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SOLUTIONS TO NONLOCAL FRACTIONAL DIFFERENTIAL EQUATIONS USING A NONCOMPACT SEMIGROUP

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Abstract. This article concerns the existence of solutions to nonlocal fractional differential equations in Banach spaces. By using a type of newly-defined measure of noncompactness, we discuss this problem in general Banach spaces without any compactness assumptions to the operator semigroup. Some existence results are obtained when the nonlocal term is compact and when is Lipschitz continuous.

1. Introduction

In this article, we study the following fractional differential equations with nonlocal conditions

$$\begin{gathered} {}^C D^\alpha u(t) = Au(t) + f(t, u(t)), \quad t \in J = [0, b], \\ u(0) = g(u), \end{gathered} \tag{1.1}$$

where $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ in a Banach space X; ${}^C D^\alpha$ is the Caputo fractional derivative operator of order α with $0 < \alpha \leq 1$; f and g are appropriate continuous functions to be specified later.

Fractional differential equations arise in many engineering and scientific problems, such as diffusion process, control theory, signal and image processing. Compared with the classical integer-order models, the fractional-order models are more realistic and practical to describe many phenomena in nature (see [5]). For some recent development on this topic, we refer to the monographs of Kilbas et al. [16], Podlubny [25], Lakshmikantham et al. [18], and [1, 7, 17, 19, 20, 30]. By using some probability density functions, El-Borai [9] introduced fundamental solutions of fractional evolution equations in a Banach space. Wang et al. [28] obtained the existence and uniqueness of α-mild solutions by means of fractional calculus and Leray-Schauder fixed point theorem with a compact analytic semigroup. Ren et al. [26, 27] established the existence of mild solutions for a class of semilinear integro-differential equations of fractional order with delays.

The study of nonlocal semilinear differential equation in Banach spaces was initiated by Byszewski [6] and the importance of the problem consists in the fact

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that it is more general and has better effect than the classical initial conditions $u(0) = u_0$ alone. For example, Deng [8] defined the function g by

$$g(u) = \sum_{j=1}^{q} c_j u(s_j),$$

where c_j are given constants and $0 < s_1 < s_2 < \cdots < s_q \leq b$. This allows measurements to be made at $t = 0, s_1, \dots, s_q$, rather than just at $t = 0$ and more information can be obtained. Subsequently several authors have investigated some different types of differential equations and integrodifferential equations in Banach spaces[23, 2, 32]. Most of the previous results are obtained with the assumption that semigroup $T(t)$ is compact. Then one of the difficulties on the nonlocal problems is how to deal with the compactness on the solution operator. By using the measure of noncompactness, Xue[29] discussed integral solutions of the nonlinear nonlocal initial value problem and Fan et al.[10], Ji et al.[14] discussed nonlocal impulsive differential equations when the semigroup $T(t)$ is equicontinuous.

From the viewpoint of theory and practice, it is natural for mathematics to combine fractional differential equations and nonlocal conditions. Mophou et al.[22] and Balachandran et al.[3] investigated the existence of solutions of fractional abstract differential equations with nonlocal conditions. Zhou and Jiao[31] discussed the problem based on Krasnoselskii's fixed point theorem with the assumption that semigroup $T(t)$ is compact and nonlocal item g is Lipschitz continuous. Very recently, Li, Peng and Gao[21] study the existence of mild solutions to the problem (1.1) by using the Hausdorff measure of noncompactness when the semigroup $T(t)$ is equicontinuous and g is compact. In this paper, we study (1.1) without assuming $T(t)$ is compact or equicontinuous and do not require additional conditions compared with those in[21]. Therefore, the existence theorems of mild solutions given here are quite general, even in the case of integer-order differential equations. The work is based on a type of newly-defined measure of noncompactness (see Lemma 3.1), which can be seen as a generalization of classical Hausdorff measure of noncompactness. Moreover, we do not need the assumption of separability on the Banach space X.

The article is organized as follows. In Section 2 we recall some preliminary facts that we need in the sequel. In Section 3 we prove our results when nonlocal item g is compact. In Section 4 we get our results when nonlocal item g is Lipschitz continuous. The conclusions and applications of the paper are given in Section 5.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space and $\mathbb{N}$ be the set of positive integers. We denote by $C([0,b];X)$ the space of X-valued continuous functions on $[0,b]$ with the norm $\|x\| = \sup\{\|x(t)\|, t \in [0,b]\}$ and by $L^1([0,b];X)$ the space of X-valued Bochner integrable functions on $[0,b]$ with the norm $\|f\|_{L^1} = \int_0^b \|f(t)\|\,\mathrm{d}t$.

Let us recall the following definitions. For basic facts about fractional derivatives and fractional calculus, one can refer to the books [16, 25].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ with the lower limit zero for a function f can be defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad t > 0,$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The Riemann-Liouville derivative of order α with the lower limit zero for a function f can be written as

$$^{R-L}D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t (t-s)^{n-\alpha-1}f(s)\,\mathrm{d}s,\quad t>0,$$

where $\alpha\in(n-1,n)$, $n\in\mathbb{N}$.

Definition 2.3. The Caputo derivative of order α with the lower limit zero for a function f can be written as

$$^{C}D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)\,\mathrm{d}s,\quad t>0,$$

where $\alpha\in(n-1,n)$, $n\in\mathbb{N}$.

