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EXISTENCE OF SOLUTION FOR ASYMPTOTICALLY LINEAR SYSTEMS IN $\mathbb{R}^N$

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Abstract. We show the existence of solution for a strongly coupled elliptic system under three different conditions, namely: the autonomous case, the radial case, and a general case. For the autonomous case, we present a characterization of the solution.

1. Introduction

Over the previous years, nonlinear optic has attracted much attention from physicists and mathematicians, due to its applicability on the knowledge of how the light transmission behaves on high velocity. More precisely, we call attention for spatial optical solitons, which have the interesting property of maintaining their shape during propagation (see [1] for more details).

The Schrödinger equation, even though it represents well the auto-interaction of the light beam, does not consider a possible interaction with the material, and the passage of a ray along different materials can produce several nonlinear effects, such as the birefringence effect, when the ray is decomposed in two. To study this situation, in [40] the authors considered a weakly coupled system of Schrödinger equations. As observed by Manakov in [32], vector solitons were generated; i.e., solitary waves with multiple components coupled together, but still with the good properties of the scalar ones. Crystals photorefractive are the material usually used for these experiments because, among other features, their refractive index change when light goes through them, and because of this the wave does not change its shape during propagation. However, it is necessary to take into account the saturation effect of the material, when the refractive index reaches an upper bound and thus ceases to increase [28, 37, 40].

The nonlinear weakly coupled system of Schrödinger equations

$$\begin{aligned} i\varphi_t + \Delta\varphi + \frac{\alpha(|\varphi|^2 + |\psi|^2)}{1 + (|\varphi|^2 + |\psi|^2)/I_0}\varphi = 0 \\ i\psi_t + \Delta\psi + \frac{\alpha(|\varphi|^2 + |\psi|^2)}{1 + (|\varphi|^2 + |\psi|^2)/I_0}\psi = 0 \end{aligned} \tag{1.1}$$

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represents the propagation of a beam with two mutually incoherent components in a bulk saturable medium in the isotropic approximation, where φ and ψ represent the amplitudes of the components of the beam, α is the strength of the nonlinearity, I_0 is the saturation parameter and the expression $(|\varphi|^2 + |\psi|^2)$ represents the total intensity created by all the incoherent components of the beam. If we consider standing wave solutions $\varphi(x,t) = \sqrt{\alpha}u(x)e^{i\lambda_1 t}$ and $\psi(x,t) = \sqrt{\alpha}v(x)e^{i\lambda_2 t}$, for u and v real functions and λ_1 and λ_2 propagation constants, we obtain the following weakly coupled elliptic system

$$\begin{aligned} -\Delta u + \lambda_1 u &= \frac{u^2+v^2}{1+s(u^2+v^2)}u \\ -\Delta v + \lambda_2 v &= \frac{u^2+v^2}{1+s(u^2+v^2)}v, \end{aligned} \tag{1.2}$$

where $s = \alpha/I_0$, $N \geq 2$, and u and v belong to the Sobolev space $H^1(\mathbb{R}^N)$.

When $s = 0$ in (1.2), it is known that under suitable parameter conditions, there exist minimal energy vector solutions [5, 8, 22, 29, 36]. In [30], the authors studied the existence of minimal energy vector solutions for such system with $s > 0$. Numerical analysis for the system (1.2) was made in [13, 19, 28, 33] for $N = 1$ and $s > 0$, while the discrete case was considered in [34].

Motivated by the work in [4] from Ambrosetti, Cerami and Ruiz, where a coupled system of nonautonomous equations with sub-critical nonlinearity was studied under several conditions, we decided here to consider the following strongly coupled system, with nonlinearity asymptotically linear at infinity, in $\mathbb{R}^N$ for $N \geq 3$:

$$\begin{aligned} -\Delta u + u &= \frac{u^2+v^2}{1+(s+a(x))(u^2+v^2)}u + \lambda v \\ -\Delta v + v &= \frac{u^2+v^2}{1+(s+a(x))(u^2+v^2)}v + \lambda u. \end{aligned} \tag{1.3}$$

Because of the coupling constant λ, if one of the entries of the vector solution (u, v) is trivial, then necessarily the other entry is also trivial. Therefore, if u_0 is a nontrivial solution for the equation

$$-\Delta u + u = \frac{u^3}{1+(s+a(x))u^2},$$

the vectors $(u_0, 0)$ and $(0, u_0)$ will not be solutions for the system (1.3).

The problem of finding solutions for the general equation in $\mathbb{R}^N$

$$-\Delta u + V(x)u = K(x)f(u), \tag{1.4}$$

with $\lim_{|x|\to\infty} u(x) = 0$, has been extensively studied, under several conditions on the potential V and the weight K. In 1983, in a pioneer work, Berestycki and Lions [9] considered the autonomous equation $-\Delta u + mu = f(u)$, in $\mathbb{R}^N$. They showed the existence of a nontrivial solution for such equation, using constrained minimization, when f has sub-critical growth at infinity.

In [39], Stuart and Zhou proved the existence of a positive radial solution for the equation $-\Delta u + \lambda u = f(|x|, u(x))u(x)$, where the nonlinearity f was asymptotically linear.

Considering V and K periodic, we recall the works of Alama and Li [2] and Coti-Zelati and Rabinowitz [18]. We also cite the work of Bartolo, Benci and Fortunato [7], where nonlinear problems with 'strong resonance' at infinity were considered. Also, we cite the works of Costa and Tehrani [17] and Li and Zhou, [26] where the equation $-\Delta u + \lambda u = f(x,u)u$ was considered, with the function f being asymptotically linear, and the work [6] of Bahri and Li, where the equation $-\Delta u + u - q(x)|u|^{p-1}u = 0$, for $0 < p < \frac{N+2}{N-2}, N \geq 3$ was considered.

As the equation (1.4), the system (1.3) and its related problems were studied in several works in the past years. In 1984, Brezis and Lieb [11], applying constrained minimization methods, proved the existence of a solution $(u,v) \neq (0,0)$ for a class of autonomous systems, including ours when $a(x) \equiv 0$. For bounded domains, Costa and Magalhães considered sub-quadratical elliptic systems and noncooperative systems ([15, 16], respectively). In [23], Furtado, Maia and Silva considered a system similar to (1.3), noncoupled and with a superlinear nonlinearity, while our nonlinearity is asymptotically linear. We also mentioned the work [3] from A. Ambrosetti, where it was showed the existence of a solution for an elliptic system in $\mathbb{R}^N$, by perturbation methods.

Instead of the usual Palais-Smale condition, we will use the Cerami condition:

> a functional $I \in C^1(X,\mathbb{R})$ satisfies the Cerami condition (Ce) if every sequence $(z_n) \subset X$ with $|I(z_n)| < M$ and $\|I'(z_n)\|(1+\|z_n\|) \to 0$ has a convergent subsequence $z_{n_k} \to z \in X$. A functional $I \in C^1(X,\mathbb{R})$ satisfies the Cerami condition at level c, $(Ce)_c$, if every sequence $(z_n) \subset X$ with $I(z_n) \to c$ and $\|I'(z_n)\|(1+\|z_n\|) \to 0$ has a convergent subsequence $z_{n_k} \to z \in X$.

This will be done since, by the structure of our system, with a nonlinearity that satisfies the nonquadraticity condition stated by Costa and Magalhães in [14], we can adapt for the system an argument presented by Stuart and Zhou in [39], in order to show that any Cerami sequence has a bounded subsequence.

The paper is structured as follows: in the second section, we take the function a identically 0 in the system (1.3). We prove the existence of a radial positive ground state solution for this autonomous system, and we make a characterization of such solution. In section 3, we consider the system (1.3) assuming that a is a radial function and we prove the existence of a radial nontrivial solution for the system. Finally, in section 4, we consider general conditions on the function a and again are able to prove, under these conditions, the existence of a nontrivial solution for the system.

