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INVERSE SCATTERING PROBLEMS FOR ENERGY-DEPENDENT STURM-LIOUVILLE EQUATIONS WITH POINT δ-INTERACTION AND EIGENPARAMETER-DEPENDENT BOUNDARY CONDITION

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In memory of M. G. Gasymov, one of the pioneers of this subject

Abstract. We consider an inverse problem of the scattering theory for energy-dependent Sturm-Liouville equations on the half line $[0,+\infty)$ with point δ-interaction and eigenparameter-dependent boundary condition. We define the scattering data of the problem first, then consider the basic equation and study an algorithm for finding the potentials with the given scattering data.

1. Introduction

We consider inverse scattering problem for the equation

$$-y'' + q(x)y = \lambda^2 y, \quad x \in (0,a) \cup (a,+\infty) \tag{1.1}$$

with the boundary condition

$$U(y) := \lambda^2(y'(0) - hy(0)) - (h_1 y'(0) - h_2 y(0)) = 0 \tag{1.2}$$

and conditions at the point $x = a$,

$$I(y) := \begin{cases} y(a+0) = y(a-0) = y(a), \\ y'(a+0) - y'(a-0) = 2i\alpha\lambda y(a), \end{cases} \tag{1.3}$$

where λ is a spectral parameter, $q(x)$ is real-valued function satisfying the condition

$$\int_0^\infty (1+x)|q(x)|dx < \infty,$$

and $\alpha < 0$, h, h_1, h_2 are real numbers such that

$$\delta := hh_1 - h_2 > 0.$$

Notice that, we can understand problem (1.1) and (1.3) as one of the study of the equation

$$y'' + (\lambda^2 - 2i\lambda p(x) - q(x))y = 0, \quad x \in (0,+\infty), \tag{1.4}$$

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when $p(x) = \alpha\delta(x-a)$, where $\delta(x)$ is the Dirac function, (see [4]).

Sturm-Liouville spectral problems with potentials depending on the spectral parameter arise in various models of quantum and classical mechanics. For instance, to this form can be reduced the corresponding evolution equations (such as the Klein-Gordon equation [12], [21]) that are used to model interactions between colliding relativistic spinless particles. Then λ^2 is related to the energy of the system, this explaining the term "energy-dependent" in (1.4).

Problems of the form (1.4) have also appeared in the physical literature in the context of scattering of waves and particles. In particular, in [9]-[11] the inverse scattering problems for energy-dependent Schrödinger operators on the line are studied, see also [2, 8, 13, 15, 19, 20].

The non-linear dependence of equation (1.4) on the spectral parameter λ should be regarded as a spectral problem for a quadratic operator pencil. The problem with $p(x) \in W_2^1(0,1)$ and $q(x) \in L_2(0,1)$ and with Robin boundary conditions was discussed in [5]. Such problems for separated and nonseparated boundary conditions were considered (see [1, 6, 22, 23] and the references therein).

The inverse scattering problem for equation (1.1) (in the case $\alpha = 0$) with boundary condition $y'(0) - hy(0) = 0$ was completely solved in [16] (in the case $h = \infty$), in [14] (for arbitrary real number h). Similar problems were dealt with in [3] and references in these works. The inverse scattering problem for equation (1.4) (in the case $\alpha = 0$) with spectral parameter in the boundary condition on the half line were examined in [17], [18].

In this paper we consider the inverse problem scattering theory on the half line $[0,\infty)$ for the (1.4), (1.2) boundary-value problem.

2. Integral representation for the Jost solution and scattering data

In this section, we will find an integral representation for the Jost solution and study some properties of scattering data of the problem.

Let us denote by $e_0(x,\lambda)$ the solution of (1.1), when $q(x) \equiv 0$, satisfying conditions (1.3) and the condition at infinity

$$\lim_{x\to\infty} e_0(x,\lambda)e^{-i\lambda x} = 1\,.$$

This function is called the Jost solution.

It is obvious that the function $e_0(x,\lambda)$ can be written as

$$e_0(x,\lambda) = \begin{cases} (1-\alpha)e^{i\lambda x} + \alpha e^{i\lambda(2a-x)}, & 0 < x < a \\ e^{i\lambda x}, & x > a \end{cases}$$

Analogously to [7] the following theorem can be proved.

