1), \quad X_{-1/2} = (H_0^1(0,1))^* = H^{-1}(0,1) \equiv H^1(0,1).$$

For $x \in (0,1)$, we define the function $f : \mathbb{R}_+ \times X_{1/2} \times X_{-1/2} \to X$ by

$$f(t, u, \psi)(x) = \tilde{H}(x, \psi) + G(t, x, u), \tag{5.2}$$

where $\tilde{H} : [0,1] \times X_{-1/2} \to X$ is given by

$$\tilde{H}(x, \psi) = \int_0^x K(x,y) \psi(y) dy, \tag{5.3}$$

and $G : \mathbb{R}_+ \times [0,1] \times X_{1/2} \to X$ satisfies

$$\|G(t, x, u)\| \le Q(x,t)(1 + \|u\|_{1/2}), \tag{5.4}$$

with $Q(.,t) \in X$ and Q continuous in its second arguments.

For $u \in D(A)$ and $\lambda \in \mathbb{R}$ with $-Au = \lambda u$, we have

$$\langle -Au, u \rangle = \langle \lambda u, u \rangle, \quad \|u'\|_{L^2} = \lambda \|u\|_{L^2};$$

so we have $\lambda > 0$. The solution u of $-Au = \lambda u$ is

$$u(x) = D_1 \cos(\sqrt{\lambda} x) + D_2 \sin(\sqrt{\lambda} x).$$

Using the boundary condition, we get $D_1 = 0$ and $\lambda = \lambda_n = n^2 \pi^2$ for $n \in \mathbb{N}$. Thus, for $n \in \mathbb{N}$, we have

$$u_n(x) = D_2 \sin(\sqrt{\lambda_n} x).$$

Also $\langle u_n, u_m \rangle = 0$, $m \neq n$ and $\langle u_n, u_n \rangle = 1$. So, for $u \in D(A)$ there exists a sequence α_n of real numbers such that $u(x) = \sum_{n \in \mathbb{N}} \alpha_n u_n(x)$ with $\sum_{n \in \mathbb{N}} (\alpha_n)^2 < \infty$ and $\sum_{n \in \mathbb{N}} (\alpha_n)^2 (\lambda_n)^2 < \infty$.

The semigroup is given by

$$S(t)u = \sum_{n\in\mathbb{N}} \exp(-n^2 t)\langle u, u_m\rangle u_m.$$

Let $T > 0$ be fixed. Now we will show that assumptions (H1) and (H2) are satisfied. For (H1), we will show that $\tilde{H} : [0,1] \times X_{-1/2} \to X$, where

$$\tilde{H}(x, \psi(x,t)) = \int_0^x K(x,y)\psi(y,t)dy,$$

with $\psi(x,t) = u(x, w_1(t, u(x,t)))$. For $x \in [0,1]$, we have

$$\begin{aligned}\|\tilde{H}(x,\psi_1(x,\cdot)) - \tilde{H}(x,\psi_2(x,\cdot))\| &\le \int_0^x |K(x,y)|\|(\psi_1-\psi_2)(y,\cdot)\|dy \\ &\le \|K\|_\infty \int_0^x |(\psi_1-\psi_2)(y,\cdot)|dy.\end{aligned}$$

Thus for $\psi_1, \psi_2 \in H^1(0,1)$ and by applying the Minkowski's integral inequality we obtain

$$\begin{aligned}\|\tilde{H}(x,\psi_1(x,\cdot)) - \tilde{H}(x,\psi_2(x,\cdot))\|^2 &\le \|K\|_\infty^2 \int_0^1\int_0^y |(\psi_1-\psi_2)(y,\cdot)|^2 dxdy \\ &\le \|K\|_\infty^2 \int_0^1 y|(\psi_1-\psi_2)(y,\cdot)|^2 dy \\ &\le \|K\|_\infty^2 \|\psi_1-\psi_2\|_{-1/2}^2.\end{aligned}$$

Since,

$$\frac{\partial}{\partial x}\tilde{H}(x,\psi(x,\cdot)) = K(x,x)\psi(x,\cdot) + \int_0^x \frac{\partial K}{\partial x}(x,y)\psi(y,\cdot),$$

we obtain, in similar way, that

$$\|\frac{\partial}{\partial x}H(x,\psi_1(x,\cdot)) - \frac{\partial}{\partial x}H(x,\psi_2(x,\cdot))\| \le (\|K\|_\infty + \|\frac{\partial K}{\partial x}\|_\infty)\|\psi_1-\psi_2\|_{-1/2}.$$