2. The Autonomous System

In this section, we will study the following autonomous system in $\mathbb{R}^N$ for $N \geq 3$,

$$\begin{aligned} -\Delta u + u &= \frac{u^2+v^2}{1+s(u^2+v^2)}u + \lambda v \\ -\Delta v + v &= \frac{u^2+v^2}{1+s(u^2+v^2)}v + \lambda u. \end{aligned} \tag{2.1}$$

where we are assuming that s and λ are constants satisfying $0 < s < 1$ and $0 < \lambda < 1$. First, working only on the subspace of radial functions $H^1_{\rm rad}(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)$, we will show that this system has a nontrivial radial solution, by the Mountain Pass Theorem (see [7]). Then, we will obtain a ground state solution (possibly nonradial), by constrained minimization and, following the ideas of Jeanjean and Tanaka [25], we will make a characterization of this ground state solution. We will conclude this section by proving the following theorem.

Theorem 2.1. *The system* (2.1) *has a positive radial ground state solution, obtained by the Mountain Pass Theorem.*

We will work on the space $E = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ with the norm

$$\|(u,v)\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 + u^2 + v^2 dx.$$

We will also use the notation

$$\|(u,v)\|_p = \Big(\int_{\mathbb{R}^N} |u|^p + |v|^p dx \Big)^{1/p}, \forall p \in [1,\infty)$$

for the usual norm on $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$.

The following functional is associated with the system (2.1):

$$I_\infty(u,v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 + u^2 + v^2 dx - \int_{\mathbb{R}^N} H_\infty(u,v)dx - \lambda \int_{\mathbb{R}^N} uvdx \quad (2.2)$$

where

$$H_\infty(u,v) = \frac{u^2+v^2}{2s} - \frac{1}{2s^2} \ln(1 + s(u^2+v^2)), \quad (2.3)$$

and the derivative of the functional I_∞ is given by

$$\begin{aligned} \nabla I_\infty(u,v)(\varphi,\psi) &= \int_{\mathbb{R}^N} \nabla u \nabla \varphi + \nabla v \nabla \psi + u\varphi + v\psi dx \\ &\quad - \int_{\mathbb{R}^N} \frac{u^2+v^2}{1+s(u^2+v^2)} (u\varphi + v\psi) - \lambda(v\varphi + u\psi) dx. \end{aligned}$$

We will look for critical points of I_∞, which will be solutions to (2.1).

Remark 2.2. First, we observe that if there exists a positive solution (u,v) of (2.1), then this solution is radial. This result follows from Theorem 2, [12]. Therefore, we are motivated to search for solutions to (2.1) on $E_{\rm rad} = H^1_{\rm rad}(\mathbb{R}^N) \times H^1_{\rm rad}(\mathbb{R}^N)$ and we will consider on $E_{\rm rad}$ the norm

$$\|(u,v)\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 + u^2 + v^2 dx.$$

We also define the function F_∞ on E by

$$F_\infty(u,v) := H_\infty(u,v) + \lambda uv. \quad (2.4)$$

Lemma 2.3. *The function F_∞ satisfies the nonquadraticity condition* (NQ)*; i.e.,*

$$\lim_{|(u,v)| \to \infty} \frac{1}{2} \nabla F_\infty(u,v)(u,v) - F_\infty(u,v) = +\infty,$$

and

$$\frac{1}{2} \nabla F_\infty(u(x),v(x))(u(x),v(x)) - F_\infty(u(x),v(x)) \geq 0, \forall x \in \mathbb{R}^N.$$

This condition was first given by Costa and Magalhães in [14] and the proof of this lemma can be found in [31], Lemma 2.1.

In this article, we will use of a version of the Mountain Pass Theorem for Cerami sequences, whose proof can be found in [7]. In what follows, B_ρ represents an open ball in $\mathbb{R}^N$, centered at the origin, with radius ρ.

Theorem 2.4 (Mountain Pass Theorem)**.** *Let X be a real Banach space and $I \in C^1(X,\mathbb{R})$ be a functional satisfying (Ce) with $I(0) = 0$. Suppose that:*

(I1) *there exist constants* α, $\rho > 0$ *such that* $I\big|_{\partial B_\rho} \geq \alpha$,

(I2) *there exists* $e \in X \setminus \partial\overline{B_\rho}$ *such that* $I(e) \leq 0$.

Also, consider the set $\Gamma = \{\gamma \in C([0,1], X); \gamma(0) = 0$ *and* $I(\gamma(1)) < 0\}$. *Then*

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t))$$

is a critical value of I.

Lemma 2.5. *The functional* I_∞ *satisfies the geometric conditions of Theorem 2.4.*

Proof. By the structure of the function H_∞, given any $\varepsilon > 0$ we can obtain a constant $M(\varepsilon) > 0$ such that for any $2 < p < 2^*$ and any $(u, v) \in E$ we have

$$|H_\infty(u, v)| \leq \frac{\varepsilon}{2}(u^2 + v^2) + M(\varepsilon)(u^2 + v^2)^{p/2}. \tag{2.5}$$

With this estimate, we can verify that (I1) is satisfied. For (I2), we take a constant $R > 0$ and consider the function φ_R, a solution for the eigenvalue problem

$$\begin{gathered} -\Delta\varphi = \alpha\varphi \quad \text{on } B_R \\ \varphi \equiv 0 \quad \text{on } \partial B_R. \end{gathered}$$

where $\alpha = \alpha(R)$ is a real constant. Given $\varepsilon > 0$ there exists $R_0(\varepsilon)$ such that if $R > R_0(\varepsilon)$ then $\alpha < \varepsilon$, since α goes to zero when R approaches infinity. We also have that φ_R belongs to $H^1_{\rm rad}(\mathbb{R}^N)$. We consider now the function $\bar{\varphi}_R$, defined by

$$\bar{\varphi}_R = \begin{cases} \varphi_R & \text{on } \bar{B}_R \\ 0 & \text{on } \mathbb{R}^N \setminus \bar{B}_R \end{cases}$$

Hence, for $t > 0$, we consider $u_R := t\bar{\varphi}_R$ and we can verify that $I(u_R, 0) < 0$, with $\|(u_R, 0)\| = t\|\bar{\varphi}_R\| > \rho$. Taking $e = (u_R, 0)$, we obtain (I2). $\square$

With this at hand, we will work with the energy level c_r, given by

$$c_r = \inf_{\gamma \in \Gamma_r} \max_{0 \leq t \leq 1} I_\infty(\gamma(t)),$$

where $\Gamma_r := \{\gamma \in C([0,1], E_{\rm rad}); \gamma(0) = 0 \text{ and } I_\infty(\gamma(1)) < 0\}$.

Now, we begin to prove that the functional I_∞ satisfies the Cerami condition (Ce). This proof will be presented in the following two lemmas:

Lemma 2.6. *Let* $(z_n) = (u_n, v_n) \subset E_{\rm rad}$ *be a bounded sequence such that*

$$|I_\infty(z_n)| \leq M \quad \textit{and} \quad \|I'_\infty(z_n)\|(1 + \|z_n\|) \to 0.$$

Then there exists $z \in E_{\rm rad}$ *such that* $\|z_n - z\| \to 0$.

Proof. In these conditions, we have $I'_\infty(z_n) \to 0$ and, up to subsequences:

(a) $u_n \rightharpoonup u, v_n \rightharpoonup v$ on $H^1_{\rm rad}(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$;

(b) $u_n \to u, v_n \to v$ on $L^p(\mathbb{R}^N)$;

(c) $u_n \to u, v_n \to v$ on $L^q_{loc}(\mathbb{R}^N)$;

by the compact immersions $H^1_{\rm rad}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $2 < p < 2^*$ and $H^1(\mathbb{R}^N) \hookrightarrow L^q_{loc}(\mathbb{R}^N)$, for $1 \leq q < 2^*$. We have that

$$I'_\infty(z_n)z_n \to 0, \tag{2.6}$$

since $|I'_\infty(z_n)z_n| \leq \|I'_\infty(z_n)\|\|z_n\| \to 0$, and by hypotheses (z_n) is a bounded sequence which satisfies $I'_\infty(z_n) \to 0$. Also, since (z_n) is a bounded sequence, it has a weak limit, indicated here by z. As $\frac{t}{1+st} < \frac{1}{s}$, (a) and (c) immediately imply that z is a critical point of I_∞.