Theorem 2.1. *Let $\int_0^\infty (1+x)|q(x)|dx < \infty$. Then the Jost solution of (1.1) has the form*

$$e(x,\lambda) = e_0(x,\lambda) + \int_x^\infty K(x,t)e^{i\lambda t}dt \quad (\operatorname{Im}\lambda \ge 0), \tag{2.1}$$

where for each fixed $x \ne a$ the kernel $K(x,\cdot)$ belong to the $L_1(x,\infty)$ and

$$\int_x^\infty |K(x,t)|dt \le e^{c\sigma_1(x)} - 1,$$

where $\sigma_1(x) = \int_0^\infty t|q(t)|dt$, $c = 1 - \frac{\alpha}{2}$.

If the function $q(x)$ is differentiable then the kernel $K(x,t)$ satisfies the following properties:

$$
\begin{aligned}
&K_{xx}(x,t) - q(x)K(x,t) = K_{tt}(x,t), \quad 0 < x < \infty,\ t > x,\\
&K(x,x) = \begin{cases} \frac{1}{2}\int_x^\infty q(s)ds, & x > a\\ \frac{1}{2}(1-\alpha)\int_x^\infty q(s)ds, & x < a \end{cases}\\
&K(x,2a-x+0) - K(x,2a-x-0)\\
&= \frac{\alpha}{2}\Big\{\int_a^\infty q(s)ds - \int_a^x q(s)ds\Big\}, \quad x < a.
\end{aligned} \tag{2.2}
$$

Now we learn some properties of scattering data of the problem. Since the function $q(x)$ is real-valued, it follows that for real λ together with $e(x,\lambda)$ the solution of (1.1) are also $\overline{e(x,\lambda)}$. Since the Wronskian of the two solutions $y_1(x)$ and $y_2(x)$ of equation (1.1),

$$W\{y_1(x), y_2(x)\} = y_1'(x)y_2(x) - y_1(x)y_2'(x),$$

is independent of x, it coincides with its own value as $x \to \infty$. Therefore

$$W\{e(x,\lambda), \overline{e(x,\lambda)}\} = \lim_{x\to+\infty}[e'(x,\lambda)\overline{e(x,\lambda)} - e(x,\lambda)\overline{e'(x,\lambda)}] = 2i\lambda. \tag{2.3}$$

Consequently for $0 \neq \lambda \in \mathbb{R}$ the pair $\{e(x,\lambda), \overline{e(x,\lambda)}\}$ is a fundamental system of solutions of (1.1).

Let $\varphi(x,\lambda)$ be the solution of (1.1) satisfying the initial condition

$$\varphi(0,\lambda) = \lambda^2 - h_1, \quad \varphi'(0,\lambda) = \lambda^2 h - h_2. \tag{2.4}$$

Lemma 2.2. *The equality*

$$\frac{2i\lambda\varphi(x,\lambda)}{(\lambda^2-h_1)e'(0,\lambda) - (\lambda^2 h - h_2)e(0,\lambda)} = \overline{e(x,\lambda)} - S(\lambda)e(x,\lambda) \tag{2.5}$$

holds for all real $\lambda \neq 0$, where

$$S(\lambda) = \frac{(\lambda^2-h_1)\overline{e'(0,\lambda)} - (\lambda^2 h - h_2)\overline{e(0,\lambda)}}{(\lambda^2-h_1)e'(0,\lambda) - (\lambda^2 h - h_2)e(0,\lambda)} \tag{2.6}$$

with

$$S(\lambda) = \overline{S(-\lambda)} = [S(-\lambda)]^{-1}$$

Proof. The functions $e(x,\lambda)$ and $\overline{e(x,\lambda)}$ form a fundamental system of solutions of (1.1) for all real $\lambda \neq 0$, then we can write

$$\varphi(x,\lambda) = c_1(\lambda)e(x,\lambda) + c_2(\lambda)\overline{e(x,\lambda)}, \tag{2.7}$$

where

$$
\begin{aligned}
c_1(\lambda) &= -\frac{(\lambda^2-h_1)\overline{e'(0,\lambda)} - (h\lambda^2 - h_2)\overline{e(0,\lambda)}}{2i\lambda},\\
c_2(\lambda) &= \frac{(\lambda^2-h_1)e'(0,\lambda) - (h\lambda^2 - h_2)e(0,\lambda)}{2i\lambda}.
\end{aligned}
$$