Thus

$$\begin{aligned}\|\tilde{H}(x,\psi_1(x,\cdot)) - \tilde{H}(x,\psi_2(x,\cdot))\| &\le (2\|K\|_\infty + \|\frac{\partial K}{\partial x}\|_\infty)\|\psi_1-\psi_2\|_{-1/2} \\ &\le (2\|K\|_\infty + \|\frac{\partial K}{\partial x}\|_\infty)\|\psi_1-\psi_2\|_{-1/2} \\ &= C_1\|\psi_1-\psi_2\|_{-1/2},\end{aligned}$$

where $C_1 = (2\|K\|_\infty + \|\frac{\partial K}{\partial x}\|_\infty)$. Again the assumption on G implies that there exist constants $C_2 > 0$ and $0 < \gamma \le 1$ such that

$$\|G(t,x,u) - G(s,x,v)\|_{H_0^1(0,1)} \le C_2(|t-s|^\gamma + \|u-v\|_{1/2}),$$

for $t, s \in [0,T]$, $x \in (0,1)$ and $u, v \in H_0^1(0,1)$. Thus $f : [0,T]\times H_0^1(0,1)\times H^1(0,1) \to L^2(0,1)$ defined by $f = \tilde{H} + G$ satisfies the assumption (H1).

Our next aim is to prove that the functions $h_i : [0,T] \times X_{-1/2} \to [0,T]$ defined by $h_i(t,\phi) = g_i(t)|\phi(x,\cdot)|$ for each $i = 1, 2, \dots, m$, satisfies (H2). For $t \in [0,T]$, we obtain

$$|h_i(t,\phi)| = |g_i(t)|\,|\phi(x,\cdot)| \le \|g_i\|_\infty \|\phi\|_{L^\infty(0,1)} \le C\|\phi\|_{-1/2},$$

where we used the embedding $H_0^1(0,1) \subset C[0,1]$ in the last inequity, and C is a constant depending on the bounds of g_i's. Since g_i's are locally Hölder continuous, for some constant $L_{g_i} > 0$ and $0 < \theta \leq 1$ we have $|g_i(t) - g_i(s)| \leq L_{g_i}|t-s|^\theta$ for $t, s \in [0, T]$. Moreover $t, s \in [0, T]$ and $\phi_1, \phi_2 \in X_{-1/2}$, we have

$$\begin{aligned} |h_i(t,\phi_1) - h_i(t,\phi_2)| &= |g_i(t)(|\phi_1(x,\cdot)| - |\phi_2(x,\cdot)|) + (g_i(t) - g_i(s))|\phi_2(x,\cdot)| \\ &\leq \|g\|_\infty \|\phi_1 - \phi_2\|_{L^\infty(0,1)} + L_{g_i}|t-s|^\theta \|\phi_2\|_{L^\infty(0,1)} \\ &\leq C\|g\|_\infty \|\phi_1 - \phi_2\|_{-1/2} + L_{g_i}|t-s|^\theta \|\phi_2\|_{-1/2} \\ &\leq \max\{C\|g\|_\infty, L_{g_i}\|\phi_2\|_\infty\}(\|\phi_1 - \phi_2\|_{-1/2} + |t-s|^\theta). \end{aligned}$$

If $u, v \in D((-A)^{1/2})$, then

$$\|I_k(u) - I_k(v)\|_{1/2} \leq \frac{2\|u-v\|_{1/2}}{\|(2+u)(2+v)\|_{1/2}} \leq \frac{1}{2}\|u-v\|_{1/2}.$$

Next for Theorem (4.1) we define

$$f(t, u(t), \psi(t)))(x) = \frac{e^{-t}(\cos(u(x,t)) + \sin(\psi(x,t)))}{(2+t^2)(e^t + e^{-t})} + e^{-t},$$

for $t \in [0, 1/2) \cup (1/2, 1]$, $u \in X$, and $x \in (0, 1)$. Clearly,

$$\|f(t, u, \psi)\| \leq \frac{e^{-t}}{(2+t^2)(e^t + e^{-t})} + e^{-t} = m(t),$$

with $m(t) \in L^\infty([0, 1]; \mathbb{R}_+)$. Thus, we can apply all the results of this section to obtain the main results.

Acknowledgements. We highly appreciate the valuable comments and suggestions of the referees on our manuscript which helped to considerably improve the quality of the manuscript. The third author would like to acknowledge the financial aid from the Department of Science and Technology, New Delhi, under its research project SR/S4/MS:796.

References

[1] D. D. Bainov, M. B. Dimitrova; *Oscillation of nonlinear impulsive differential equations with deviating argument.* Bol. Soc. Parana. Mat. (2) 16 (1996), no. 1-2, 9-21 (1997).

[2] D. D. Bainov, M. B. Dimitrova; *Sufficient conditions for oscillations of all solutions of a class of impulsive differential equations with deviating argument.* J. Appl. Math. Stochastic Anal. 9 (1996), no. 1, 33-42.

[3] M. Benchohra, J. Henderson, S. K. Ntouyas; *Existence results for impulsive semilinear neutral functional differential equations in Banach spaces.* Differential Equations Math. Phys. 25 (2002), 105-120.