We observe that

$$\|z_n - z\|^2 = \langle z_n - z, z_n - z\rangle = \|z_n\|^2 - 2\langle z_n, z\rangle + \|z\|^2. \tag{2.7}$$

From (2.6) it follows that

$$\begin{aligned}
o_n(1) &= I'_\infty(z_n)(z_n - z)\\
&= I'_\infty(z_n)z_n - I'_\infty(z_n)z\\
&= \int_{\mathbb{R}^N} \nabla z_n \nabla z_n + z_n z_n dx - \int_{\mathbb{R}^N} \nabla F_\infty(z_n) z_n dx\\
&\quad - \int_{\mathbb{R}^N} \nabla z_n \nabla z + z_n z dx + \int_{\mathbb{R}^N} \nabla F_\infty(z_n) z dx\\
&= \|z_n\|^2 - \langle z_n, z\rangle - \int_{\mathbb{R}^N} \nabla F_\infty(z_n)(z_n - z)dx,
\end{aligned}$$

and hence

$$\|z_n\|^2 = o_n(1) + \langle z_n, z\rangle + \int_{\mathbb{R}^N} \nabla F_\infty(z_n)(z_n - z)dx. \tag{2.8}$$

Since $I'_\infty(z) = 0$, we obtain

$$\begin{aligned}
o_n(1) &= I'_\infty(z)(z_n - z)\\
&= \int_{\mathbb{R}^N} \nabla z \nabla z_n + z z_n dx - \int_{\mathbb{R}^N} \nabla F_\infty(z)(z_n - z)dx - \int_{\mathbb{R}^N} \nabla z \nabla z + z z dx\\
&= \langle z_n, z\rangle - \|z\|^2 - \int_{\mathbb{R}^N} \nabla F_\infty(z)(z_n - z)dx
\end{aligned}$$

and therefore

$$\|z\|^2 = \langle z_n, z\rangle - \int_{\mathbb{R}^N} \nabla F_\infty(z)(z_n - z)dx - o_n(1). \tag{2.9}$$

Replacing the expressions (2.8) and (2.9) into (2.7), we obtain

$$\begin{aligned}
\|z_n - z\|^2 &= \Big[o_n(1) + \langle z_n, z\rangle + \int_{\mathbb{R}^N} \nabla F_\infty(z_n)(z_n - z)dx\Big] - 2\langle z_n, z\rangle\\
&\quad + \Big[\langle z_n, z\rangle - \int_{\mathbb{R}^N} \nabla F_\infty(z)(z_n - z)dx - o_n(1)\Big]\\
&= o_n(1) + \int_{\mathbb{R}^N} \nabla F_\infty(z_n)(z_n - z)dx - \int_{\mathbb{R}^N} \nabla F_\infty(z)(z_n - z)dx.
\end{aligned}$$

Therefore,

$$\|z_n - z\|^2 = o_n(1) + \int_{\mathbb{R}^N} \nabla F_\infty(z_n)(z_n - z)dx - \int_{\mathbb{R}^N} \nabla F_\infty(z)(z_n - z)dx. \tag{2.10}$$

Since $\nabla F_\infty(z) \in L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ and $z_n \rightharpoonup z$, it follows that

$$\int_{\mathbb{R}^N} \nabla F_\infty(z)(z_n - z)dx \to 0. \tag{2.11}$$

From (2.10) and (2.11), we obtain

$$\|z_n - z\|^2 = o_n(1) + \int_{\mathbb{R}^N} \nabla F_\infty(z_n)(z_n - z)dx. \tag{2.12}$$

On the other hand,

$$\begin{aligned}
\Big|\int_{\mathbb{R}^N}\nabla F_\infty(z_n)(z_n-z)dx\Big| &\le \int_{\mathbb{R}^N}\Big|\nabla F_\infty(z_n)(z_n-z)\Big|dx\\
&= \int_{\mathbb{R}^N}\Big|\frac{{u_n}^2+{v_n}^2}{1+s({u_n}^2+{v_n}^2)}u_n+\lambda v_n\Big||u_n-u|dx\\
&\quad+\int_{\mathbb{R}^N}\Big|\frac{{u_n}^2+{v_n}^2}{1+s({u_n}^2+{v_n}^2)}v_n+\lambda u_n\Big||v_n-v\|dx\\
&\le \int_{\mathbb{R}^N}\Big|\frac{{u_n}^2+{v_n}^2}{1+s({u_n}^2+{v_n}^2)}\Big||u_n|\,|u_n-u|dx\\
&\quad+\int_{\mathbb{R}^N}\Big|\frac{{u_n}^2+{v_n}^2}{1+s({u_n}^2+{v_n}^2)}\Big||v_n|\,|v_n-v|dx\\
&\quad+\lambda\int_{\mathbb{R}^N}|u_n|\,|v_n-v|+|v_n|\,|u_n-u|dx\\
&\le \int_{\mathbb{R}^N}\Big(\frac{u_n^2}{1+su_n^2}+\frac{v_n^2}{1+s{v_n}^2}\Big)|u_n|\,|u_n-u|dx\\
&\quad+\int_{\mathbb{R}^N}\Big(\frac{v_n^2}{1+s{v_n}^2}+\frac{u_n^2}{1+s{u_n}^2}\Big)|v_n||v_n-v|dx\\
&\quad+\lambda\int_{\mathbb{R}^N}\big(|u_n||v_n-v|+|v_n||u_n-u|\big)dx.
\end{aligned}$$

We will use [31, Lemma 2.2], which states that for any q such that $0\le q\le 2$ there exists $C=C(q)$ such that $\frac{w^2}{1+sw^2}\le C(q)|w|^q$, for all $w\in\mathbb{R}$. Hence, taking $0<q<2$, we obtain

$$\begin{aligned}
\Big|\int_{\mathbb{R}^N}\nabla F_\infty(z_n)(z_n-z)dx\Big| &\le \int_{\mathbb{R}^N}\left(C|u_n|^q+C|v_n|^q\right)|u_n||u_n-u|dx\\
&\quad+\int_{\mathbb{R}^N}\left(C|v_n|^q+C|u_n|^q\right)|v_n||v_n-v|dx\\
&\quad+\lambda\int_{\mathbb{R}^N}\left(|u_n||v_n-v|+|v_n||u_n-u|\right)dx.
\end{aligned}$$

Considering the first integral on the right hand side

$$C\int_{\mathbb{R}^N}|u_n|^q|u_n||u_n-u|+|v_n|^q|u_n||u_n-u|dx,$$

we can apply Hölder's inequality with $p'=q+2$ and $p=\frac{p'}{1+q}$ in the first term to obtain

$$\begin{aligned}
\int_{\mathbb{R}^N}|u_n|^{1+q}|u_n-u|dx &\le \Big[\int_{\mathbb{R}^N}|u_n|^{p'}dx\Big]^{\frac{1+q}{p'}}\Big[\int_{\mathbb{R}^N}|u_n-u|^{p'}dx\Big]^{1/p'}\\
&=\|u_n-u\|_{p'}\|u_n\|_{p'}^{1+q},
\end{aligned}$$

which converges to zero since $u_n\to u$ on $L^p(\mathbb{R}^N)$, $2<p<2^*$ and $\|u_n\|_{p'}\le\tilde{C}\|u_n\|<M$. For the other term, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}|v_n|^q|u_n||u_n-u|dx\\
&\le\Big[\int_{\mathbb{R}^N}|v_n|^{p'}dx\Big]^{q/p'}\Big[\int_{\mathbb{R}^N}|u_n|^{p'}dx\Big]^{1/p'}\Big[\int_{\mathbb{R}^N}|u_n-u|^{p'}dx\Big]^{1/p'}
\end{aligned}$$

$$= C\|v_n\|_{p'}^q \|u_n\|_{p'} \|u_n - u\|_{p'}$$

which, by the same argument, converges to zero, but here, we used Hölder's inequality with $1/p' + 1/p' + q/p' = 1$, since $p' = q + 2$. In the same way, we can show that

$$\int_{\mathbb{R}^N} (|v_n|^q + |u_n|^q) |v_n||v_n - v| dx \to 0.$$

Therefore, we are left with

$$\Big| \int_{\mathbb{R}^N} \nabla F_\infty(z_n)(z_n - z)\Big| dx \leq \lambda \int_{\mathbb{R}^N} |u_n||v_n - v| + |v_n||u_n - u| dx + o_n(1).$$