To prove that $\psi(\lambda) \equiv (\lambda^2-h_1)e'(0,\lambda) - (\lambda^2 h - h_2)e(0,\lambda) \neq 0$ for all real $\lambda \neq 0$, we assume that $\lambda_0 \in (-\infty,+\infty)\backslash\{0\}$ such that

$$(\lambda_0^2 - h_1)e'(0,\lambda_0) - (\lambda_0^2 h - h_2)e(0,\lambda_0) = 0$$

or

$$e'(0,\lambda_0) = \frac{(\lambda_0^2 h - h_2)}{\lambda_0^2 - h_1} e(0,\lambda_0)$$

From (2.3), we obtained the equation

$$|e(0,\lambda_0)|^2 \frac{0}{|\lambda_0^2 - h_1|^2} = 2i\lambda_0;$$

i.e., we have a contradiction. Hence $\psi(\lambda) \neq 0$ for all real $\lambda \neq 0$. Dividing both sides of (2.7) by $\psi(\lambda)/2i\lambda$ we obtain (2.5) where $S(\lambda)$ is defined with (2.6). Since

$$\psi(\lambda) = \overline{\psi(-\lambda)} = [\psi(-\lambda)]^{-1}$$

we have

$$S(\lambda) = \overline{S(-\lambda)} = [S(-\lambda)]^{-1}.$$

□

The function $S(\lambda)$ is called *the scattering function* of the problem (1.1)-(1.2)-(1.3).

Lemma 2.3. *The function $\psi(\lambda)$ may have only a finite number of zeros in the half-plane* $\operatorname{Im}\lambda > 0$. *Moreover, all these zeros are simple and lie in the imaginary axis.*

Proof. Since for real values of $\lambda \neq 0$ the inequality $\psi(\lambda) \neq 0$ holds, the only possible real zero of the function $\psi(\lambda)$ is $\lambda = 0$. Since the function $\psi(\lambda)$ is analytic on the upper half-plane it follows that its zeros form a (finite or countable) set.

Let us show that this set is bounded. Assume the converse and suppose that there exist λ_k such that $|\lambda_k| \to \infty$, with Im $\lambda_k > 0$ and

$$(\lambda_k^2 - h_1)e'(0,\lambda_k) = (\lambda_k^2 h - h_2)e(0,\lambda_k).$$

Then as $|\lambda_k| \to \infty$ yields $\lim_{k\to\infty} e(0,\lambda_k) = 0$. On the other hand, it follows from (2.3) that $\lim_{k\to\infty} e(0,\lambda_k) = 1$. The resulting contradiction shows that the set $\{\lambda_k\}$ is bounded. Thus, the zeros of the function $\psi(\lambda)$ form a bounded finite or countable set whose unique limiting point can only be zero.

Now let us show that all the zeros of the function $\psi(\lambda)$ lie on the imaginary axis. Suppose that λ_1 and λ_2 are some zeros of the function $\psi(\lambda)$. Since

$$-e''(x,\lambda_1) + q(x)e(x,\lambda_1) = \lambda_1^2 e(x,\lambda_1), \quad -\overline{e''(x,\lambda_2)} + q(x)\overline{e(x,\lambda_2)} = \lambda_2^2 \overline{e(x,\lambda_2)}.$$

If the first equation is multiplied by $\overline{e(x,\lambda_2)}$, second equation is multiplied by $e(x,\lambda_1)$ and subtracting them side by side and finally integrating over the interval $[0,+\infty)$, the equality

$$(\lambda_1^2 - \overline{\lambda}_2^2)\int_0^\infty e(x,\lambda_1)\overline{e(x,\lambda_2)}dx + W\{e(x,\lambda_1),\overline{e(x,\lambda_2)}\}(|_0^a + |_a^{+\infty}) = 0$$

is attained.