If f takes values in a Banach space X, the integrals which appear in the above three definitions are taken in Bochner's sense. Especially, when $0<\alpha<1$, we have

$$^{C}D^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(s)}{(t-s)^{\alpha}}\,\mathrm{d}s.$$

We firstly recall the concept of mild solutions to equation (1.1) developed in [9, 31].

Definition 2.4. A function $u\in C(J;X)$ is said to be a mild solution of (1.1) if u satisfies

$$u(t)=S_{\alpha}(t)g(u)+\int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s)f(s,u(s))\,\mathrm{d}s,$$

for $t\in J$, where

$$S_{\alpha}(t)=\int_0^{\infty}\xi_{\alpha}(\theta)T(t^{\alpha}\theta)\,\mathrm{d}\theta,\quad T_{\alpha}(t)=\alpha\int_0^{\infty}\theta\xi_{\alpha}(\theta)T(t^{\alpha}\theta)\,\mathrm{d}\theta,$$

$$\xi_{\alpha}(\theta)=\frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi_{\alpha}(\theta^{-\frac{1}{\alpha}})\ge 0,$$

$$\varpi_{\alpha}(\theta)=\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),\quad \theta\in(0,\infty).$$

Here $\xi_{\alpha}(\theta)$ is a probability density function defined on $(0,\infty)$ satisfying

$$\int_0^{\infty}\xi_{\alpha}(\theta)\,\mathrm{d}\theta=1,\quad \int_0^{\infty}\theta^{v}\xi_{\alpha}(\theta)\,\mathrm{d}\theta=\frac{\Gamma(1+v)}{\Gamma(1+\alpha v)},\quad v\in[0,1].$$

Lemma 2.5 ([28]). *For any fixed $t\ge 0$, the operators $S_{\alpha}(t)$ and $T_{\alpha}(t)$ are linear and bounded operators; i.e., for any $x\in X$, $\|S_{\alpha}(t)x\|\le M\|x\|$ and $\|T_{\alpha}(t)x\|\le \frac{M\alpha}{\Gamma(1+\alpha)}\|x\|$, where M is the constant such that $\|T(t)\|\le M$ for all $t\in[0,b]$.*

Now we give some facts on measure of noncompactness, see Banas and Goebel[4].

Definition 2.6. Let E^{+} be the positive cone of an ordered Banach space $(E,\le)$. A function Φ defined on the set of all bounded subsets of the Banach space X with values in E^{+} is called a measure of noncompactness (in short MNC) on X if $\Phi(\overline{\mathrm{co}}\Omega)=\Phi(\Omega)$ for all bounded subsets $\Omega\subset X$, where $\overline{\mathrm{co}}\Omega$ stands for the closed convex hull of Ω. A measure of noncompactness Φ is said to be:

(1) *monotone* if for all bounded subsets Ω_1, Ω_2 of X we have: $(\Omega_1 \subseteq \Omega_2) \Rightarrow (\Phi(\Omega_1) \leq \Phi(\Omega_2))$;
(2) *nonsingular* if $\Phi(\{a\} \cup \Omega) = \Phi(\Omega)$ for every $a \in X$, $\Omega \subset X$;
(3) *regular* if $\Phi(\Omega) = 0$ if and only if Ω is relatively compact in X.

One of the most important examples of MNC is the Hausdorff measure of noncompactness $\beta(\cdot)$ defined by

$$\beta(B) = \inf\{\varepsilon > 0 : B \text{has a finite } \varepsilon\text{-net in } X\},$$

for each bounded subset B in a Banach space X.

It is well known that the Hausdorff measure of noncompactness β enjoys the above properties.

Lemma 2.7 ([4]). *Let X be a real Banach space and $B, C \subseteq X$ be bounded. Then the following properties are satisfied:*

(1) *B is relatively compact if and only if $\beta(B) = 0$;*
(2) *$\beta(B) = \beta(\overline{B}) = \beta(\operatorname{conv} B)$, where $\overline{B}$ and $\operatorname{conv} B$ mean the closure and convex hull of B, respectively;*
(3) *$\beta(B) \leq \beta(C)$ when $B \subseteq C$;*
(4) *$\beta(B + C) \leq \beta(B) + \beta(C)$, where $B + C = \{x + y : x \in B, y \in C\}$;*
(5) *$\beta(B \cup C) \leq \max\{\beta(B), \beta(C)\}$;*
(6) *$\beta(\lambda B) \leq |\lambda|\beta(B)$ for any $\lambda \in R$;*
(7) *If the map $Q : D(Q) \subseteq X \to Z$ is Lipschitz continuous with constant k, then $\beta_Z(QB) \leq k\beta(B)$ for any bounded subset $B \subseteq D(Q)$, where Z is a Banach space.*
(8) *If $\{W_n\}_{n=1}^{\infty}$ is a decreasing sequence of bounded closed nonempty subsets of X and $\lim_{n\to\infty} \beta(W_n) = 0$, then $\cap_{n=1}^{\infty} W_n$ is nonempty and compact in X.*

We will also use the sequential MNC β_0 generated by β, that is, for any bounded subset $B \subset X$, we define

$$\beta_0(B) = \sup\big\{\beta(\{x_n : n \geq 1\}) : \{x_n\}_{n=1}^{\infty} \text{ is a sequence in } B\big\}.$$

It follows that

$$\beta_0(B) \leq \beta(B) \leq 2\beta_0(B). \tag{2.1}$$

If X is a separable space, we have $\beta_0(B) = \beta(B)$.