Now, in the expression (2.12), we have

$$\begin{aligned} \|z_n - z\|^2 &\leq o_n(1) + \lambda \int_{\mathbb{R}^N} |u_n||v_n - v| + |v_n||u_n - u| dx \\ &\leq o_n(1) + \lambda \int_{\mathbb{R}^N} |u_n - u||v_n - v| + |u||v_n - v| dx \\ &\quad + \lambda \int_{\mathbb{R}^N} |v_n - v||u_n - u| + |v||u_n - u| dx \end{aligned}$$

Since $|\int_{\mathbb{R}^N} u(v_n - v) dx|$ and $|\int_{\mathbb{R}^N} v(u_n - u) dx|$ converge to zero, we are left with

$$\begin{aligned} \|z_n - z\|^2 &\leq o_n(1) + 2\lambda \int_{\mathbb{R}^N} |u_n - u||v_n - v| dx \\ &\leq o_n(1) + \lambda \int_{\mathbb{R}^N} |u_n - u|^2 + |v_n - v|^2 dx \\ &= o_n(1) + \lambda \|u_n - u\|_2^2 + \lambda \|v_n - v\|_2^2 \\ &= o_n(1) + \lambda \|z_n - z\|_2^2 \\ &\leq o_n(1) + \lambda \|z_n - z\|^2. \end{aligned}$$

Therefore,

$$\|z_n - z\|^2 (1 - \lambda) \leq o_n(1),$$

and $z_n \to z$ on E_{rad}, since $0 < \lambda < 1$. □

Remark 2.7. We observe that such argument is only true on E_{rad}, by the use of the compact immersions.

Lemma 2.8. *Suppose $(z_n) \subset E_{\mathrm{rad}}$ is such that $I_\infty(z_n) \to c_r$ and $\|I'_\infty(z_n)\|(1 + \|z_n\|) \to 0$. Then (z_n) has a bounded subsequence.*

Proof. By contradiction, we suppose that $\|z_n\| \to \infty$. We define $\hat{z}_n := \frac{z_n}{\|z_n\|}$. Hence $(\hat{z}_n)$ is a bounded sequence with $\|\hat{z}_n\| = 1$ and, up to subsequences, $\hat{z}_n \rightharpoonup \hat{z}$. Therefore, one of the cases below must happen:
case 1:

$$\limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |\hat{z}_n|^2 dx > 0,$$

case 2:

$$\limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |\hat{z}_n|^2 dx = 0.$$

We will show that none of these cases can occur, obtaining a contradiction for each one. We will start from case 2, adapting for the system the ideas presented in [39] for the scalar case.

In the same way as presented in [30, Lemma 3.30], we obtain the following inequality, that holds for n sufficiently large and $t > 0$,

$$I_\infty(tz_n) \le \frac{1+t^2}{2n} + I_\infty(z_n),$$

which can be rewritten as

$$\frac{t^2}{2}\|z_n\|^2 \le \frac{1+t^2}{2n} + c_r + o_n(1) + \int_{\mathbb{R}^N} F_\infty(tz_n)dx \tag{2.13}$$

since $I_\infty(z_n) = c_r + o_n(1)$. We define $t_n := \frac{2}{\sqrt{1-\lambda}}\frac{\sqrt{c_r}}{\|z_n\|}$. We have that $t_n \to 0$ when $n \to \infty$ and, replacing t_n on (2.13), we obtain

$$\frac{2c_r}{1-\lambda} \le c_r + o_n(1) + \int_{\mathbb{R}^N} H_\infty(t_n z_n)dx + \frac{\lambda}{2}t_n^2\int_{\mathbb{R}^N} z_n^2 dx. \tag{2.14}$$

Considering the estimate (2.5) with $\varepsilon = \frac{1-\lambda}{4}$, we obtain

$$|H_\infty(z)| \le z^2\big(\frac{1-\lambda}{8}\big) + M(\varepsilon)z^p, 2 < p < 2^*.$$

Replacing this into (2.14), we obtain

$$\frac{2c_r}{1-\lambda} \le o_n(1) + c_r + \frac{c_r(1+3\lambda)}{2(1-\lambda)}\int_{\mathbb{R}^N} \hat{z_n}^2 dx + M\Big(\frac{2\sqrt{c_r}}{\sqrt{1-\lambda}}\Big)^p \int_{\mathbb{R}^N} |\hat{z_n}|^p dx.$$

However, if case 2 occurs, by Lions Lemma (Lemma 1.21 in [41]), we would have $\hat{z}_n \to 0$ on $L^p(\mathbb{R}^N), 2 < p < 2^*$, and hence

$$\int_{\mathbb{R}^N} |\hat{z}_n|^p dx \to 0.$$

Besides, since

$$\int_{\mathbb{R}^N} \hat{z}_n^2 dx \le \|\hat{z}_n\|^2 \le 1,$$

we would have

$$\frac{2c_r}{1-\lambda} \le o_n(1) + \frac{3c_r + c_r\lambda}{2(1-\lambda)},$$

and therefore $\frac{c_r}{2} \le o_n(1)$, which is a contradiction, since $c_r > 0$. Therefore, case 2 cannot occur.

Now, we suppose that case 1 occurs. Hence, there exists a sequence $n_j \to \infty$ such that

$$\int_{B_1(y)} |\hat{u}_{n_j}|^2 dx > \frac{\delta}{2} \quad \text{or} \quad \int_{B_1(y)} |\hat{v}_{n_j}|^2 dx > \frac{\delta}{2}.$$

Indeed, otherwise, it would exists an $n_0 \in \mathbb{N}$ such that if $n \ge n_0$, then

$$\int_{B_1(y)} |\hat{u}_{n_j}|^2 dx < \frac{\delta}{2} \quad \text{and} \quad \int_{B_1(y)} |\hat{u}_{n_j}|^2 dx < \frac{\delta}{2}, \forall n \ge n_0.$$

Therefore, we would have

$$\int_{B_1(y)} |\hat{z}_{n_j}|^2 dx < \delta,$$

which contradicts the hypotheses. Therefore, we will assume

$$\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\hat{u}_n|^2 dx = \frac{\delta}{2} > 0.$$

The argument would be the same in case we assume the similar hypotheses for the sequence $(\hat{v}_n)$.