If conditions (1.3) and

$$W\{e(x,\lambda_1),\overline{e(x,\lambda_2)}\}_{x=a-0} = W\{e(x,\lambda_1),\overline{e(x,\lambda_2)}\}_{x=a+0}$$

are considered, then

$$(\lambda_1^2 - \overline{\lambda}_2^2)\int_0^\infty e(x,\lambda_1)\overline{e(x,\lambda_2)}dx + W\{e(x,\lambda_1),\overline{e(x,\lambda_2)}\}_{x=0} = 0 \tag{2.8}$$

is obtained. On the other hand, according to the definition of the function $\psi(\lambda)$, the following relation holds

$$\psi(\lambda_j) = (\lambda_j^2 - h_1)e'(0,\lambda_j) - (\lambda_j^2 h - h_0)e(0,\lambda_j) = 0, \quad j = 1,2.$$

Also, by (1.2) we can write

$$W\{e(x,\lambda_1), \overline{e(x,\lambda_2)}\}_{x=0} = \frac{[e'(0,\lambda_1) - he(0,\lambda_1)][\overline{e'(0,\lambda_2)} - h\overline{e(0,\lambda_2)}]}{\delta}(\lambda_2^{-2} - \lambda_1^2). \tag{2.9}$$

Therefore, using (2.8) and (2.9) we have

$$(\lambda_1^2 - \lambda_2^{-2})\{\int_0^\infty e(x,\lambda_1)\overline{e(x,\lambda_2)}dx + \frac{1}{\delta}[e'(0,\lambda_1) - he(0,\lambda_1)][\overline{e'(0,\lambda_2)} - h\overline{e(0,\lambda_2)}]\} = 0. \tag{2.10}$$

In particular, the choice $\lambda_2 = \lambda_1$ at (2.10) implies that $\lambda_1 = ik_1$, where $k_1 \geq 0$. Therefore, zeros of the function $\psi(\lambda)$ can lie only on the imaginary axis. Now, let us prove that function $\psi(\lambda)$ has zeros in finite numbers. Since we can give an estimate for the distance between the neighboring zeros of the function $\psi(\lambda)$, it follows that the number of zeros is finite, (see [16, p. 186]). $\square$

Let

$$m_p^{-2} \equiv \int_0^\infty |e(x,ik_p)|^2 dx + \frac{1}{\delta}|e'(0,ik_p) - he(0,ik_p)|^2, \quad p = 1,2,\dots,n. \tag{2.11}$$

These numbers are called the *norming constants* for the boundary problem (1.4), (1.2).

The collections $\{S(\lambda), (-\infty < \lambda < +\infty); k_p, m_p (p = 1,2,\dots,n)\}$ are called the *scattering data* of the boundary value problem (1.1)-(1.2)-(1.3) or (1.4)-(1.2).

3. Basic equation of the inverse scattering problem

To derive the basic equation for the kernel $K(x,t)$ of the solution (2.1), we use equality (2.5). Substituting expression (2.1) for $e(x,\lambda)$ into this equality, we get

$$\begin{aligned} &\frac{2i\lambda\varphi(x,\lambda)}{\psi(\lambda)} - \overline{e_0(x,\lambda)} + S_0(\lambda)e_0(x,\lambda) \\ &= \int_x^\infty K(x,t)e^{-i\lambda t}dt + [S_0(\lambda) - S(\lambda)]e_0(x,\lambda) - S_0(\lambda)\int_x^\infty K(x,t)e^{i\lambda t}dt \\ &\quad + (S_0(\lambda) - S(\lambda))\int_x^\infty K(x,t)e^{i\lambda t}dt. \end{aligned} \tag{3.1}$$

Multiplying this equality by $(1/2\pi)e^{i\lambda r}$ and integrating over λ from $-\infty$ to $+\infty$, for $r > x$, at the right-hand side we obtain

$$\begin{aligned} &K(x,r) + \frac{1}{2\pi}\int_{-\infty}^\infty [S_0(\lambda) - S(\lambda)]e_0(x,r)e^{i\lambda r}d\lambda \\ &- \int_x^\infty K(x,t)\Big\{\frac{1}{2\pi}\int_{-\infty}^\infty S_0(\lambda)e^{i\lambda(t+r)}d\lambda\Big\}dt \\ &+ \int_x^\infty K(x,t)\Big\{\frac{1}{2\pi}\int_{-\infty}^\infty [S_0(\lambda) - S(\lambda)]e^{i\lambda(t+r)}d\lambda\Big\}dt. \end{aligned} \tag{3.2}$$