Lemma 2.8 ([15]). *If $\{u_n\}_{n=1}^{\infty} \subset L^1(J; X)$ satisfies $\|u_n(t)\| \leq \varphi(t)$ a.e. on $[0, b]$ for all $n \geq 1$ with some $\varphi \in L^1(J; \mathbb{R}_+)$, then for $t \in [0, b]$, we have*

$$\beta\Big(\big\{ \int_0^t u_n(s)\,\mathrm{d}s\big\}_{n=1}^{\infty}\Big) \leq 2 \int_0^t \beta(\{u_n(s)\}_{n=1}^{\infty})\,\mathrm{d}s.$$

Lemma 2.9 ([12]). *Suppose $b \geq 0$, $\sigma > 0$ and $a(t)$ is nonnegative function locally integrable on $0 \leq t < b$ ($b \leq +\infty$), and suppose $c(t)$ is nonnegative and locally integrable on $0 \leq t < b$ with*

$$c(t) \leq a(t) + b \int_0^t (t-s)^{\sigma-1} c(s)\,\mathrm{d}s$$

on this interval. Then

$$c(t) \leq a(t) + \mu \int_0^t E'_\sigma(\mu(t-s)) a(s)\,\mathrm{d}s, \quad 0 \leq t \leq b,$$

where $\mu = (b\Gamma(\sigma))^{1/\sigma}$, $E_\sigma(z) = \sum_{n=0}^{\infty} z^{n\sigma}/\Gamma(n\sigma + 1)$, $E'_\sigma(z) = \frac{\mathrm{d}}{\mathrm{d}z} E_\sigma(z)$.

3. $T(t)$ strongly continuous and g compact

Let r be a positive constant and $B_r = \{x \in X : \|x\| \le r\}$, $W_r = \{x \in C(J;X) : x(t) \in B_r, t \in J\}$. We define the solution operator $G : C(J;X) \to C(J;X)$ by

$$Gu(t) = S_\alpha(t)g(u) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s,u(s))\,\mathrm{d}s$$

with

$$G_1u(t) = S_\alpha(t)g(u),$$
$$G_2u(t) = \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s,u(s))\,\mathrm{d}s,$$

for all $t \in J$. It is easy to see that u is the mild solution of the problem (1.1) if and only if u is a fixed point of the map G.

Now we introduce some noncompact measures. For any bounded set $B \subset C(J;X)$, we define

$$\chi_1(B) = \sup_{t\in J} \beta(B(t)),$$

where β is the Hausdorff MNC on X and it is easy to see that χ_1 coincides with the the Hausdorff MNC β_C on equicontinuous sets. We also define

$$\chi_2(B) = \sup_{t\in J} \mathrm{mod}_C(B(t)),$$

where $\mathrm{mod}_C(B(t))$ is the modulus of equicontinuity of the set B at $t \in J$, given by the formula

$$\mathrm{mod}_C(B(t)) = \lim_{\delta\to 0} \sup\{\|x(t_1) - x(t_2)\| : t_1, t_2 \in (t-\delta, t+\delta), x \in B\}.$$

Then the MNC χ_1, χ_2 are well-defined and are both monotone and nonsingular, but in general they are not necessarily regular. Similar definitions with χ_1, χ_2 can be found in Kamenskii [15] or Fan[11]. Now we define

$$\chi(B) = \chi_1(B) + \chi_2(B).$$

Then we have the following result.

Lemma 3.1. *χ is a monotone, nonsingular and regular measure of noncompactness defined on bounded subsets of $C(J;X)$.*

Proof. It is easy to check that χ is well-defined, monotone and nonsingular. Now we shall show that χ is regular. If B is relatively compact in $C(J;X)$, then by the abstract version of the Ascoli-Arzela theorem, we have $\chi(B) = 0$.

On the other hand, if $\chi(B) = 0$, by the definition of χ, we have

$$\chi_1(B) = 0,\ \chi_2(B) = 0.$$

That is, for every $t \in J$, $B(t)$ is precompact in X. Then it remains to prove that B is equicontinuous on J. Suppose not, then there exist $\varepsilon_0 > 0$ and sequences $\{u_n\} \subseteq B$, $\{t_n\}$, $\{\bar{t}_n\} \subseteq [0,b]$, such that $t_n \to t_0$, $\bar{t}_n \to t_0$ as $n \to \infty$ and

$$\|u_n(t_n) - u_n(\bar{t}_n)\| \ge \varepsilon_0,$$

for all $n \ge 1$. Note that

$$\|u_n(t_n) - u_n(\bar{t}_n)\| \le \sup\{\|u(t_n) - u(\bar{t}_n)\| : u \in B\}.$$