[4] Y. K. Chang, J. J. Nieto; *Existence of solutions for impulsive neutral integro-differential inclusions with nonlocal initial conditions via fractional operators.* Funct. Anal. Optim. 30 (2009), no. 3-4, 227-244.

[5] Y. K. Chang, A. Anguraj, M. Arjunan; *Existence results for impulsive neutral functional differential equations with infinite delay.* Nonlinear Anal. Hybrid Syst. 2 (2008), no. 1, 209-218.

[6] W. Ding, Y. Wang; *New result for a class of impulsive differential equation with integral boundary conditions.* Commun. Nonlinear Sci. Numer. Simul. 18 (2013), no. 5, 1095-1105.

[7] L. E. El'sgol'ts, S. B. Norkin; *Introduction to the theory of differential equations with deviating arguments*, Academic Press, 1973.

[8] C. G. Gal; *Nonlinear abstract differential equations with deviated argument*, J. Math. Anal. Appl. 333 (2007) 971-983.

[9] R. Haloi, D. Bahuguna, D. N. Pandey; *Existence and uniqueness of solutions for quasi-linear differential equations with deviating arguments.* Electron. J. Differential Equations 2012, No. 13, 1-10.

[10] R. Haloi, D. Bahuguna, D. N. Pandey; *Existence, uniqueness and asymptotic stability of solutions to a non-autonomous semi-linear differential equation with deviated argument*, Nonlinear Dynamics and System Theory, 12 (2) (2012), 179–191.
[11] R. Haloi, D. Bahuguna, D. N. Pandey; *Existence and Uniqueness of a Solution for a Non-Autonomous Semilinear Integro-Differential Equation with Deviated Argument*, Differntial Equation and Dynamical systems 20(1),(2012), 1-16.
[12] R. Haloi, D. Bahuguna, D. N. Pandey; *Existence of solutions to a non-autonomous abstract neutral differential equation with deviated argument*, J. Nonl. Evol. Equ. Appl.5 (2011), 75-90.
[13] T. Jankowski; *Three positive solutions to second-order three-point impulsive differential equations with deviating arguments.* Int. J. Comput. Math. 87 (2010), no. 1-3, 215-225.
[14] T. Jankowski; *Existence of solutions for second order impulsive differential equations with deviating arguments.* Nonlinear Anal. 67 (2007), no. 6, 1764-1774.
[15] J. H. Liu; *Nonlinear impulsive evolution equations.* Dynam. Contin. Discrete Impuls. Systems 6 (1999), no. 1, 77-85.
[16] Z. Liu, J. Liang; *A class of boundary value problems for first-order impulsive integro-differential equations with deviating arguments.* J. Comput. Appl. Math. 237 (2013), no. 1, 477-486.
[17] G. M. Mophou; *Existence and uniqueness of mild solutions to impulsive fractional differential equations.* Nonlinear Anal. 72 (2010), no. 3-4, 1604-1615.
[18] A. Pazy; *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.
[19] M. N. Rabelo, M. Henrique, G. Siracusa; *Existence of integro-differential solutions for a class of abstract partial impulsive differential equations.* J. Inequal. Appl. 2011, (2011) 19 pp.
[20] S. Stevic'; *Solutions converging to zero of some systems of nonlinear functional differential equations with iterated deviating arguments.* Appl. Math. Comput. 219 (2012), no. 8, 4031-4035.
[21] S. Stevic'; *Globally bounded solutions of a system of nonlinear functional differential equations with iterated deviating argument.* Appl. Math. Comput. 219 (2012), no. 4, 2180-2185.
[22] X. B. Shu, Y. Lai, Y. Chen; *The existence of mild solutions for impulsive fractional partial differential equations.* Nonlinear Anal. 74 (2011), no. 5, 2003-2011.
[23] J. Wang, M. Fečkan, Y. Zhou; *On the new concept of solutions and existence results for impulsive fractional evolution equations.* Dyn. Partial Differ. Equ. 8 (2011), no. 4, 345-361.
[24] W. Wang, X. Chen; *Positive solutions of multi-point boundary value problems for second order impulsive differential equations with deviating arguments.* Differ. Equ. Dyn. Syst. 19 (2011), no. 4, 375-387.
[25] G. P. Wei; *First-order nonlinear impulsive differential equations with deviating arguments.* Acta Anal. Funct. Appl. 3 (2001), no. 4, 334-338.
[26] G. Ye, J. Shen, J. Li; *Periodic boundary value problems for impulsive neutral differential equations with multi-deviation arguments.* Comput. Appl. Math. 29 (2010), no. 3, 507-525.

Pradeep Kumar
Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Pin 208016, India
E-mail address: prdipk@gmail.com

Dwijendra N. Pandey
Department of Mathematics, Indian Institute of Technology Roorkee, Pin 247667, India
E-mail address: dwij.iitk@gmail.com

Dhirendra Bahuguna
Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Pin 208016, India
E-mail address: dhiren@iitk.ac.in, Tel. +91-512-2597053, Fax +91-512-2597500