Hence, if (y_n) is a sequence such that $|y_n| \to \infty$ and

$$\int_{B_1(y_n)} |\hat{u}_n|^2 dx > \delta/2,$$

considering that $\hat{u}_n(x+y_n) \rightharpoonup \bar{u}(x)$, we obtain

$$\int_{B_1(0)} |\hat{u}_n(x+y_n)|^2 dx > \delta/2,$$

and hence

$$\int_{B_1(0)} |\bar{u}(x)|^2 dx > \delta/2;$$

i.e., $\bar{u} \not\equiv 0$. Hence, there exists $\Omega \subset B_1(0)$, with $|\Omega| > 0$, where $|\Omega|$ denotes the Lebesgue measure of the set Ω, such that

$$0 < |\bar{u}(x)| = \lim_{n\to\infty} |\hat{u}_n(x+y_n)| = \lim_{n\to\infty} \frac{|u_n(x+y_n)|}{\|z_n\|}, \forall\, x \in \Omega.$$

Since we have that $\|z_n\| \to \infty$, then necessarily

$$u_n(x+y_n) \to \infty, \quad \forall x \in \Omega \subset B_1(0).$$

Hence, by condition (NQ) and Fatou's Lemma, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{1}{2} \nabla F_\infty(z_n(x+y_n))(z_n(x+y_n)) - F_\infty(z_n(x+y_n)) dx \\
&\geq \liminf_{n\to\infty} \int_{\Omega} \frac{1}{2} \nabla F_\infty(z_n(x+y_n))(z_n(x+y_n)) - F_\infty(z_n(x+y_n)) dx \\
&\geq \int_{\Omega} \liminf_{n\to\infty} \frac{1}{2} \nabla F_\infty(z_n(x+y_n))(z_n(x+y_n)) - F(z_n(x+y_n)) dx \\
&= +\infty.
\end{aligned} \tag{2.15}$$

On the other hand, we know that $|I'_\infty(z_n)z_n| \leq \|I'_\infty(z_n)\| \|z_n\| \to 0$, if $n \to \infty$ and hence, $I'_\infty(z_n)z_n = o_n(1)$. Therefore,

$$\int_{\mathbb{R}^N} \frac{1}{2} \nabla F_\infty(z_n)(z_n) - F_\infty(z_n) dx = I_\infty(z_n) - \frac{1}{2} I'_\infty(z_n)z_n \leq c_r + 1,$$

contradicting (2.15).

If $|y_n| \leq R$, $R > 1$, we obtain

$$\frac{\delta}{2} \leq \int_{B_1(0)} |\hat{u}_n(x+y_n)|^2 dx \leq \int_{B_{2R}(0)} |\hat{u}_n(x+y_n)|^2 dx,$$

and since $\hat{u}_n(x+y_n) \to \bar{u}$ on $B_{2R}(0)$, it follows that

$$\frac{\delta}{2} \leq \int_{B_1(0)} |\bar{u}(x)|^2 dx.$$

As in the previous case, there exists $\Omega \subset B_1(0)$, $|\Omega| > 0$ such that

$$\lim_{n\to\infty} \frac{|u_n(x+y_n)|}{\|z_n\|} = \lim_{n\to\infty} |\hat{u}_n(x+y_n)| = |\bar{u}(x)| \neq 0, \quad \forall x \in \Omega.$$

Therefore, the argument follows analogously to the case when $|y_n| \to \infty$. Hence, case 1 cannot occurs as well and the existence of bounded subsequences is guaranteed. $\square$

Remark 2.9. We observe that on the proof of the previous lemma is it not necessary that (z_n) is a sequence of radial functions.

Remark 2.10. We can show that the functional I_∞ satisfies the *Principle of symmetric criticality*, found as [41, Theorem 1.28]; i.e., if (u,v) is a critical point of I_∞ in $E_{\rm rad}$, then (u,v) is a critical point of I_∞ in E.

Now, we can prove the following result.

Lemma 2.11. *The system* (2.1) *has a nontrivial solution* $(u,v)\in E_{\rm rad}$.

Proof. By Lemma 2.5, the functional I_∞, defined on $E_{\rm rad}$, satisfies the geometric conditions of Theorem 2.4. Hence, by Ekeland's Variational Principle [20], it is guaranteed the existence of a Cerami sequence at level c_r. In Lemmas 2.6 and 2.8 we showed that the Cerami condition is satisfied at level c_r on $E_{\rm rad}$. Then, by Theorem 2.4, we obtain that c_r is a critical value of I_∞ and therefore the strong limit z of the Cerami sequence (z_n) is a critical point of I_∞ on $E_{\rm rad}$. Therefore, since $c_r>0$, $z=(u,v)$ is a nontrivial solution of (2.1) and this solution is radial, because it belongs to $E_{\rm rad}$. By Remark 2.10, this solution (u,v) found in $E_{\rm rad}$ is also a solution for (2.1) on E. $\square$

Now, we consider the functional I_∞ written as

$$I_\infty(u,v)=\frac{1}{2}\|(\nabla u,\nabla v)\|^2-\int_{\mathbb{R}^N}G_\infty(u,v)dx,$$

with

$$G_\infty(u,v)=-\frac{u^2+v^2}{2}+\frac{u^2+v^2}{2s}-\frac{1}{2s^2}\ln(1+s(u^2+v^2))+\lambda uv. \tag{2.16}$$

Lemma 2.12. *System* (2.1) *has a ground state solution.*

Proof. The proof is presented in [11], and we sketch it here for the sake of clearness. The function G_∞ satisfies the hypotheses of Theorem 2.1 of [11], and therefore there exists $(\hat u,\hat v)\in E$ such that

$$\frac{1}{2}\int_{\mathbb{R}^N}|\nabla\hat u|^2+|\nabla\hat v|^2dx=m \quad\text{and}\quad \int_{\mathbb{R}^N}G_\infty(\hat u,\hat v)dx=1,$$

where m is defined by

$$m:=\inf_{(u,v)\in E}\big\{\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2+|\nabla v|^2dx,\int_{\mathbb{R}^N}G_\infty(u,v)dx\geq 1\big\}.$$

Also, Theorem 2.2 of [11] states that after some appropriate scaling with $\theta>0$, the functions $\bar u(x):=\hat u(\frac{x}{\theta})$ and $\bar v(x):=\hat v(\frac{x}{\theta})$ are solutions of (2.1), with

$$0<I_\infty(\bar u,\bar v)\leq I_\infty(u,v),\forall\ (u,v)\in E.$$

Therefore, we have a ground state solution of (2.1); i.e.,

$$I_\infty(\bar u,\bar v)=m:=\inf\big\{I_\infty(u,v);(u,v)\in E\setminus\{(0,0)\}\text{ and }(u,v)\text{ solves (2.1)}\big\}.$$

$\square$

Remark 2.13. We do not know if this solution is radial, neither have information about its sign.

Remark 2.14. By Lemma 2.4 of [11], any solution of (2.1) on E belongs to the Pohozaev manifold:

$$\mathcal{P} := \big\{(u,v) \in E \setminus \{(0,0)\}; \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 dx = N \int_{\mathbb{R}^N} G_\infty(u,v)dx\big\} \tag{2.17}$$

Our next step will be to show that this solution $(\bar{u},\bar{v})$ obtained in Lemma 2.12 coincides with the solution belonging to $E_{\rm rad}$ obtained by the Mountain Pass Theorem, in Lemma 2.11. First, we will show that the solution $(\bar{u},\bar{v})$ coincides with a solution obtained by the Mountain Pass Theorem applied on the whole space E, not just on $E_{\rm rad}$. After this, we will prove that these solutions obtained by the Mountain Pass Theorem coincide.

For this, we will make use of these symbols:

$$c_\infty := \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} I_\infty(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1],E); \gamma(0) = 0, I_\infty(\gamma(1)) < 0\}$;

$$c_r := \inf_{\gamma\in\Gamma_r} \max_{0\le t\le 1} I_\infty(\gamma(t)),$$

where $\Gamma_r := \{\gamma \in C([0,1],E_{\rm rad}); \gamma(0) = 0, I_\infty(\gamma(1)) < 0\}$. We observe that the level c_∞ exists, by the geometry of the functional I_∞, but we cannot verify the Cerami condition for this level on E, only on $E_{\rm rad}$.

Lemma 2.15. $m = c_\infty$.