We take the upper limit for n and get that

$$\overline{\lim_{n}}\|u_n(t_n)-u_n(\bar{t}_n)\| \le \mathrm{mod}_C(B(t_0)) \le \chi_2(B)=0,$$

which gives the contradiction $0<\varepsilon_0\le 0$. Thus $B\subseteq C(J;X)$ is equicontinuous on J. This completes the proof. $\square$

We will use the following hypotheses:

- (HA) The operator A generates a strongly continuous semigroup $\{T(t)\}_{t\ge 0}$ in X. Moreover, there exists a positive constant $M>0$ such that $M=\sup_{0\le t\le b}\|T(t)\|$ (see Pazy[24]).
- (HG1) $g: C(J;X)\to X$ is continuous and compact. There exists a positive constant N such that $\|g(u)\|\le N$ for all $u\in C(J;X)$.
- (HF1) $f:[0,b]\times X\to X$ is continuous.
- (HF2) there exists a constant $L>0$, such that for any bounded set $D\subset X$, $\beta(f(t,D))\le L\beta(D)$, for a.e. $t\in J$.

The following lemma is useful for our proofs.

Lemma 3.2. *Suppose that the semigroup $\{T(t)\}_{t\ge 0}$ is strongly continuous and hypotheses* (HF1), (HF2) *are satisfied. Then for any bounded set $B\subset C(J;X)$, we have*

$$\beta(G_2B(t))\le \frac{4\alpha ML}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\beta(B(s))\,\mathrm{d}s,$$

for $t\in[0,b]$.

Proof. For $t\in[0,b]$, due to the inequality (2.1), we obtain that for arbitrary $\varepsilon>0$, there exists a sequence $\{v_k\}_{k=1}^{\infty}\subset B$ such that

$$\beta(G_2B(t))\le 2\beta(\{G_2v_k(t)\}_{k=1}^{\infty})+\varepsilon. \tag{3.1}$$

It follows from Lemma 2.8 and hypotheses (Hf_1), (Hf_2) that

$$\begin{aligned}
\beta(\{G_2v_k(t)\}_{k=1}^{\infty}) &\le 2\int_0^t (t-s)^{\alpha-1}\beta(\{T_\alpha(t-s)f(s,v_k(s))\}_{k=1}^{\infty})\,\mathrm{d}s\\
&\le 2\int_0^t (t-s)^{\alpha-1}\frac{\alpha M}{\Gamma(1+\alpha)}L\beta(\{v_k(s)\}_{k=1}^{\infty})\,\mathrm{d}s\\
&\le \frac{2\alpha ML}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\beta(B(s))\,\mathrm{d}s.
\end{aligned}$$

According to e3.1, we can derive that

$$\beta(G_2B(t))\le \frac{4\alpha ML}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\beta(B(s))\,\mathrm{d}s+\varepsilon.$$

Since the above inequality holds for arbitrary $\varepsilon>0$, it follows that

$$\beta(G_2B(t))\le \frac{4\alpha ML}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\beta(B(s))\,\mathrm{d}s.$$

This completes the proof. $\square$

Now, we give the main existence result of this section.

Theorem 3.3. *Assume that the hypotheses* (HA), (HG1), (HF1), (HF2) *are satisfied. Then the nonlocal fractional differential system* (1.1) *has at least one mild solution on* $[0, b]$*, provided that there exists a constant* $r > 0$ *such that*

$$MN + \frac{Mb^\alpha}{\Gamma(1+\alpha)} \sup_{s\in[0,b],u\in W_r} \|f(s, u(s))\| \le r. \tag{3.2}$$

Proof. We shall prove this result by using the Schauder's fixed point theorem.

Step 1. We shall prove that G is continuous on $C(J;X)$. Let $\{u_m\}_{m=1}^\infty$ be a sequence in $C(J;X)$ with $\lim_{m\to\infty} u_m = u$ in $C(J;X)$. By the continuity of f, we deduce that for each $s \in [0, b]$, $f(s, u_m(s))$ converges to $f(s, u(s))$ in X uniformly for $s \in [0, b]$. And we have

$$\begin{aligned}
\|Gu_m - Gu\| &\le \|S_\alpha(t)g(u_m) - g(u)\| \\
&\quad + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)[f(s,u_m(s)) - f(s,u(s))]\| \, ds \\
&\le M\|g(u_m) - g(u)\| + \frac{Mb^\alpha}{\Gamma(1+\alpha)} \sup_{s\in[0,b]} \|f(s,u_m(s)) - f(s,u(s))\|.
\end{aligned}$$

Then by the continuity of g, we get $\lim_{m\to\infty} Gu_m = Gu$ in $C(J;X)$, which implies that G is continuous on $C(J;X)$.

Step 2. We construct a bounded convex and closed set $W \subset C(J;X)$ such that G maps W into itself. Let $W_0 = \{u \in C(J;X) : \|u(t)\| \le r, t \in J\}$, where r satisfies the condition (3.2). For any $u \in W_0$, by hypotheses (HG1), (HF1) and (3.2), we have

$$\begin{aligned}
\|Gu(t)\| &\le \|S_\alpha(t)g(u)\| + \|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,u(s))\, ds \\
&\le MN + \frac{Mb^\alpha}{\Gamma(1+\alpha)} \sup_{s\in[0,b],u\in W_0} \|f(s,u(s))\| \le r,
\end{aligned}$$

for $t \in [0, b]$, which implies that $GW_0 \subseteq W_0$.