Now we will compute the integral $(1/2\pi)\int_{-\infty}^{\infty} S_0(\lambda)e^{i\lambda(t+r)}d\lambda$. By elementary transforms we obtain

$$S_0(\lambda) = \frac{\tau^2-1}{1+\tau e^{2i\lambda a}} + \tau e^{-2i\lambda a} = (\tau^2-1)\sum_{k=0}^{\infty}(-1)^k\tau^k e^{2i\lambda ak} + \tau e^{-2i\lambda a}$$

where $\tau = \alpha/(\alpha-1)$. Thus we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_0(\lambda)e^{i\lambda(t+r)}d\lambda = (\tau^2-1)\sum_{k=0}^{\infty}(-1)^k\tau^k\delta(t+r+2ak) + \tau\delta(t+r-2a).$$

Consequently (3.2) can be written as

$$\begin{aligned}&K(x,r) + F_S(x,r) + \int_x^{\infty} K(x,t)F_{0S}(t+r)dt - \tau K(x,2a-r)\\ &- (\tau^2-1)\sum_{k=0}^{\infty}(-1)^k\tau^k K(x,-2ak-r),\end{aligned}$$

where

$$\begin{gathered}F_{0S}(x) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}[S_0(\lambda)-S(\lambda)]e^{i\lambda x}d\lambda,\\ F_S(x,r) \equiv (1-\alpha)F_{0S}(x+r) + \alpha F_{0S}(2a-x+r).\end{gathered}$$

We note that $K(x,r)=0$ for $r<x$ (see [7]). Therefore, for $r>x$, (2.4) takes the form

$$K(x,r) + F_S(x,r) + \int_x^{\infty} K(x,r)F_{0S}(t+r)dt - \tau K(x,2a-r). \tag{3.3}$$

On the left-hand side of (3.1) with help of Jordan's lemma and the residue theorem and by taking Lemma 2.3 into account for $r>x$, we obtain

$$-\sum_{p=1}^{n}\frac{2ik_p\varphi(x,ik_p)}{\psi'(ik_p)}e^{-k_pr}. \tag{3.4}$$

From the definition of norming constants m_p ($p=1,2,\dots,n$) in (2.11) we have

$$\begin{aligned}&-\sum_{p=1}^{n}\frac{2ik_p\varphi(x,ik_p)}{\psi'(ik_p)}e^{-k_pr}\\ &= -\sum_{p=1}^{n}\frac{2ik_pe^{-k_pr}e(x,ik_p)}{[e'(0,ik_p)-he(0,ik_p)]\psi'(ik_p)}\\ &= -\sum_{p=1}^{n}m_p^2e(x,ik_p)e^{-k_pr}\\ &= -\sum_{p=1}^{n}m_p^2\{e(x,ik_p)e^{-k_p(x+r)} + \int_x^{\infty}K(x,t)e^{-k_p(x+r)}dt\}.\end{aligned} \tag{3.5}$$

Thus, for $x<\min(r,2a-r)$, by taking (3.3) and (3.5) into account, from (3.2) we derive the relation

$$K(x,r) + F(x,r) + \int_x^{\infty} K(x,t)F_0(t+r)dt + \frac{\alpha}{1-\alpha}K(x,2a-r) = 0 \tag{3.6}$$

where

$$
\begin{aligned}
F_0(x) &= F_{0S}(x) + \sum_{p=1}^{n} m_p^2 e^{-k_p r}, \\
F(x,r) &= F_S(x,r) + \sum_{p=1}^{n} m_p^2 e_0(x, ik_p) e^{-k_p(x+r)}.
\end{aligned}
\tag{3.7}
$$

Equation (3.6) is called the *basic equation* of the inverse problem of the scattering theory for the boundary problem (1.1)-(1.2)-(1.3) or (1.4)-(1.2). The basic equation is different from the classical equation of Marchenko (see [16]). Thus, we have proved the following theorem.

Theorem 3.1. *For each $x \geq 0$, the kernel $K(x,r)$ of the solution* (2.1) *satisfies the basic equation* (3.6).

Obviously, to form the basic equation (3.6), it suffices to know the functions $F_0(x)$ and $F(x,r)$. In turn, in order to find the functions $F_0(x)$, $F(x,r)$, it suffices to know only the scattering data $\{S(\lambda)\,(-\infty < \lambda < \infty); k_p, m_p\,(p = 1, 2, \dots, n)\}$. Given the scattering data, we can use formulas (3.7) to construct the functions $F_0(x)$, $F(x,r)$ and write down the basic equation (3.6) for the unknown solution $K(x,r)$, the function $q(x)$ may be found from formula (2.2).