Proof. As presented in [25, Lemma 2.1], there exists a path $\gamma \in \Gamma$ such that the ground state solution $\bar{z} = (\bar{u},\bar{v})$ belongs to $\gamma([0,1])$ and

$$\max_{t\in[0,1]} I_\infty(\gamma(t)) = I_\infty(\bar{z}) = m.$$

Hence, we have that $c_\infty \le m$. On the other hand, given any $\gamma \in \Gamma$, we have that $\gamma([0,1]) \cap \mathcal{P} \neq \emptyset$ [25, Lemma 4.1]; i.e., there exists $t_0 \in (0,1]$ such that $\gamma(t_0) \in \mathcal{P}$. But, since $m = \inf_{z\in\mathcal{P}} I_\infty(z)$ [25, Lemma 3.1], we obtain

$$\max_{t\in[0,1]} I_\infty(\gamma(t)) \ge I_\infty(\gamma(t_0)) \ge \inf_{z\in\mathcal{P}} I_\infty(z) = m,$$

and hence $c_\infty \ge m$. Therefore, $m = c_\infty$. □

Remark 2.16. The ground state solution is the solution obtained by the Mountain Pass Theorem, if this solution exists.

Now we are ready to present the proof of Theorem 2.1:

Proof. By Lemmas 2.12 and 2.15 and Remark 2.14, we can guarantee the existence of a pair $(\bar{u},\bar{v}) \in \mathcal{P}$ satisfying

$$\int_{\mathbb{R}^N} G_\infty(\bar{u},\bar{v})dx > 0$$

and

$$I_\infty(\bar{u},\bar{v}) = c_\infty = m = \inf_{(u,v)\in\mathcal{P}} I_\infty(u,v).$$

Since $\bar{u}\bar{v} \le |\bar{u}||\bar{v}|$, we obtain

$$G_\infty(|\bar{u}|,|\bar{v}|) \ge G_\infty(\bar{u},\bar{v}) > 0.$$

Hence, by [31, Lemma 3.1], we can project $(|\bar{u}|, |\bar{v}|)$ onto $\mathcal{P}$; i.e., there exists $t_0 > 0$ such that

$$\Big(|\bar{u}(\frac{\cdot}{t_0})|, |\bar{v}(\frac{\cdot}{t_0})|\Big) \in \mathcal{P}.$$

Hence,

$$\begin{aligned} I_\infty\Big(|\bar{u}(\frac{\cdot}{t_0})|, |\bar{v}(\frac{\cdot}{t_0})|\Big) &= \frac{t_0^{N-2}}{2}\|(\nabla|\bar{u}|, \nabla|\bar{v}|)\|_2^2 - t_0^N \int_{\mathbb{R}^N} G_\infty(|\bar{u}|, |\bar{v}|)dx \\ &\le \frac{t_0^{N-2}}{2}\|(\nabla\bar{u}, \nabla\bar{v})\|_2^2 - t_0^N \int_{\mathbb{R}^N} G_\infty(\bar{u}, \bar{v})dx \\ &= \frac{1}{2}\|(\nabla\bar{u}(\frac{\cdot}{t_0}), \nabla\bar{v}(\frac{\cdot}{t_0}))\|_2^2 - \int_{\mathbb{R}^N} G_\infty(\bar{u}(\frac{\cdot}{t_0}), \bar{v}(\frac{\cdot}{t_0}))dx \\ &= I_\infty\Big(\bar{u}(\frac{\cdot}{t_0}), \bar{v}(\frac{\cdot}{t_0})\Big) \\ &\le I_\infty(\bar{u}, \bar{v}) \end{aligned}$$

since $(\bar{u}, \bar{v}) \in \mathcal{P}$ and $I_\infty(\bar{u}, \bar{v}) = \max_{t>0} I_\infty\Big(\bar{u}(\frac{\cdot}{t}), \bar{v}(\frac{\cdot}{t})\Big)$.

Also, since the infimum over $\mathcal{P}$ is attained, we have

$$I_\infty(\bar{u}, \bar{v}) = \min_{z\in\mathcal{P}} I_\infty(z) \le I_\infty\Big(|\bar{u}(\frac{\cdot}{t_0})|, |\bar{v}(\frac{\cdot}{t_0})|\Big) \le I_\infty(\bar{u}, \bar{v}).$$

Hence, $I_\infty\big(|(\bar{u}\frac{\cdot}{t_0})|, |\bar{v}(\frac{\cdot}{t_0})|\big) = m$; i.e., $\big(|\bar{u}(\frac{\cdot}{t_0})|, |\bar{v}(\frac{\cdot}{t_0})|\big)$ is a critical point of I_∞ onto $\mathcal{P}$. Since the Pohozaev manifold $\mathcal{P}$ is a natural constraint for the problem, (see [35]), we have that $\big(|\bar{u}(\frac{\cdot}{t_0})|, |\bar{v}(\frac{\cdot}{t_0})|\big)$ is also a critical point over E, and therefore a non-negative solution of (2.1). Applying the Maximum Principle [24, Theorem 3.5], for each entry u and v separately, we conclude that they are positive, and by Theorem 2 of [12], radial.

On Remark 2.16, we did not know if there exits a solution by the Mountain Pass Theorem but, since the positive ground state solution is radial, with $m = c_\infty$, and in Lemma 2.11 we obtained a radial solution for the problem on E, by the Mountain Pass Theorem, it follows that $c_\infty = c_r$ and the solutions are the same. □

3. The Radial System

Now we will modify the autonomous system in order to obtain a radial system, which still has the problem (2.1) as its limit problem. We will consider the following system in $\mathbb{R}^N$, for $N \ge 3$ and $0 < \lambda < 1$:

$$\begin{aligned} -\Delta u + u &= \frac{u^2 + v^2}{1 + (s + a(|x|))(u^2 + v^2)} u + \lambda v \\ -\Delta v + v &= \frac{u^2 + v^2}{1 + (s + a(|x|))(u^2 + v^2)} v + \lambda u. \end{aligned} \tag{3.1}$$

where a is a function satisfying the following conditions:

- (R1) $a : \mathbb{R}^N \to \mathbb{R}$, with $a(x) = a(|x|)$, i.e., a is a radial function;
- (R2) $\lim_{|x|\to\infty} a(x) = 0$;
- (R3) there exist constants $a_0, a_1 > 0$ such that $-a_0 < a(x) < a_1$ for all $x \in \mathbb{R}^N$, but still satisfying

$$0 < s - a_0 < s + a(x) < s + a_1 < 1.$$

Associated with this system is the functional

$$I_r(u,v) = \frac{1}{2}\|(u,v)\|^2 - \int_{\mathbb{R}^N} H_r(x,u,v)dx - \lambda \int_{\mathbb{R}^N} uvdx,$$

where

$$H_r(x,u,v) = \frac{u^2+v^2}{2(s+a(x))} - \frac{1}{2(s+a(x))^2}\ln(1+(s+a(x))(u^2+v^2)),$$

and we define

$$F_r(x,u,v) := H_r(x,u,v) + \lambda uv.$$

Remark 3.1. By condition (R3), there is not a clear relation between the functionals I_∞ and I_r, i.e., given any pair $(u,v) \in E$, we may have either $I_\infty(u,v) \le I_r(u,v)$ or $I_r(u,v) \le I_\infty(u,v)$, or even an oscillation between the previous two cases, when we change the pair (u,v). We will show in the next section that when the function a is no longer radial, we need to impose new conditions in order to have a fixed relation between the functionals related to the general problem and the autonomous limit problem and hence be able to obtain a solution.

Remark 3.2. We will work on $E_{\rm rad}$ again since by the Principle of symmetric criticality, any solution (u,v) in $E_{\rm rad}$ is a solution in E.

The proof of the following proposition is straightforward.

Proposition 3.3. *Consider the function*

$$L(t) := \frac{z}{2t} - \frac{1}{2t^2}\ln(1+tz),$$

where z is a positive constant and $t \in (0,\infty)$. Then L is a strictly decreasing function.