Define $W_1 = \overline{\mathrm{conv}}\{G(W_0), u_0\}$, where $\overline{\mathrm{conv}}$ means the closure of convex hull, $u_0 \in W_0$. Then $W_1 \subset W_0$ is nonempty bounded closed and convex. We define $W_n = \overline{\mathrm{conv}}\{G(W_{n-1}), u_0\}$ for $n \ge 1$. It is easy to know that $\{W_n\}_{n=0}^\infty$ is a decreasing sequence of $C(J;X)$. Moreover, set

$$W = \cap_{n=0}^\infty W_n,$$

then W is a nonempty, convex, closed and bounded subset of $C(J;X)$ and $GW \subseteq W$.

Step 3. We claim that W is compact in $C(J;X)$ by using the newly-defined MNC χ. As $\{W_n\}$ is a decreasing sequence of $C(J;X)$, then $\{\beta(W_n(t))\}_{n=0}^\infty$ is nonnegative decreasing sequence for any $t \in [0, b]$. From the compactness of g and Lemma 3.2, we get

$$\begin{aligned}
\beta(W_{n+1}(t)) &\le \beta(\{S_\alpha(t)g(u) : u \in W_n\}) \\
&\quad + \beta(\{\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,u(s))\, ds : u \in W_n\}) \\
&\le \frac{4\alpha ML}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1}\beta(W_n(s))ds.
\end{aligned} \tag{3.3}$$

Taking $n \to \infty$ to both sides of (3.3), we have

$$\beta(W(t)) \leq \frac{4\alpha ML}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \beta(W(s)) \mathrm{d}s.$$

By Lemma 2.9 we obtain that $\beta(W(t)) = 0$ for any $t \in [0, b]$. Then according to the definition of χ_1, we have

$$\chi_1(W_n) \to 0, \quad \text{as } n \to \infty. \tag{3.4}$$

Next, we will estimate $\chi_2(W_n)$. Fix $t_0 \in (0, b)$, $0 < p < \alpha$. Then for given $n \in \mathbb{N}$, according to the continuity of f, we take δ_0, $0 < \delta_0 < \min\{t_0, b - t_0\}$ such that

$$\Big(\int_{t_0-\delta_0}^{t_0+\delta_0} \|f(s, u(s))\|^{1/p} \, \mathrm{d}s \Big)^p \leq \frac{1}{n}, \quad \text{for any } u \in W_r. \tag{3.5}$$

From the definition of Hausdorff MNC β, there exist finite points $x_1, x_2, \dots, x_k \in X$ such that

$$W_n(t_0 - \delta_0) \subset \cup_{i=1}^k B(x_i, 2\beta(W_n(t_0 - \delta_0))).$$

Moreover, for each $u \in GW_{n-1}$, there exists $v \in W_{n-1}$ such that

$$\begin{aligned} u(t_1) &= S_\alpha(t_1 - (t_0 - \delta_0)) u(t_0 - \delta_0) \\ &\quad + \int_{t_0-\delta_0}^{t_1} (t_1 - s)^{\alpha-1} T_\alpha(t_1 - s) f(s, v(s)) \, \mathrm{d}s, \end{aligned} \tag{3.6}$$

$$\begin{aligned} u(t_2) &= S_\alpha(t_2 - (t_0 - \delta_0)) u(t_0 - \delta_0) \\ &\quad + \int_{t_0-\delta_0}^{t_2} (t_2 - s)^{\alpha-1} T_\alpha(t_2 - s) f(s, v(s)) \, \mathrm{d}s, \end{aligned} \tag{3.7}$$

for $t_1, t_2 \in (t_0 - \delta, t_0 + \delta)$. For the above given u, there exists x_j, $1 \leq j \leq k$, such that

$$\|u(t_0 - \delta_0) - x_j\| \leq 2\beta(W_n(t_0 - \delta_0)). \tag{3.8}$$

By the strong continuity of $S_\alpha(t)$, there exists δ, $0 < \delta < \delta_0$, such that

$$\|S_\alpha(t_1 - (t_0 - \delta_0)) x_i - S_\alpha(t_2 - (t_0 - \delta_0)) x_i\| \leq M\beta(W_n(t_0 - \delta_0)). \tag{3.9}$$

where $i = 1, \dots, k$ and $t_1, t_2 \in (t_0 - \delta, t_0 + \delta)$.

On the other hand, for $0 < p < \alpha < 1$, by the Hölder's inequality and (3.5), we have