Theorem 3.2. *Equation* (3.6) *has a unique solution $K(x,\cdot) \in L_1(x,+\infty)$ for each fixed $x \geq 0$.*

Proof. It is easy to show that for each fixed $x \geq 0$ the operator

$$
(M_x f)(r) = \begin{cases} f(r), & x > a \\ f(r) + \frac{\alpha}{1-\alpha} f(2a - r), & x < a, \end{cases}
$$

acting in the space $L_1(x,+\infty)$ (and also in $L_2(x,+\infty)$) is invertible. Therefore, basic equation (3.6) is equivalent to

$$
K(x,r) + (M_x)^{-1} F(x,r) + (M_x)^{-1}(F_0 K(x,\cdot))(r) = 0;
$$

i.e. to the equation with a compact operator $(M_x)^{-1}F$ (for the compactness of F, see [16, Lemma 3.3.1]). To prove the theorem, it is sufficient to show that the homogeneous equation

$$
f_x(r) + \frac{\alpha}{1-\alpha} f_x(2a - r) + \int_x^\infty f_x(t) F_0(t + r)\,dt = 0, \quad x < \min(r, 2a - r), \tag{3.8}
$$

has only the trivial solution $f_x(r) \in L_1(x,+\infty)$. We can show that (see [16]) the function $F_0(r)$ belongs to the space $L_2[0,+\infty)$, is absolutely continuous on all the intervals not containing the point $2a$; and for all $\beta \geq 0$

$$
\int_\beta^\infty |F_0(r)|\,dr < +\infty, \quad \int_\beta^\infty (1 + r)|F'(r)|\,dr < +\infty.
$$

Therefore, the function $F(r)$ and the solution $f_x(r)$ are together bounded on the semi-axis $x \leq y < +\infty$. Consequently, $f_x(r) \in L_2(x,+\infty)$.

Now let us multiply (3.8) by $\overline{f_x(r)}$ and integrate with respect to y over the interval $(x,+\infty)$. Using (3.6), (3.7) and Parseval's identity

$$
\int_x^\infty |f_x(r)|^2\,dr = \frac{1}{2\pi} \int_{-\infty}^\infty |\widetilde{f}_x(\lambda)|^2\,d\lambda,
$$

$$\frac{\alpha}{1-\alpha}\int_x^\infty f_x(2a-r)\overline{f_x(r)}dr = \frac{1}{2\pi}\int_{-\infty}^\infty S_0(\lambda)\overline{\widetilde{f}_x(\lambda)}\widetilde{f}_x(-\lambda)d\lambda,$$

where

$$\widetilde{f}_x(\lambda) = \int_x^\infty f_x(t)e^{-i\lambda t}dt,$$

we obtain

$$\frac{1}{2\pi}\int_{-\infty}^\infty |\widetilde{f}_x(\lambda)|^2 d\lambda + \sum_{p=1}^n m_p|\widetilde{f}_x(-ik_p)|^2 + \frac{1}{2\pi}\int_{-\infty}^\infty S(\lambda)\widetilde{f}_x(-\lambda)\overline{\widetilde{f}_x(\lambda)}d\lambda = 0.$$

Since $|S(\lambda)| = |S(-\lambda)|$, we obtain the estimate

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^\infty |\widetilde{f}_x(\lambda)|^2 d\lambda &\le \frac{1}{2\pi}\int_{-\infty}^\infty |S(\lambda)||\overline{\widetilde{f}_x(-\lambda)}||\widetilde{f}_x(\lambda)|d\lambda \\ &\le \frac{1}{2\pi}\int_{-\infty}^\infty |S(\lambda)|\frac{|\widetilde{f}_x(-\lambda)|^2 + |\widetilde{f}_x(\lambda)|^2}{2}d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty |S(\lambda)||\widetilde{f}_x(\lambda)|^2 d\lambda\end{aligned}$$

or

$$\frac{1}{2\pi}\int_{-\infty}^\infty \{1-|S(\lambda)|\}|\widetilde{f}_x(\lambda)|^2 d\lambda \le 0.$$

It follows from the above that $\widetilde{f}_x(\lambda) \equiv 0$ since $1-|S(\lambda)| > 0$ for all $\lambda \neq 0$. Thus, the basic equation (3.6) is uniquely solvable. □

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