Lemma 3.4. *The functional I_r satisfies the geometric conditions of Theorem 2.4.*

Proof. The proof that (I1) holds is similar to the autonomous case, made in Lemma 2.5, since in this case we can obtain an estimate similar to (2.5). For (I2), we observe that from (R3) and Proposition 3.3, we have $I_r(u,v) \le I_1(u,v)$, where I_1 is the functional associated with the autonomous problem obtained by replacing s by $s+a_1$ in (2.1). As made in Lemma 2.5, we can find $e \in E_{\rm rad}$ such that $I_1(e,0) < 0$, and therefore, $I_r(e,0) < 0$. □

The next proposition, which can be easily proved, states how the function a affects the nonquadraticity condition (NQ):

Proposition 3.5. *Consider the function $Q(t) := \frac{1}{2}\nabla F_r(t,z)z - F_r(t,z)$, for $t > 0$ and a given $z = (u,v) \in E$. Then $Q(t)$ is a decreasing function.*

Remark 3.6. By the above proposition, the function $F_r(x,u,v)$ satisfies the nonquadraticity condition (NQ), since by $(R3)$, we have $s+a(x) < s+a_1$ and therefore $Q(s+a_1) \le Q(s+a(x))$, with $Q(s+a_1)$ satisfying (NQ) by the same argument for the function F_∞.

Remark 3.7. The Cerami condition is satisfied by the functional I_r at the min-max level d_r given by

$$d_r := \inf_{\gamma\in\Gamma_r} \max_{0\le t\le 1} I_r(\gamma(t)),$$

where $\Gamma_r := \{\gamma \in C([0,1], E_{\rm rad}); \gamma(0) = 0, I_r(\gamma(1)) < 0\}$. This is proved in the same way as in the previous Lemmas 2.6 and 2.8, by the use of the compact immersions, since we are working on $E_{\rm rad}$.

Theorem 3.8. *System* (3.1) *has a nontrivial radial solution* (u, v).

Proof. By Lemma 3.4, the functional I_r satisfies the geometric conditions of the Mountain Pass Theorem. By Ekeland's Variational Principle [20] the existence of a Cerami sequence (z_n) at level d_r is guaranteed. By Remark 3.7, the functional I_r satisfies the Cerami condition at this level. Therefore, the strong limit of this sequence, z, is a critical point of I_r, belonging to $E_{\rm rad}$, and is a weak solution of (3.1). Since $d_r > 0$, this solution is nontrivial. Again, by the principle of symmetric criticality [41], z is a critical point on E, and therefore (3.1) has a nontrivial solution on E. □

4. The general system

Now we will consider the system (1.3) in $\mathbb{R}^N$, for $N \geq 3$, with $0 < \lambda < 1$, $0 < s < 1$ and suppose the function a satisfies:

- (A1) $a : \mathbb{R}^N \to \mathbb{R}$, with $a(x) < 0$ for all $x \in \mathbb{R}^N$;
- (A2) $\lim_{|x|\to\infty} a(x) = 0$;
- (A3) there exists a constant $s_0 > 0$ such that $0 < s_0 < s + a(x) < s$ for all $x \in \mathbb{R}^N$.

The functional associated with this problem is

$$I(u,v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 + u^2 + v^2 dx - \int_{\mathbb{R}^N} H(x,u,v)dx - \lambda \int_{\mathbb{R}^N} uv\, dx,$$

where

$$H(x,u,v) = \frac{u^2+v^2}{2(s+a(x))} - \frac{1}{2(s+a(x))^2}\ln(1 + (s+a(x))(u^2+v^2)),$$

and we define

$$F(x,u,v) := H(x,u,v) + \lambda uv.$$

Remark 4.1. We want to consider here the more general case for the function a. Since the function a is no longer radial, we cannot make use of the Sobolev immersions in order to obtain the strong convergence of the Cerami sequence. To overcome this lack of compactness, we will use a concentration compactness result, introduced by Lions [27], in a better version known as 'Splitting', presented by Struwe [38], which describes the behaviour of a Cerami sequence at a level c. Therefore, on a specific interval $(0, c_\infty)$, we will have compactness. For this, we need condition (A2), which assures that problem (2.1) is the limit problem for problem (1.3). We also need to impose that $0 < s_0 < a(x) + s$ in (A3) to have some integrability conditions during the proof of the 'Splitting'. We will then show that the min-max level c, from the Mountain Pass Theorem, belongs to this interval, by the construction of a specific path, generated from the ground state solution of the problem (2.1), and for this we need conditions (A1) and (A3), which together guarantee the relation $I(u,v) \leq I_\infty(u,v)$ for all $(u,v) \in E$, which is essential for the construction of the path.

Remark 4.2. The functional I also satisfies the geometric conditions of the Mountain Pass Theorem, since by the structure of the function H we have, for any $2 < p < 2^*$, and any $(u, v) \in E$,

$$|H(x, u, v)| \leq \frac{\varepsilon}{2}(u^2 + v^2) + M(\varepsilon)(u^2 + v^2)^{p/2},$$

which proves that (I1) is satisfied. Now we consider $\bar{z}$, the radial positive ground state solution of (2.1). In the proof of Lemma 2.2 of [31], we see that there exists a $L > 0$ such that $I_\infty\left(\bar{z}(\frac{x}{L})\right) < 0$. We define $z_1(x) := \bar{z}(\frac{x}{L})$ and then we have $I(z_1) < I_\infty(z_1) < 0$, which proves (I2).

We define $z_0 = (0, 0)$ and then the min-max level of the Mountain Pass Theorem is given by

$$c := \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0, 1], E) : \gamma(0) = z_0; \gamma(1) = z_1\}$. With this, we can prove the following result.

Lemma 4.3. *There exists a sequence $(z_n) = (u_n, v_n) \subset E$ satisfying $I(z_n) \to c$ and $\|I'(z_n)\|(1 + \|z_n\|) \to 0$.*

Proof. The proof consists in applying the Ghoussoub-Preiss Theorem [21, Theorem 6] for the set $\mathcal{F} := \{z \in E; I_\infty(z) \geq 0\}$. □

Lemma 4.4. *Suppose $(z_n) \subset E$ is a sequence such that $I(z_n) \to c$ and $\|I'(z_n)\|(1+ \|z_n\|) \to 0$. Then (z_n) has a bounded subsequence.*

The proof of this lemma follows the same ideas of the proof of Lemma 2.8; i.e., Lions Lemma [41], the nonquadraticity condition (NQ) and Fatou Lemma.

Lemma 4.5 (Splitting)**.** *Consider $z_n = (u_n, v_n) \subset E$ a bounded sequence such that $I(z_n) \to c$ and $\|I'(z_n)\|(1 + \|z_n\|) \to 0$. Then, replacing (z_n) by a subsequence, if necessary, there exists a solution $\hat{z} = (\hat{u}, \hat{v})$ of (1.3), a number $k \in \mathbb{N} \cup \{0\}$, k pairs of functions $(u_1, v_1), \dots, (u_k, v_k)$ and k sequences of points $\{y_n^j\}$, $y_n^j \in \mathbb{R}^N, 1 \leq j \leq k$, satisfying*

(i) *$(u_n, v_n) \to (\hat{u}, \hat{v})$ on E or*
(ii) *$|y_n^j| \to \infty$, $|y_n^j - y_n^i| \to \infty$, if $j \neq i$;*
(iii) *$(u_n, v_n) - \sum_{j=1}^k \left(u_j(x - y_n^j), v_j(x - y_n^j)\right) \to (\hat{u}, \hat{v})$ on E;*
(iv) *$I(z_n) \to I(\hat{z}) + \sum_{j=1}^k I_\infty(z_j)$;*
(v) *$z_j = (u_j, v_j)$ are nontrivial weak solutions of (2.1).*

The proof of this lemma is standard nowadays and is a version (for systems) of the concentration compactness of Lions [27] and presented in [38]. The main ingredients are Brezis-Lieb Lemma [10] and Lions Lemma [41]. We refer to [4], where a similar lemma was proved for a coupled system of nonautonomous equations, with subcritical nonlinearity.