$$\begin{aligned}
&\| \int_{t_0-\delta_0}^{t_1} (t_1 - s)^{\alpha-1} T_\alpha(t_1 - s) f(s, v(s))\, \mathrm{d}s \\
&- \int_{t_0-\delta_0}^{t_2} (t_2 - s)^{\alpha-1} T_\alpha(t_2 - s) f(s, v(s))\, \mathrm{d}s\| \\
&\le \frac{\alpha M}{\Gamma(1+\alpha)} \int_{t_0-\delta_0}^{t_1} (t_1 - s)^{\alpha-1} \|f(s, v(s))\|\, \mathrm{d}s \\
&\quad + \frac{\alpha M}{\Gamma(1+\alpha)} \int_{t_0-\delta_0}^{t_2} (t_2 - s)^{\alpha-1} \|f(s, v(s))\|\, \mathrm{d}s \\
&\le \frac{\alpha M}{\Gamma(1+\alpha)} \Big(\int_{t_0-\delta_0}^{t_1} (t_1 - s)^{\frac{\alpha-1}{1-p}}\, \mathrm{d}s \Big)^{1-p} \Big(\int_{t_0-\delta_0}^{t_1} \|f(s, v(s))\|^{1/p}\, \mathrm{d}s \Big)^p \\
&\quad + \frac{\alpha M}{\Gamma(1+\alpha)} \big(\int_{t_0-\delta_0}^{t_2} (t_2 - s)^{\frac{\alpha-1}{1-p}}\, \mathrm{d}s \big)^{1-p} \Big(\int_{t_0-\delta_0}^{t_2} \|f(s, v(s))\|^{1/p}\, \mathrm{d}s \Big)^p \\
&\le \frac{2\alpha M}{\Gamma(1+\alpha)} \big(\frac{1-p}{\alpha-p} \big)^{1-p} b^{\alpha-p} \Big(\int_{t_0-\delta_0}^{t_0+\delta_0} \|f(s, v(s))\|^{1/p}\, \mathrm{d}s \Big)^p \\
&\le \frac{2\alpha M}{\Gamma(1+\alpha)} \big(\frac{1-p}{\alpha-p} \big)^{1-p} b^{\alpha-p} \frac{1}{n}.
\end{aligned} \tag{3.10}$$

Therefore, for any given $u \in GW_{n-1}$, it follows from (3.6)–(3.10) that

$$\begin{aligned}
&\|u(t_1) - u(t_2)\| \\
&\le \|S_\alpha(t_1 - (t_0 - \delta_0)) u(t_0 - \delta_0) - S_\alpha(t_1 - (t_0 - \delta_0)) x_j\| \\
&\quad + \|S_\alpha(t_1 - (t_0 - \delta_0)) x_j - S_\alpha(t_2 - (t_0 - \delta_0)) x_j\| \\
&\quad + \|S_\alpha(t_2 - (t_0 - \delta_0)) x_j - S_\alpha(t_2 - (t_0 - \delta_0)) u(t_0 - \delta_0)\| \\
&\quad + \| \int_{t_0-\delta_0}^{t_1} (t_1 - s)^{\alpha-1} T_\alpha(t_1 - s) f(s, v(s))\, \mathrm{d}s \\
&\quad - \int_{t_0-\delta_0}^{t_2} (t_2 - s)^{\alpha-1} T_\alpha(t_2 - s) f(s, v(s))\, \mathrm{d}s\| \\
&\le 5M\beta(W_n(t_0 - \delta_0)) + \frac{2\alpha M}{\Gamma(1+\alpha)} \big(\frac{1-p}{\alpha-p} \big)^{1-p} b^{\alpha-p} \frac{1}{n},
\end{aligned}$$

for $t_1, t_2 \in (t_0 - \delta_0, t_0 + \delta_0)$. Then we have

$$\begin{aligned}
\mathrm{mod}_C(GW_{n-1}(t_0)) &\le 5M\beta(W_n(t_0 - \delta_0)) + \frac{2\alpha M}{\Gamma(1+\alpha)} \big(\frac{1-p}{\alpha-p} \big)^{1-p} b^{\alpha-p} \frac{1}{n} \\
&\le 5M\chi_1(W_n) + \frac{2\alpha M}{\Gamma(1+\alpha)} \big(\frac{1-p}{\alpha-p} \big)^{1-p} b^{\alpha-p} \frac{1}{n}.
\end{aligned}$$

For $t_0 = 0$ or b, we can also verify the above inequality. By the definition of MNC χ_2, we have

$$\chi_2(GW_{n-1}) \le 5M\chi_1(W_n) + \frac{2\alpha M}{\Gamma(1+\alpha)} \big(\frac{1-p}{\alpha-p} \big)^{1-p} b^{\alpha-p} \frac{1}{n}. \tag{3.11}$$

According to the property of MNC, we obtain

$$\chi_2(W_n) = \chi_2(\overline{\mathrm{conv}}\{G(W_{n-1}), u_0\}) = \chi_2(GW_{n-1}). \tag{3.12}$$

Thus from the definition of χ and (3.4), (3.11), (3.12), it follows that

$$\chi(W_n) = \chi_1(W_n) + \chi_2(W_n) \le (5M+1)\chi_1(W_n) + \frac{2\alpha M}{\Gamma(1+\alpha)}\big(\frac{1-p}{\alpha-p}\big)^{1-p} b^{\alpha-p}\frac{1}{n} \to 0,$$

as $n \to \infty$. Therefore, the set $W = \cap_{n=1}^{\infty} W_n$ is a nonempty, convex, compact subset of $C(J, X)$, since χ is a monotone, nonsingular and regular MNC (see Lemma 3.1). Moreover, G maps W into W.

Therefore, due to the Schauder's fixed point theorem, G has at lest one fixed point $u \in C(J; X)$, which is just the mild solution to the problem (1.1). The proof of Theorem 3.3 is complete. ∎

Remark 3.4. Li et al. [21] discussed (1.1) when $T(t)$ is equicontinuous, g is compact and the Banach space X is separable. Our existence results are more general than many previous results in this field, where the compactness of $T(t)$ and f are needed. If f is compact or Lipschitz continuous, then hypothesis (HF2) is obviously satisfied. The regular MNC χ defined by us plays a key role in the proof.

4. $T(t)$strongly continuous and g Lipschitz continuous

In this section, two existence results are given when g is Lipschitz continuous. The map $Q : D \subseteq X \to X$ is said to be β-condensing if Q is continuous, bounded and for any nonprecompact bounded subset $B \subset D$, we have $\beta(QB) < \beta(B)$, where X is a Banach space.

Lemma 4.1 (See Darbo-Sadovskii [4]). *If $D \subset X$ is bounded closed and convex, the continuous map $Q : D \to D$ is β-condensing, then Q has at least one fixed point in D.*

We will use the following hypotheses:

(HG2) $g : C([0, b]; X) \to X$ and there exists a constant $l_g > 0$ such that $\|g(x) - g(y)\| \le l_g\|x - y\|$, $x, y \in C([0, b]; X)$.

(HF3) $f : [0, b] \times X \to X$ is continuous and compact.

Theorem 4.2. *Assume that* (HA), (HG2), (HF3) *are satisfied. Then the nonlocal fractional differential system* (1.1) *has at least one mild solution on* $[0, b]$*, provided that* (3.2) *and*

$$Ml_g < 1. \tag{4.1}$$

Proof. Define the solution operator $G : C(J; X) \to C(J; X)$ by

$$Gu(t) = S_\alpha(t)g(u) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, u(s))\,\mathrm{d}s$$

with

$$G_1u(t) = S_\alpha(t)g(u),$$

$$G_2u(t) = \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, u(s))\,\mathrm{d}s,$$

for all $t \in J$. From the proof of Theorem 3.3, we have got that the solution operator G is continuous and maps W_r into itself. It remains to show that G is β-condensing.

For $u, v \in W_r$, we have

$$\|G_1u - G_1v\| = \|S_\alpha(t)g(u) - S_\alpha(t)g(v)\| \le Ml_g\|u - v\|,$$

which implies that G_1 is Lipschitz continuous with the constant Ml_g. From Lemma 2.7 (7), we have

$$\beta(G_1W_r) \le Ml_g\beta(W_r). \tag{4.2}$$

Next, we shall show that G_2 is a compact operator. From the Ascoli-Arzela theorem, we need prove that G_2W_r is equicontinuous and $G_2W_r(t)$ is precompact in X for $t \in [0,b]$. For $u \in W_r$ and $0 \le t_1 < t_2 \le b$, we have

$$\begin{aligned}
&\|G_2u(t_2) - G_2u(t_1)\| \\
&\le \Big\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}T_\alpha(t_2-s)f(s,u(s))\,\mathrm{d}s\Big\| \\
&\quad + \Big\|\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]T_\alpha(t_2-s)f(s,u(s))\,\mathrm{d}s\Big\| \\
&\quad + \Big\|\int_0^{t_1}(t_1-s)^{\alpha-1}[T_\alpha(t_2-s)-T_\alpha(t_1-s)]f(s,u(s))\,\mathrm{d}s\Big\| \\
&:= I_1+I_2+I_3,
\end{aligned} \tag{4.3}$$

where

$$\begin{gathered}
I_1 = \Big\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}T_\alpha(t_2-s)f(s,u(s))\,\mathrm{d}s\Big\|, \\
I_2 = \Big\|\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]T_\alpha(t_2-s)f(s,u(s))\,\mathrm{d}s\Big\|, \\
I_3 = \Big\|\int_0^{t_1}(t_1-s)^{\alpha-1}[T_\alpha(t_2-s)-T_\alpha(t_1-s)]f(s,u(s))\,\mathrm{d}s\Big\|.
\end{gathered}$$

By direct calculation we obtain

$$\begin{aligned}
I_1 &\le \frac{\alpha M}{\Gamma(1+\alpha)}\frac{1}{\alpha}(t_2-t_1)^\alpha \sup_{t\in J,u\in W_r} f(s,u(s)) \\
&\le \frac{M}{\Gamma(1+\alpha)}\sup_{t\in J,u\in W_r} f(s,u(s))\cdot(t_2-t_1)^\alpha.
\end{aligned} \tag{4.4}$$

For $t_1 = 0$, $0 < t_2 \le b$, it is easy to see that $I_2 = I_3 = 0$. For $t_1 > 0$, we have

$$\begin{aligned}
I_2 &\le \frac{\alpha M}{\Gamma(1+\alpha)}\sup_{t\in J,u\in W_r} f(s,u(s))\cdot\int_0^{t_1}(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\,\mathrm{d}s \\
&\le \frac{M}{\Gamma(1+\alpha)}\sup_{t\in J,u\in W_r} f(s,u(s))\cdot(t_2^\alpha - t_1^\alpha).
\end{aligned} \tag{4.5}$$

Since f is compact, then $\|[T_\alpha(t_2-s)-T_\alpha(t_1-s)]f(s,u(s))\| \to 0$, as $t_1 \to t_2$, uniformly for $s \in J$ and $u \in W_r$. This implies that, for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\|[T_\alpha(t_2-s)-T_\alpha(t_1-s)]f(s,u(s))\| < \varepsilon,$$

for $0 < t_2 - t_1 < \delta$ and $u \in W_r$. Then we have

$$I_3 \le \varepsilon\int_0^{t_1}(t_1-s)^{\alpha-1}\,\mathrm{d}s \le \frac{b^\alpha}{\alpha}\varepsilon. \tag{4.6}$$

Thus, combining the above inequalities (4.3)–(4.6), we obtain the equicontinuity of GW_r on $[0,b]$.