Corollary 4.6. *The functional I satisfies $(Ce)_c$ for all c such that $0 < c < c_\infty$.*

Proof. We have that $I_\infty(u^j, v^j) \geq c_\infty$ for all pair (u^j, v^j) of nontrivial solution of (2.1). Taking (u_n, v_n) a Cerami sequence at level $\beta > 0$ such that $\beta < c_\infty$ and applying it to Lemma 4.5, we obtain that $k = 0$, since $I(u_n, v_n) < c_\infty$, and therefore $(u_n, v_n) \to (\hat{u}, \hat{v})$ on E. □

Lemma 4.7. $0 < c < c_\infty$.

Proof. Recall that

$$G(u,v) = -\frac{u^2+v^2}{2} + \frac{u^2+v^2}{2(s+a(x))} - \frac{1}{2(s+a(x))^2}\ln(1+(s+a(x))(u^2+v^2)) + \lambda uv,$$

and

$$G_\infty(u,v) = -\frac{u^2+v^2}{2} + \frac{u^2+v^2}{2s} - \frac{1}{2s^2}\ln(1+s(u^2+v^2)) + \lambda uv.$$

Hence the functions G and G_∞ differ by the function $L(t) = \frac{z}{2t} - \frac{1}{2t^2}\ln(1+tz)$, with $t > 0$. But, by Proposition 3.3, $L(t)$ is a strictly decreasing function, and since $s + a(x) < s, \forall x \in \mathbb{R}^N$, we obtain $L(s) < L(s + a(x))$, which implies

$$G_\infty(z) < G(z), \forall z \in E \setminus \{(0,0)\}.$$

Consider $\bar{z}$, the radial positive ground state solution of (2.1), and define $z_y(x) := \bar{z}(x-y)$, for some fix $y \in \mathbb{R}^N$. Then

$$\begin{aligned}
\int_{\mathbb{R}^N} G(z_y(x))dx &> \int_{\mathbb{R}^N} G_\infty(z_y(x))dx \\
&= \int_{\mathbb{R}^N} G_\infty(\bar{z}(x-y))dx \\
&= \int_{\mathbb{R}^N} G_\infty(\bar{z}(x))dx > 0,
\end{aligned}$$

where we used the translation invariance of integrals and the fact that, since $\bar{z}$ is solution of (2.1), it follows that $\bar{z}$ satisfies the Pohozaev identity and hence $\int_{\mathbb{R}^N} G_\infty(\bar{z}(x))dx > 0$.

Since $\int_{\mathbb{R}^N} G(z_y)dx > 0$, it follows from the proof of [31, Lemma 3.1] that there exists $0 \le t_y \le 1$ such that

$$\max_{0\le t\le 1} I\Big(z_y\big(\frac{x}{t}\big)\Big) = I\Big(z_y\big(\frac{x}{t_y}\big)\Big) = I\Big(\bar{z}\big(\frac{x-y}{t_y}\big)\Big).$$

But

$$\begin{aligned}
I\Big(z_y\big(\frac{x}{t_y}\big)\Big) &< I_\infty\Big(z_y\big(\frac{x}{t_y}\big)\Big) \\
&= I_\infty\Big(\bar{z}\big(\frac{x-y}{t_y}\big)\Big) \\
&= I_\infty\Big(\bar{z}\big(\frac{x}{t_y} - \frac{y}{t_y}\big)\Big) \\
&= I_\infty\Big(\bar{z}\big(\frac{x}{t_y}\big)\Big) \\
&\le I_\infty\Big(\bar{z}\big(\frac{x}{1}\big)\Big) = c_\infty
\end{aligned}$$

where we used again the translation invariance of integrals and the fact that $\bar{z}$ is the ground state solution of (2.1), hence the maximum on the path $\bar{z}(x/t)$ is attained on $t = 1$. We need to construct a path $\gamma \in \Gamma$ such that

$$\max_{0\le t\le 1} I(\gamma(t)) = I\Big(z_y\big(\frac{x}{t_y}\big)\Big) < c_\infty,$$

recalling that

$$\Gamma := \big\{\gamma \in C([0,1],E); \gamma(0) = 0, \gamma(1) = z_1(x) = \bar{z}\big(\frac{x}{L}\big)\big\}.$$

We also have that, when $t = L$,

$$I\left(z_y\big(\frac{x}{L}\big)\right) < I_\infty\left(z_y\big(\frac{x}{L}\big)\right) = I_\infty\left(\bar{z}\big(\frac{x}{L}\big)\right) = I_\infty(z_1(x)) < 0.$$

Therefore

$$I\left(z_y\big(\frac{x}{L}\big)\right) < 0, \quad \text{with } L > t_y.$$

Consider

$$\beta(t) := \bar{z}\left(\frac{x}{L}t + (1-t)\left(\frac{x}{L} - \frac{y}{L}\right)\right).$$

We have $\beta(0) = z_y\big(\frac{x}{L}\big)$ and

$$\beta(1) = \bar{z}\big(\frac{x}{L}\big) = z_1(x).$$

Hence, $\beta(t)$ is a path connecting $z_y\big(\frac{x}{L}\big)$ to $z_1(x)$. Besides

$$\begin{aligned}
I(\beta(t)) &= I\left(\bar{z}\left(\frac{x}{L}t + (1-t)\big(\frac{x}{L} - \frac{y}{L}\big)\right)\right) \\
&< I_\infty\left(\bar{z}\left(\frac{x}{L}t + (1-t)\left(\frac{x}{L} - \frac{y}{L}\right)\right)\right) \\
&= I_\infty\left(\bar{z}\left(\frac{x}{L} - \frac{y}{L}(1-t)\right)\right) \\
&= I_\infty\left(\bar{z}\big(\frac{x}{L}\big)\right) \\
&= I_\infty(z_1) < 0.
\end{aligned}$$

Therefore, the functional I is always negative over the path $\beta(t)$. Let $\tilde{\alpha}(t)$ be the path

$$\tilde{\alpha}(t) := \begin{cases} 0, & t = 0 \\ z_y\big(\frac{x}{t}\big), & 0 < t \le L \end{cases}$$

and consider $\alpha(t) := \tilde{\alpha}(Lt)$, a path connecting $z_0 = 0$ to $z_y\big(\frac{x}{L}\big)$, passing by $z_y\big(\frac{x}{t_y}\big)$, since $0 < t_y < L$. Hence, considering $\gamma(t)$ the composition between the paths $\alpha(t)$ and $\beta(t)$, we obtain $\gamma(t) \in \Gamma$ and

$$\max_{0 \le t \le 1} I(\gamma(t)) = I\Big(z_y\big(\frac{x}{t_y}\big)\Big) < c_\infty,$$

and therefore $c < c_\infty$. $\square$

Theorem 4.8. *Suppose* (A1)–(A3) *are satisfied. Then the system* (1.3) *has a nontrivial solution* $(u, v) \in E$.

Proof. By the previous results, we have the existence and boundness of a Cerami sequence at level c. By Corollary 4.6, the functional I satisfies the Cerami condition at level c, since $c < c_\infty$, as proved in Lemma 4.7. Therefore, we can apply Theorem 2.4 and guarantee the existence of a nontrivial solution for problem (1.3). $\square$

Remark 4.9. We believe the results presented here are also true for $N = 2$. However, the proofs for some of the results used in this paper are different from the $N \ge 3$ case (see [11, 25]). We did not consider the $N = 1$ case. We also want to observe that the condition $0 < \lambda < 1$ is not needed in the proof of the geometric conditions of Theorem 2.4. However, it is extremely necessary in other proofs, such as Lemma 2.6 and in the use of [12, Theorem 2], for example. With the use of the Pohozaev Identity, we can conclude that the autonomous system would have a solution only if the inequality $0 < \lambda + 2\big(\frac{1}{s} - 1\big)$ is satisfied, which explains our choice for s. Further studies are required in order to find the optimal interval of the parameters.

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