The set $\{T_\alpha(t-s)f(s,u(s)) : t,s \in J, u \in W_r\} \subset X$ is precompact in X as f is compact and $T(t)$ is continuous. Similarly with the proof of Lemma 3.2, for arbitrary $\varepsilon > 0$, there exists a sequence $\{v_k\}_{k=1}^\infty \subset W_r$, such that

$$\begin{aligned}\beta(G_2W_r(t)) &\le 2\beta(\{G_2v_k(t) : k \ge 1\}) + \varepsilon \\ &\le 4\int_0^t (t-s)^{\alpha-1}\beta(\{T_\alpha(t-s)f(s,v_k(s)) : k \ge 1\})\,\mathrm{d}s + \varepsilon.\end{aligned}$$

Noticing that

$$\beta(\{T_\alpha(t-s)f(s,v_k(s)) : k \ge 1\}) \le \beta(\{T_\alpha(t-s)f(s,u(s)) : t,s \in J, u \in W_r\}) = 0,$$

we have $\beta(G_2W_r(t)) = 0$ for $t \in J$. By the Ascoli-Arzela theorem, we have that G_2 is a compact operator, which implies

$$\beta(G_2W_r) = 0. \tag{4.7}$$

So, according to (4.2) and (4.7), we can conclude that

$$\beta(GW_r) \le \beta(G_1W_r) + \beta(G_2W_r) \le Ml_g\beta(W_r).$$

From the condition $Ml_g < 1$, G is $\beta-$condensing in W_r. By the Darbo-Sadovskii's fixed point theorem, G has at least a fixed point u in W_r, which is just a mild solution of the problem (1.1). The proof is complete. □

By using Banach contraction principle, we also give the existence theorem when f, g are uniformly Lipschitz continuous. We give the following hypothesis on f.

(HF4) $f : C(J;X) \to X$ is continuous and there exists a constant $l_f > 0$, such that
$$\|f(t,x_1) - f(t,x_2)\| \le l_f\|x_1 - x_2\|, \quad x_1, x_2 \in X.$$

Theorem 4.3. *Assume that the hypotheses* (HA), (HG2), (HF4) *are satisfied. Then the nonlocal fractional system* (1.1) *has a unique mild solution on* $[0,b]$, *provided that*

$$Ml_g + \frac{Mb^\alpha}{\Gamma(1+\alpha)}l_f < 1. \tag{4.8}$$

Proof. For $u, v \in C(J;x)$, $t \in J$, we have that

$$\begin{aligned}\|Gu(t) - Gv(t)\| &\le \|S_\alpha(t)[g(u) - g(v)]\| \\ &\quad + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)[f(s,u(s)) - f(s,v(s))]\|\,\mathrm{d}s \\ &\le Ml_g\|u-v\|_C + \frac{\alpha M}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s \cdot l_f\|u-v\|_C \\ &\le \big(Ml_g + \frac{Mb^\alpha}{\Gamma(1+\alpha)}l_f\big)\|u-v\|_C.\end{aligned}$$

Then

$$\|Gu - Gv\|_C \le \big(Ml_g + \frac{Mb^\alpha}{\Gamma(1+\alpha)}l_f\big)\|u-v\|_C.$$

According to (4.8), we find that G is a contraction operator in $C(J;X)$. Thus G has a unique fixed point u, which is the unique mild solution to the problem (1.1). The proof is complete. □

Conclusions. This article is motivated by some recent papers [22, 31, 21], where some fractional nonlocal differential equations are discussed when $T(t)$ is compact or equicontinuous. Since it is difficult to determine whether an operator semigroup is compact (see Pazy[24]), we do not assume that A generates a compact semigroup. It allow us to discuss some differential equations which contain a linear operator that generates a noncompact semigroup. We give a simple example. Let $X = L^2(-\infty, +\infty)$. The ordinary differential operator $A = \mathrm{d}/\mathrm{d}x$ with $D(A) = H^1(-\infty, +\infty)$, generates a semigroup $T(t)$ defined by $T(t)u(s) = u(t+s)$, for every $u \in X$. The C_0-semigroup $T(t)$ is not compact on X.

Another motivation of this paper is the control problem of fractional differential system. Exact controllability for fractional order systems have been discussed by many authors. Some controllability results are obtained with the assumptions that the associated semigroup $T(t)$ is compact and the inverse of control operator is bounded. However, Hernández and O'Regan [13] have pointed out that in this case the application of controllability result is restricted to the finite dimensional space. Here we can also deal with the fractional control problem in the similar way and give a way to remove the compactness assumptions for the nonlocal control problems.

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